

Quaternionic-perfect forms in dimension 12

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1 General facts and credits

The group Q of unit quaternions (Hurwitz) in dimension 4 is attached to the form D_4 , which is the unique Q -perfect form in dimension 4. In dimension 8, it is rather easy to check that E_8 is the unique Q^2 -perfect form.

The situation is more complicated in dimension 12, where one finds 12 Q^3 -perfect forms. This list was known to me before appearance of the paper [BMS], and was not published at the time. A few years later, the complete computation (with orbits under the various automorphism groups) was done by my then PhD-student David-Olivier Jaquet-Chiffelle, again without publication. He deserves therefore also credit for this classification.

I am grateful to Achille Schuermann, who cross-checked the whole computation (2007 – 2008). He should also be part of this story. I shall give below my matrices, resp. his (together with his notation, which keeps track of Voronoi's algorithm), to allow comparison with his database.

2 The Q^3 -perfect forms

Forms are listed with decreasing density. Apart from JMAX and JMIN, which are laminated lattices [CS], the given names are my own invention. All are perfect in the ordinary sense.

No=1;

Name=JMAX;

Matrices (S vs Sch)

JMAX=

[4,2,2,2,0,-2,2,0,2,1,1,1;2,4,0,0,2,0,2,2,1,2,0,0;2,0,4,0,-2,-2,0,0,1,0,2,0;
2,0,0,4,0,-2,0,0,1,0,0,2;0,2,-2,0,4,2,2,2,0,0,0,0;-2,0,-2,-2,2,4,0,0,0,0,0,0;
2,2,0,0,2,0,4,0,0,0,0,0;0,2,0,0,2,0,0,4,0,0,0,0;2,1,1,1,0,0,0,0,4,2,2,2;
1,2,0,0,0,0,0,0,2,4,0,0;1,0,2,0,0,0,0,0,2,0,4,0;1,0,0,2,0,0,0,0,2,0,0,4];

f5=

[4,0,1,2,2,2,2,0,1,2,0,1;0,4,0,-2,2,0,0,2,0,0,2,0;1,0,4,-1,0,2,0,0,2,0,0,2;
2,-2,-1,4,0,1,2,-2,0,2,0,0;2,2,0,0,4,0,2,2,0,0,2,0;2,0,2,1,0,4,1,0,2,1,0,2;
2,0,0,2,2,1,4,0,1,0,2,1;0,2,0,-2,2,0,0,4,0,-2,0,0;1,0,2,0,0,2,1,0,4,-1,0,0;
2,0,0,2,0,1,0,-2,-1,4,0,1;0,2,0,0,2,0,2,0,0,0,4,0;1,0,2,0,0,2,1,0,0,1,0,4];

MINIMUM=4; DETERMINANT=1024; GAMMA=2.2449; KISSING NO=324;
NEIGHBOURS=["JMIN",[384];"QMAX",[384]];

No=2;
Name="JMIN";
Matrices (S vs Sch)
JMIN=

[4,2,2,2,2,0,2,1,0,0,0,1;2,4,0,0,2,2,1,1,0,0,-1,1;2,0,4,0,0,-1,2,0,0,1,0,0;
2,0,0,4,1,-1,0,2,-1,-1,0,0;2,2,0,1,4,2,2,2,2,1,1,1;0,2,-1,-1,2,4,0,0,1,2,0,0;
2,1,2,0,2,0,4,0,1,0,2,0;1,1,0,2,2,0,0,4,1,0,0,2;0,0,0,-1,2,1,1,1,4,2,2,2;
0,0,1,-1,1,2,0,0,2,4,0,0;0,-1,0,0,1,0,2,0,2,0,4,0;1,1,0,0,1,0,0,2,2,0,0,4];

f0=

[4,0,0,2,2,2,2,0,1,2,0,1;0,4,0,-2,2,0,0,2,-1,0,2,1;0,0,4,-2,0,2,-1,1,2,-1,-1,2;
2,-2,-2,4,0,0,2,-2,0,2,0,-1;2,2,0,0,4,0,2,2,0,0,2,1;2,0,2,0,0,4,0,0,2,1,-1,2;
2,0,-1,2,2,0,4,0,0,2,1;0,2,1,-2,2,0,0,4,0,-2,0,1;1,-1,2,0,0,2,0,0,4,-1,-1,0;
2,0,-1,2,0,1,0,-2,-1,4,0,0;0,2,-1,0,2,-1,2,0,-1,0,4,0;1,1,2,-1,1,2,1,1,0,0,0,4];

