

Data on cyclotomic quadratic forms :

Part I*

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1 Introduction

Denote by $\Phi_m(x) = 1 + c_1x + \dots + c_{n-1}x^{n-1} + x^n$ the m -th cyclotomic polynomial, of degree $n = \phi(m)$. Let $G \in GL(n, \mathbb{Z})$ be the matrix

$$\begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & -1 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & -c_1 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 & -c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & -c_{n-1} \end{pmatrix}$$

generating the standard n -dimensional representation of the cyclic group C_m . A *cyclotomic form* is a real quadratic form $q(x) = \sum a_{ij}x_i x_j$ in n variables, invariant under this action of C_m : the symmetric matrix $A = (a_{ij})$ satisfies $G^t A G = A$.

An easy computation shows that the matrix A has then constant diagonals ($a_{i+k, j+k} = a_{i, j}$), and is completely known from the first half of its first row : the space of cyclotomic forms is a linear subspace of $Sym_n(\mathbb{R})$ of dimension $n/2$. For convenience, we always shall give a cyclotomic matrix by its full first row.

A positive definite real quadratic form has

$$\text{a minimum } \underline{m} = \min(x^t A x | x \in \mathbb{Z}^n - \{0\}).$$

a set $S = \{\pm v_1, \pm v_2, \dots, \pm v_s\}$ of pairs of *minimal vectors* in \mathbb{Z}^n , satisfying $v_i^t A v_i = \underline{m}$. $2s$ is called the *kissing number* of the form.

*In relation with the paper [?]

a *Hermite invariant* $\gamma_n(A) = \underline{m} \cdot (\det A)^{-1/n}$.

There is a familiar dictionary linking positive definite forms and Euclidean lattices, via the Gram matrix. A Euclidean lattice has an associated sphere packing, and the packing density δ is related to the Hermite invariant by the relation $\delta = \text{Const}(n) \cdot \gamma_n(A)^{n/2}$: a dense form means a high Hermite invariant.

A quadratic form is *perfect* if it is the unique quadratic form having minimum \underline{m} and set S of minimal vectors. Similarly, a cyclotomic form is said to be *C_m -perfect* if it is the unique *cyclotomic form* with \underline{m} and S as before. A quadratic form is *extreme* if its density is locally maximal in the space of quadratic forms. A cyclotomic form is *C_m -extreme* if its density is locally maximal in the space of *cyclotomic forms*.

An extreme form is always perfect. For the converse, a theorem of Voronoï domain : oï requires a condition called eutaxy (cf [?]). These properties carry over verbatim to the case of C_m -forms [?]. A perfect form is always proportional to an integral form, as is a C_m -perfect form.

There is a second useful dictionary linking rational cyclotomic forms to the cyclotomic field $\mathbb{Q}(\zeta_m)$. If $x^t Ax$ is a cyclotomic form, there is a unique real element $\alpha \in \mathbb{Q}(\zeta_m)$ such that $x^t Ax = \text{Trace}(\alpha x \bar{x})$. The practical computation runs as follows : Write \vec{a} for the first column of the matrix A , and associate to $\alpha = \alpha_0 + \alpha_1 \zeta + \dots + \alpha_{n-1} \zeta^{n-1}$ the column $\vec{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})^t$. Then $\vec{a} = D \vec{\alpha}$ with matrix $D = (d_{ij}) = \text{Trace}(\zeta^{i-j})$. The determinant of D is the discriminant of the field $\mathbb{Q}(\zeta_m)$, and is therefore nonzero : A and α determine each other. If a cyclotomic form is positive definite, the element α is totally positive ($\alpha \gg 0$) : all its conjugates are positive. If now u is a unit, the cyclotomic form corresponding to $u \cdot \bar{u} \cdot \alpha$ is clearly C_m -equivalent to the original form. This method is extremely useful for finding G -equivalences.

Taking determinants in $x^t Ax = \text{Trace}(\alpha x \bar{x})$, one gets (using the fact that α is real) :

$$\det(A) = \det(D) \cdot \text{Norm}_{\mathbb{Q}(\zeta_m)/\mathbb{Q}}(\alpha) = \text{Disc}(\mathbb{Q}(\zeta_m)) \cdot (\text{Norm}_{\mathbb{Q}(\zeta_m+\zeta_m^{-1})/\mathbb{Q}}(\alpha))^2.$$

Using the detailed study of decomposition of primes in cyclotomic fields, E. Bayer-Fluckiger and J. Martinet obtained the following useful

Lemma 1 [?]. The determinant of an integral cyclotomic form is equal to $k(m)d^2$, where $k(m)$ is the square-free part of the discriminant of the cyclotomic field $\mathbb{Q}(\zeta_m)$. Further, if q is a prime power dividing d , and prime to m , then $q \equiv \pm 1 \pmod{m}$.

