Sphere Packing

François Sigrist

Editor's Note: *The following article is the second of three reports by the winners of the French Museum competition. These articles are based on their winning entries and are intended to provide some ideas for exhibiting mathematics to a general audience.*

Introduction

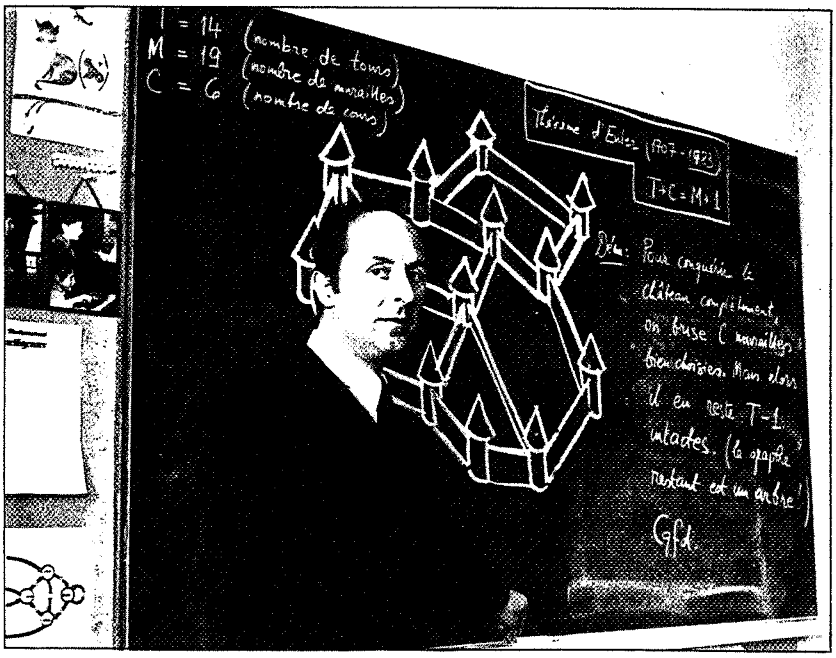
I was kindly asked by *The Mathematical Intelligencer* to discuss my entry in the French Museum competition. Before commenting, let me first make some preliminary remarks.

Mathematics shares with museums a common problem: how to attract visitors. (Of course, they ought to know in advance that they will face a challenge.) Some hints about a solution can be found in *The Intelligencer.*

John Ewing: Hilbert's first criterion for a good problem is that it can be explained "to the first person one meets in the street."
Wolfgang Haken: How can we expect the non-specialist to know that there are still open problems left in mathematics if we usually present mathematics as a closed subject?

To these statements, I wish to add a more controversial point. The basic facts of everyday life are Euclidean, but familiarity with geometry belongs less and less to the culture of, say, Mr. and Mrs. Everybody. On bad days, I say it's Bourbaki's fault; at other times, I think all mathematicians are responsible for this. How many of my fellow mathematicians correctly guess the density of a sphere inscribed in a cube? In my experience, accurate answers are rare. How many know that the volume of the n-dimensional unit ball first increases and then decreases very rapidly with n? Ironically, the computation has been done by every calculus student. How many know about "the problem of the 13 spheres"? This is a marvelous thriller, beginning at Oxford in the seventeenth century, for which the answer was found in 1953 by Schütte and van der Waerden. I recommend it.

These remarks are not intended to suggest that I



François Sigrist is shown proving a well-known theorem of Euler in an authentic eighteenth-century setting. (This particular proof was a tour de force.)

know why the professor can't teach. But I think we mathematicians face a "museumlike" situation; we do not always do our best to challenge our students in the right way. This problem is more important and deserves more of our energy than the "pure versus applied" controversy.

What We Do Not Know

Take a fixed, large number of equal size balls and pour them into a cubical container. By shaking gently, the situation can be rapidly stabilized, and the level in the container evidently enables us to compute the packing density. A repetition of the experiment shows that this density is remarkably stable. This fact is well-known in everyday life: Not only grocers but their customers, too, believe that the volume of a box containing one pound of coffee can be computed by "the first mathematician one meets in the street." Unfortunately, no mathematician on earth knows how!

We are not, at present, capable of predicting the packing density of a "random close packing". (The experimental value is not very different from $2/\pi$.) More distressing is the fact that we do not have a convenient mathematical description of what "random packing really is." The fact that the density is very stable is not much help; the experience of walking on wet sand shows that density has a local maximum and is therefore stationary. (Coxeter's chapter on the subject is required reading!)