MINIMUM=4; DETERMINANT=1024; GAMMA=2.2449; KISSING NO=312;
NEIGHBOURS=["JMAX",[16];"JMIN",[384,128,128,128,128,96,16,2];"QMAX",[128,128,32];
"QMIN",[384,128,128];"S",[384];"T",[384,96]];

No=3;
Name="QMAX";
Matrices (S vs Sch)
QMAX=

[6,3,3,3,-2,-3,1,-1,0,0,0,2;3,6,0,0,1,-2,2,2,0,0,-2,2;3,0,6,0,-3,-2,-2,0,0,2,0,0;
3,0,0,6,-1,-2,0,-2,-2,-2,0,0;-2,1,-3,-1,6,3,3,3,0,-3,0,0;-3,-2,-2,-2,3,6,0,0,3,0,3,0;
1,2,-2,0,3,0,6,0,0,-3,0,3;-1,2,0,-2,3,0,0,6,0,0,-3,0;0,0,0,-2,0,3,0,0,6,3,3,3;
0,0,2,-2,-3,0,-3,0,3,6,0,0;0,-2,0,0,0,3,0,-3,3,0,6,0;2,2,0,0,0,0,3,0,3,0,0,6];

f3=

[10,0,0,5,5,5,5,1,2,5,-1,3;0,8,0,-5,4,0,-1,4,-3,1,4,2;0,0,8,-5,0,4,-2,3,4,-3,-2,4;
5,-5,-5,10,0,0,5,-5,0,5,-1,-2;5,4,0,0,8,0,5,4,-1,1,4,3;5,0,4,0,0,8,0,1,4,2,-3,4;
5,-1,-2,5,5,0,10,0,0,4,3;1,4,3,-5,4,1,0,8,0,-4,0,3;2,-3,4,0,-1,4,0,0,8,-3,-3,0;
5,1,-3,5,1,2,0,-4,-3,10,0,0;-1,4,-2,-1,4,-3,4,0,-3,0,8,0;3,2,4,-2,3,4,3,3,0,0,8];

MINIMUM=6; DETERMINANT=202500; GAMMA=2.1675; KISSING NO=228;
NEIGHBOURS=["JMAX",[1];"JMIN",[8,8,2];"QMAX",[2];"QMIN",[8];"S",[12]];

No=4;
Name="QMIN";
Matrices (S vs Sch)
QMIN=

[6,3,3,3,-2,-3,1,-1,1,0,1,3;3,6,0,0,1,-2,2,2,1,1,-2,3;3,0,6,0,-3,-2,-2,0,0,2,1,0;
3,0,0,6,-1,-2,0,-2,-2,-3,0,1;-2,1,-3,-1,6,3,3,3,0,-2,-1,0;-3,-2,-2,-2,3,6,0,0,2,0,2,-1;
1,2,-2,0,3,0,6,0,1,-2,0,3;-1,2,0,-2,3,0,0,6,0,1,-3,0;1,1,0,-2,0,2,1,0,6,3,3,3;
0,1,2,-3,-2,0,-2,1,3,6,0,0;1,-2,1,0,-1,2,0,-3,3,0,6,0;3,3,0,1,0,-1,3,0,3,0,0,6];

f1=
 [8,-1,-1,4,4,4,4,-1,2,4,0,1;-1,8,1,-5,4,1,0,4,-1,-1,4,3;-1,1,8,-5,0,4,-3,2,4,-2,-2,4;
 4,-5,-5,8,-1,-1,4,-5,-1,4,0,-3;4,4,0,-1,8,1,4,4,1,-1,4,2;4,1,4,-1,1,8,0,0,4,2,-1,4;
 4,0,-3,4,4,0,8,-1,-1,0,5,2;-1,4,2,-5,4,0,-1,8,1,-5,0,2;2,-1,4,-1,1,4,-1,1,8,-2,-2,0;
 4,-1,-2,4,-1,2,0,-5,-2,8,-1,-1;0,4,-2,0,4,-1,5,0,-2,-1,8,1;1,3,4,-3,2,4,2,2,0,-1,1,8];
 MINIMUM=6; DETERMINANT=202500; GAMMA=2.1675; KISSING NO=216;
 NEIGHBOURS=["JMIN",[18,6,6];"QMAX",[6];"QMIN",[9,6,6];"R",[3];"S",[18,9]];