Up to equivalence, the classification of perfect quadratic forms is finite.

Voronoi's algorithm, devised in 1908 [?], constitutes a method of explicit complete classification. The densest quadratic form(s) in each dimension can be found with an entirely mechanical computation, the only problem being the gigantic computational complexity. The classification of seven-dimensional perfect forms (there are 33 of them), done in [?], required massive computer time. Explorations in dimension 8, along manageable paths of the algorithm, have been made by various authors ([?],[?],[?],[?]), bringing already more than 10000 perfect forms. It is practically impossible to send the algorithm through the form E_8 , knowing that already in dimension 7, the Voronoi domain : oi domain of E_7 has more than 70 million faces [?].

For cyclotomic forms, it was shown in [?] that the orthogonal projection of Voronoi's algorithm onto the (dual) space of cyclotomic forms provides an exhaustive classification. Later, it was proved [?] that the number of C_m -inequivalent C_m -perfect forms is finite. The practical advantage lies in the dimension of the space of forms ($n/2$ instead of $n(n+1)/2$), bringing the computing time to almost nothing for $n \leq 12$, and to "tolerable" values for the groups C_{17} and C_{32} ($n = 16$). There are 1344 C_{17} -perfect forms, and 4624 C_{32} -perfect forms. Both densest forms are completely new, and provide a new invariant for these cyclic groups.

As is customary in algebraic number theory, we take m to be either odd or divisible by 4.

For many values of m , the conjectured (or proved) densest form ($D_4, E_6, E_8, K_{12}, \Lambda_{16}$) appears as a C_m -form. This gives new matrix representations of these well-known forms, and is particularly suitable for further computation of their algebraic invariants.

The other values of m provide a new invariant for the group C_m , namely the Hermite invariant of the densest C_m -form, lower than the absolute Hermite constant γ_n . Curiously perhaps, the densest C_m -form is not always perfect (C_{16}, C_{36} , probably C_{32}). The proportion of perfect forms among C_m -perfect forms follows the general trend : very high if m is prime (0.99 for C_{17}), drastically low if C_m has lower-dimensional \mathbb{Q} -irreducible representations (0.25 for C_{32} -perfect forms).

The book by Jacques Martinet [?] systematically looks at almost all methods and ingredients used here. This reference is therefore compulsory, as is, evidently, the book by Conway and Sloane [?].

We refer to the paper [?] for a detailed description of Voronoi's algorithm.

2 Complete description of the C_{16} -case

A quick computation with $\Phi_{16}(x) = 1 + x^8$ shows that the 4-dimensional space of invariant C_{16} -forms is spanned by the 4 matrices whose first rows are

$$T1 = (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$$

$$T2 = (0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ -1)$$

$$T3 = (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ -1 \ 0)$$

$$T4 = (0 \ 0 \ 0 \ 1 \ 0 \ -1 \ 0 \ 0)$$

We abbreviate $aT_1 + bT_2 + cT_3 + dT_4$ by (a, b, c, d) . As a starting form for the algorithm, we take the matrix

$$(2, -1, 0, 0) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} = D_8.$$

The minimum is 2, and the determinant 4. There are $s = 56$ pairs of minimal vectors. The columns of the matrix below give one minimal vector in each of the 7 orbits under the group C_{16} :

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The dual C_{16} -Voronoi domain : oï domain is the set of matrices (a, b, c, d) whose evaluation on the minimal vectors is ≥ 0 . The seven equations are :

$$O1: a \geq 0$$

$$O2: 2a + 2b \geq 0$$

$$O3: 3a + 4b + 2c \geq 0$$

$$O4: 4a + 6b + 4c + 2d \geq 0$$

$$O5: 5a + 8b + 6c + 4d \geq 0$$

$$O6: 6a + 10b + 8c + 4d \geq 0$$

$$O7: 7a + 12b + 8c + 4d \geq 0$$

The quadratic form D_8 is known to be extreme, hence perfect, hence C_{16} -perfect. Here is, for completeness, the test of C_{16} -perfection : The system of equations

$$O1 = O2 = O3 = O4 = O5 = O6 = O7 = 2$$

has a unique solution $(a, b, c, d) = (2, -1, 0, 0)$.