Another side of this question shows that the stakes are high. Random close packing could plausibly give a description of the structure of liquids. For face-centered cubic metals (gold, silver, aluminum, copper, etc.) the ratio of the densities near the fusion point (solid to liquid) is approximately the same as the ratio of packing densities (face-centered cubic versus random). The metallic atoms being not absolutely spherical, there is some discrepancy. But physicists have performed high-precision experiments with rare gases whose atoms show good spherical symmetry and, the concordance is then much better.

The next problem arises when you are allowed to put the balls one by one into the container. Can you then beat the world record for density, which is held by face-centered cubic packing? Again nobody knows, and this situation is, quoting John Milnor, "scandalous." Instead of taking sides, I shall conclude this comment with two legendary statements.

H. S. M. Coxeter: It is conceivable that some irregular packing might be still denser.
C. A. Rogers: Many mathematicians believe, and all physicists know, that the density cannot exceed $\pi/\sqrt{18}$.

What We Have Learned

One of the stunning achievements of mathematics is the geometry of numbers. As an introduction to this subject, I warmly recommend Scharlau's chapter on Minkowski, as well as the relevant parts of Milnor-Husemoller's book. The topic has three stars in the Michelin Guide to Mathematics (*vaut le voyage*) and few visitors.

I shall take a very simple example (evidently known to Gauss) to illustrate the thread which led Minkowski to his celebrated convex body theorem. Suppose that a, b and c are integers satisfying $a > 0$ and $ac - b^2 = 1$. Then the quadratic equation $ax^2 + 2bxy + cy^2 = 1$ always has an integral solution. Warning: Do not try

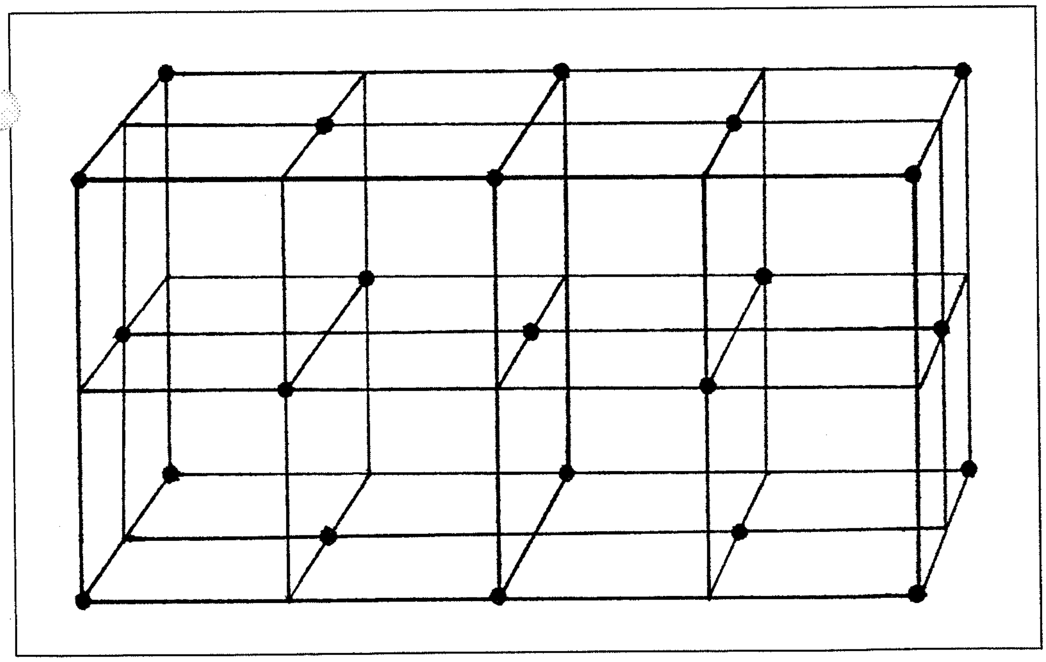


Figure 1. The face-centered cubic lattice in $\mathbf{R}^3$.

to prove it—read further. Minkowski's solution is as follows. Take the coordinate system in the Euclidean plane having $ax_1x_2 + bx_1y_2 + bx_2y_1 + cy_1y_2$ as a scalar product. The integral lattice of this coordinate system is then unimodular. Now call d the Euclidean distance to the origin of the next integral lattice point (x,y). Then $ax^2 + 2bxy + cy^2 = d^2$, and we therefore must show that $d^2 = 1$. For this, we center a sphere of radius $d/2$ at each lattice point. The result is a sphere packing, and its density is easily seen (in a good picture) to be $\pi d^2/4$. Now observe that d^2 is an integer smaller than $4/\pi$. Voilà!

Many readers may tell this story to their students. If they do, they should add the following pretty cascade of consequences.