No=5;
 Name="R";
 Matrices (S vs Sch)

R=
 [10,5,5,5,-2,-5,3,-1,0,-2,2,4;5,10,0,0,3,-2,4,4,2,0,-2,6;5,0,10,0,-5,-4,-2,0,-2,2,0,0;
 5,0,0,10,-1,-4,0,-2,-4,-6,0,0;-2,3,-5,-1,10,5,5,5,2,-4,0,2;-5,-2,-4,-4,5,10,0,0,6,2,4,0;
 3,4,-2,0,5,0,10,0,2,-4,2,6;-1,4,0,-2,5,0,0,10,0,0,-6,2;0,2,-2,-4,2,6,2,0,12,6,6,6;
 -2,0,2,-6,-4,2,-4,0,6,12,0,0;2,-2,0,0,0,4,2,-6,6,0,12,0;4,6,0,0,2,0,6,2,6,0,0,12];

f6=
 [12,-2,-2,6,6,6,6,-2,2,6,0,2;-2,14,1,-8,7,1,0,7,-2,-2,7,4;-2,1,14,-8,0,7,-4,3,7,-4,-3,7;
 6,-8,-8,12,-2,-2,6,-8,-2,6,0,-4;6,7,0,-2,14,1,6,7,1,-2,7,3;6,1,7,-2,1,14,0,0,7,2,-2,7;
 6,0,-4,6,6,0,12,-2,-2,0,8,4;-2,7,3,-8,7,0,-2,14,1,-8,0,3;2,-2,7,-2,1,7,-2,1,14,-4,-3,0;
 6,-2,-4,6,-2,2,0,-8,-4,12,-2,-2;0,7,-3,0,7,-2,8,0,-3,-2,14,1;2,4,7,-4,3,7,4,3,0,-2,1,14];

MINIMUM=10; DETERMINANT=107495424; GAMMA=2.1415; KISSING NO=180;
 NEIGHBOURS=["QMIN",[6];"S",[9]];

No=6; Name="S";
 Matrices (S vs Sch)

S=
 [8,4,4,4,-2,-4,2,-1,0,-1,1,3;4,8,0,0,2,-2,3,3,1,0,-2,4;4,0,8,0,-4,-3,-2,0,-1,2,0,0;
 4,0,0,8,-1,-3,0,-2,-3,-4,0,0;-2,2,-4,-1,8,4,4,4,1,-3,0,1;-4,-2,-3,-3,4,8,0,0,4,1,3,0;
 2,3,-2,0,4,0,8,0,1,-3,1,4;-1,3,0,-2,4,0,8,0,0,-4,1;0,1,-1,-3,1,4,1,0,8,4,4,4;
 -1,0,2,-4,-3,1,-3,0,4,8,0,0;1,-2,0,0,0,3,1,-4,4,0,8,0;3,4,0,0,1,0,4,1,4,0,0,8];

f2=
 [10,-1,-1,5,5,5,5,-1,2,5,0,2;-1,10,1,-6,5,1,0,5,-2,-1,5,3;-1,1,10,-6,0,5,-3,3,5,-3,-2,5;
 5,-6,-6,10,-1,-1,5,-6,-1,5,0,-3;5,5,0,-1,10,1,5,5,0,-1,5,3;5,1,5,-1,1,10,0,1,5,2,-2,5;
 5,0,-3,5,5,0,10,-1,-1,0,6,3;-1,5,3,-6,5,1,-1,10,1,-6,0,3;2,-2,5,-1,0,5,-1,1,10,-3,-3,0;
 5,-1,-3,5,-1,2,0,-6,-3,10,-1,-1;0,5,-2,0,5,-2,6,0,-3,-1,10,1;2,3,5,-3,3,5,3,3,0,-1,1,10];

MINIMUM=8; DETERMINANT=7496644; GAMMA=2.1389; KISSING NO=180;
 NEIGHBOURS=["JMIN",[4];"QMAX",[2];"QMIN",[4,2];"R",[1];"T",[2]];