Observe here that the equation $O4$ is the arithmetic mean of the 6 others. This means that the 4-dimensional Voronoï domain is the convex hull of the normals to the 6 other half-spaces, the edge $O4$ being interior to the domain. Intersecting with an appropriate hyperplane furnishes an octahedron, with opposite pairs of vertices $(1, 7)$, $(2, 6)$ and $(3, 5)$. A routine computation gives then the faces and the neighbouring forms :

	Vertices	Face vector	Neighbouring form	s	Determinant
$F1$	1, 2, 3	$(0, 0, 0, 1)$	$(2, -1, 0, 1/2)$	40	$289/64 = 17^2 \cdot 2^{-6}$
$F2$	1, 2, 5	$(0, 0, 2, -3)$	$(2, -1, 1/2, -3/4)$	40	$58081/16384 = 241^2 \cdot 2^{-14}$
$F3$	1, 3, 6	$(0, 2, -4, 3)$	$(2, -1/2, -1, 3/4)$	40	$58081/16384 = 241^2 \cdot 2^{-14}$
$F4$	1, 5, 6	$(0, 2, -2, -1)$	$(2, 0, -1, -1/2)$	40	$289/64 = 17^2 \cdot 2^{-6}$
$F5$	2, 3, 7	$(4, -4, 2, 1)$	$(3, -2, 1/2, 1/4)$	40	$58081/16384 = 241^2 \cdot 2^{-14}$
$F6$	2, 5, 7	$(4, -4, 4, -3)$	$(4, -3, 2, -3/2)$	40	$289/64 = 17^2 \cdot 2^{-6}$
$F7$	3, 6, 7	$(4, -2, -2, 3)$	$(4, -2, -1, 3/2)$	40	$289/64 = 17^2 \cdot 2^{-6}$
$F8$	5, 6, 7	$(4, -2, 0, -1)$	$(3, -3/2, 0, -1/4)$	40	$58081/16384 = 241^2 \cdot 2^{-14}$

The tools for finding G -equivalences are : reduction, identification $x^t Ax = \text{Trace}(\alpha x \bar{x})$, and action of the Galois group.

A cyclotomic form is said to be *reduced* if its matrix has the minimum of the form on the main diagonal. The example of the form $F8 = (3, -3/2, 0, -1/4)$ can illustrate a reduction method. Take a minimal vector v , and build the

matrix S whose columns are $v, Gv, G^2v, \dots, G^{m-1}v$. Then the matrix S^tAS is cyclotomic, reduced, and is therefore the matrix of a cyclotomic sublattice. In our example, the two minimal vectors $(1, 2, 2, 2, 2, 2, 1, 0)^t$ and $(1, 1, 1, 1, 1, 0, 0, 0)^t$ transform $F8$ into $(2, 3/4, 0, -1/4)$, a sublattice of index 4, respectively $(2, -3/4, -1/2, 1)$, an equivalent reduced form.

In our case (C_{16}), the form (a, b, c, d) has invariant

$$\alpha = \frac{1}{8}(a + b(\zeta + \zeta^{-1}) + c(\zeta^2 + \zeta^{-2}) + d(\zeta^3 + \zeta^{-3})) := \frac{1}{8}(a + b\gamma_1 + c\gamma_2 + d\gamma_3).$$

The Galois group is cyclic of order 4 (real subfield!) and produces the chain

$$(a, b, c, d) \rightarrow (a, -d, -c, b) \rightarrow (a, -b, c - d) \rightarrow (a, d, -c, -b)$$

The reduced form just found above $(2, -3/4, -1/2, 1)$ is therefore equivalent to $F2 = (2, -1, 1/2, -3/4)$.

Remark : Strictly speaking, the action of the Galois group is not a G -equivalence, because it changes the group G to a conjugate subgroup. But the extended classification has no interest in our context.

To find an equivalence between $F1$ and $F7$, first transform $F1$ into $(2, -1/2, 0, -1)$ with the Galois group, and then compute

$$\frac{2 - \frac{1}{2}\gamma_1 - \gamma_3}{4 - 2\gamma_1 - \gamma_2 + \frac{3}{2}\gamma_3} = (1 + \gamma_1)^2, \text{ and}$$

$$\frac{4 - 2\gamma_1 - \gamma_2 + \frac{3}{2}\gamma_3}{2 - \frac{1}{2}\gamma_1 - \gamma_3} = (1 - \gamma_2 + \gamma_3)^2.$$

The quotient of the representative elements is therefore the square of a real unit. This provides explicit G -equivalences between the two forms.

Entirely similar computations show first that the form D_8 has two G -inequivalent neighbours, for example $F1$ and $F2$.