Any positive divisor a of $b^2 + 1$ is a sum of two squares.
Proof: Take $c = (b^2 + 1)/a$, solve $ax^2 + 2bxy + cy^2 = 1$, giving $(ax + by)^2 + y^2 = a$.
Any prime $p \equiv 1 \pmod 4$ is a sum of two squares.
Proof: For such a prime, Wilson's theorem can be written $\{[(p - 1)/2]!\}^2 + 1 \equiv 0 \pmod p$.

Lattice packings correspond to positive definite quadratic forms, as indicated above. Their study was significantly boosted by the development of the geometry of numbers. In view of this fact, one has to wonder that 80 years earlier, in Chapter 266 of his *Disquisitiones Arithmeticae*, Gauss made this extraordinary remark (free translation): "For more than three variables, the study of forms has to be pursued by geometers, who will find there a very good opportunity for applying their skills." If anything can still be added to the memory of Gauss, this seems to me to be worth consideration.

A Mystery

Sphere packing problems can be formulated in any dimension. One of the most tantalizing is the following: Is the absolute record for density shared by a lattice packing? The answer is yes in dimension 2, and believed to be yes in dimension 3. But the physical argument seems to lose ground as the dimension increases.

From a (10,38,4) binary code, Leech and Sloane constructed in 1970 a 10-dimensional nonlattice packing denser than any presently known lattice packing. The situation "worsened" in 1978 when a (10,40,4) code was discovered by Best. But, since the maximal lattice density is known only up to dimension 8, the existence of a still denser lattice packing cannot be excluded. We are looking at what seems to be an abnormality of nature, but the possiblity remains that our patient is perfectly healthy.

The connection between coding theory and packings was first shown by John Leech, who constructed, from the famous (24,4096,8) Golay code, an incredibly dense 24-dimensional lattice packing (a unimodular integral quadratic form with a minimum of 4). Performed on nonlinear codes, his construction provides nonlattice

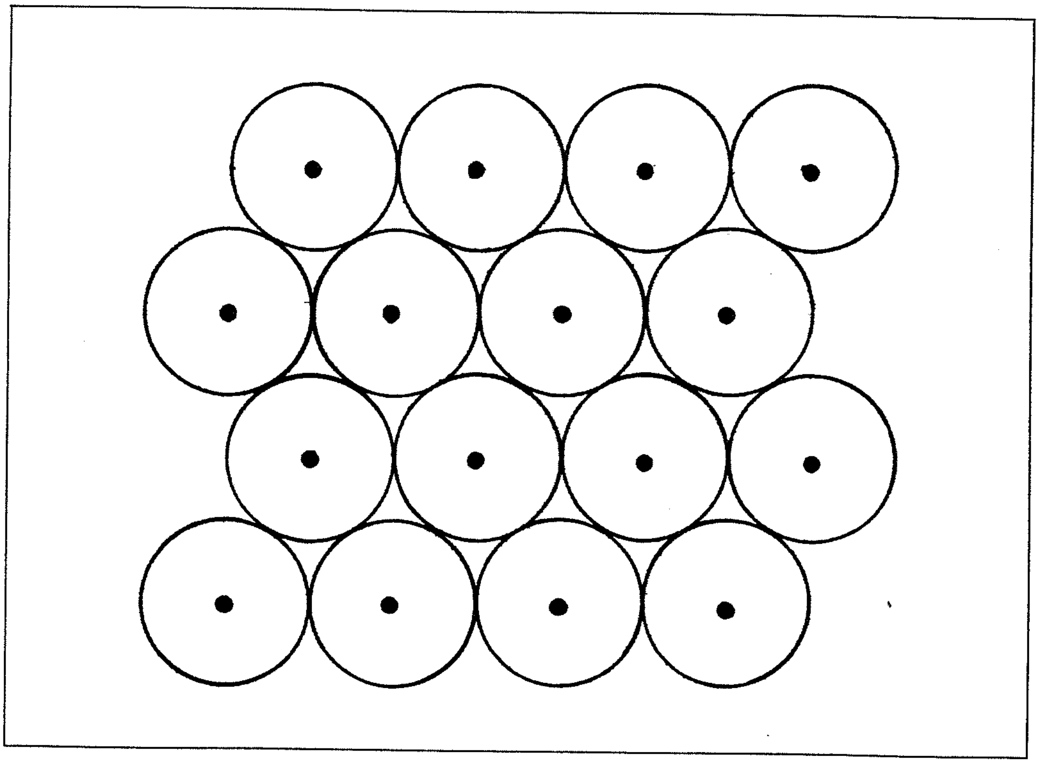


Figure 2. The familiar hexagonal packing in the plane.