No=7;
 Name="T";
 Matrices (S vs Sch)
 T=

[6,3,3,3,-1,-3,2,-1,0,-1,1,2;3,6,0,0,2,-1,3,2,1,0,-1,3;3,0,6,0,-3,-3,-1,0,-1,1,0,0;
 3,0,0,6,0,-2,0,-1,-2,-3,0,0;-1,2,-3,0,6,3,3,3,1,-2,0,1;-3,-1,-3,-2,3,6,0,0,3,1,2,0;
 2,3,-1,0,3,0,6,0,1,-2,1,3;-1,2,0,-1,3,0,0,6,0,0,-3,1;0,1,-1,-2,1,3,1,0,6,3,3,3;
 -1,0,1,-3,-2,1,-2,0,3,6,0,0;1,-1,0,0,0,2,1,-3,3,0,6,0;2,3,0,0,1,0,3,1,3,0,0,6];

f4=

[10,0,0,5,5,6,5,1,2,5,-1,3;0,8,2,-5,4,1,-1,4,-2,1,4,3;0,2,8,-6,1,4,-2,4,4,-3,-1,4;
 5,-5,-6,10,0,0,5,-5,-1,5,-1,-2;5,4,1,0,8,2,5,4,0,1,4,4;6,1,4,0,2,8,1,2,4,2,-2,4;
 5,-1,-2,5,5,1,10,0,0,0,4,4;1,4,4,-5,4,2,0,8,2,-4,0,3;2,-2,4,-1,0,4,0,2,8,-4,-3,0;
 5,1,-3,5,1,2,0,-4,-4,10,0,0;-1,4,-1,-1,4,-2,4,0,-3,0,8,2;3,3,4,-2,4,4,4,3,0,0,2,8];

MINIMUM=6; DETERMINANT=252144; GAMMA=2.1282; KISSING NO=180;
 NEIGHBOURS=["JMIN",[8,2];"S",[4];"U",[1]];

No=8;

Name="U";

Matrices (S vs Sch)

U=

[4,2,2,2,-1,-2,1,-1,0,-1,1,1;2,4,0,0,1,-1,2,1,1,0,0,2;2,0,4,0,-2,-2,-1,0,-1,0,0,0;
 2,0,0,4,0,-1,0,-1,-1,-2,0,0;-1,1,-2,0,4,2,2,2,1,-1,0,1;-2,-1,-2,-1,2,4,0,0,2,1,1,0;
 1,2,-1,0,2,0,4,0,1,-1,1,2;-1,1,0,-1,2,0,0,4,0,0,-2,1;0,1,-1,-1,1,2,1,0,4,2,2,2;
 -1,0,0,-2,-1,1,-1,0,2,4,0,0;1,0,0,0,0,1,1,-2,2,0,4,0;1,2,0,0,1,0,2,1,2,0,0,4];

f7=

[8,0,0,4,4,5,4,0,2,4,0,2;0,6,2,-4,3,1,0,3,-1,0,3,3;0,2,6,-5,1,3,-2,3,3,-2,-1,3;
 4,-4,-5,8,0,0,4,-4,-1,4,0,-2;4,3,1,0,6,2,4,3,1,0,3,3;5,1,3,0,2,6,1,1,3,2,-1,3;
 4,0,-2,4,4,1,8,0,0,0,4,3;0,3,3,-4,3,1,0,6,2,-4,0,2;2,-1,3,-1,1,3,0,2,6,-3,-2,0;
 4,0,-2,4,0,2,0,-4,-3,8,0,0;0,3,-1,0,3,-1,4,0,-2,0,6,2;2,3,3,-2,3,3,3,2,0,0,2,6];

MINIMUM=4; DETERMINANT=2500; GAMMA=2.0840; KISSING NO=180;
 NEIGHBOURS=["T",[15]];

References

- [BMS] A.-M. Bergé, J. Martinet, and F. Sigrist. *Une généralisation de l'algorithme de Voronoï pour les formes quadratiques*. Astérisque **209** (1992), 137-158.
- [CS] J.H. Conway and N.J.A. Sloane. *Sphere Packings, Lattices, and Groups*. Springer-Verlag, 1992.