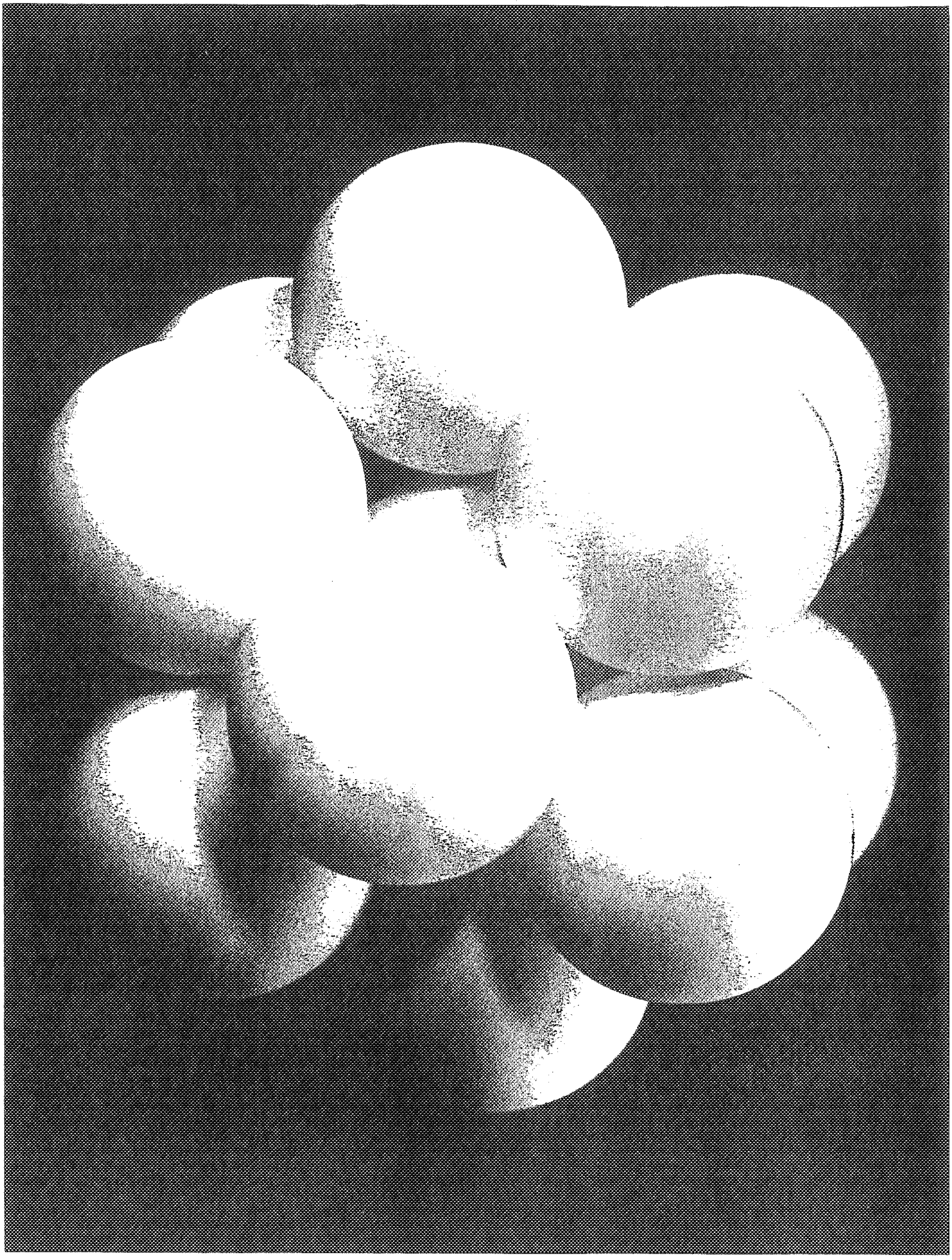


Figure 3. A fragment of the face-centered cubic lattice packing.

packings with, however, a rather high degree of symmetry. In this sense, one can consider the "criminal" 10-dimensional packings mentioned above as "geometrically almost innocent." But the mystery remains.

Final Comment

By measuring the packing density σ_n inside a regular n-dimensional simplex (that is, locally), one obtains an upper bound for the density of all n-dimensional packings. Although this sounds like a result of Euler, it was proved only in 1958 by C. A. Rogers, who deserves great credit. I cannot resist adding here a short story. Although its definition is totally elementary, Rogers' constant σ_n is extraordinarily difficult to compute in general (skeptics should try $n = 3$, barehanded). Moreover, accurate numerical values are hard to obtain, even with modern methods. But in 1959, Coxeter, Few, and Rogers were able to determine the exact value of σ_4: they found the ingredients in the complete works of the nineteenth-century Swiss geometer Ludwig Schläfli. There is at least one obvious reason for my liking this anecdote!

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