Further, $F1$ has 4 neighbours, one is D_8 , and the three others are G -equivalent to $F2$.

Finally, $F2$ has 4 neighbours, three of them being G -equivalent to $F1$.

We shall display the most relevant informations on cyclotomic forms under the following scheme, given here with the C_{16} -forms :

Discriminant 2^{24} , determinant d^2 , complete list : 3 forms.

No	Δ_8	s	i	E	Nbr	min	d	Name
2	3,544	40	3	5	$4 : 1, 3^3$	8	$2 \cdot 241$	D_8
1	4.0	56	0	7	$8 : 2^4, 3^4$	2	2	
3	4.515	40	1	5	$4 : 1, 2^3$	4	17	

Matrix of the densest form : $(8 \ 2 \ -4 \ -3 \ 0 \ 3 \ 4 \ -2)$

Corresponding α : $\frac{1}{8}(8 + 2\gamma_1 - 4\gamma_2 - 3\gamma_3) = \frac{1}{8}(8 - 4\sqrt{2} + \sqrt{2 + \sqrt{2}}(5 - 3\sqrt{2}))$.

No : Numbering of the form in the path followed by the enumeration algorithm. Used to locate neighbours.

Δ_n : Instead of Hermite's invariant, we use the determinant of the form, calibrated to minimum 2. Δ_n is rational, and will be given with a few decimal places. Forms are listed with decreasing density, hence *increasing* values of Δ_n .

s : Number of pairs of minimal vectors (half kissing number).

i : Default of perfection (A C_m -perfect form is not always perfect).

E : Minimal number of edges of the C_m -Voronoi domain : oï domain. This number is often smaller than the number of orbits, the information is not redundant.

Nbr : Number of neighbours (= Number of faces of the C_m -Voronoi domain : oï domain, in absence of "culs-de-sac"). If not too long to display, this number will be followed by the list of neighbours, with multiplicity.

min : Minimum of the form, in its smallest integral representative.

d : The determinant of the integral representative is $k(m)d^2$, with k the square-free part of the discriminant of $\mathbb{Q}(\zeta_m)$. (The discriminant of $\mathbb{Q}(\zeta_{16})$ is 2^{24} , so $k(8) = 1$)

Name: The previously known forms will appear with their standard name, notations from [?] or [?].

The matrix of the densest C_m -form is given by its first row, followed by the corresponding value of α .

Comment on C_{16} -forms : The densest form is not perfect, and its density is rather low. An explanation could be the orthogonality of the cyclotomic representation, forcing Euclidean symmetries in the lattices. Also observe that $241 \equiv 1 \pmod{16}$ in agreement with Lemma 1.

3 The "critical" forms $D_4, E_6, E_8, K_{12}, \Lambda_{16}$

These famous forms are the densest in their dimension (at least presumably for K_{12} and Λ_{16}). Their automorphism groups contain cyclotomic representations, and their cyclotomic matrices are of interest (compare [?] p.148, where a C_{35} -cyclotomic matrix is given for the Leech lattice Λ_{24}).

3.1 C_8 -perfect forms

Discriminant 2^8 , determinant d^2 , complete list : one form.

No	Δ_4	s	i	E	Nbr	min	d	Name
1	4.000	12	0	3	2 : 1^2	2	2	D_4

Matrix of D_4 : $\begin{pmatrix} 2 & -1 & 0 & 1 \end{pmatrix}$

Corresponding α : $\frac{1}{4}(2 - \gamma_1) = \frac{1}{4}(2 - \sqrt{2})$

3.2 C_{12} -perfect forms

Discriminant $2^4 \cdot 3^2$, determinant d^2 , complete list : one form.

No	Δ_4	s	i	E	Nbr	min	d	Name
1	4.000	12	0	2	2 : 1^2	2	2	D_4

Matrix of D_4 : $\begin{pmatrix} 2 & -1 & 1 & 0 \end{pmatrix}$

Corresponding α : $\frac{1}{6}(3 - \gamma_1) = \frac{1}{6}(3 - \sqrt{3})$

3.3 C_9 -perfect forms

Discriminant 3^9 , determinant $3d^2$, complete list : 2 forms.

No	Δ_6	s	i	E	Nbr	min	d	Name
1	3.000	36	0	4	3 : 2^3	2	1	E_6
2	3.796	27	0	3	3 : 1^3	4	3^2	E_6^*

Matrix of E_6 : $\begin{pmatrix} 2 & 1 & 0 & -1 & -1 & -1 \end{pmatrix}$

Corresponding α : $\frac{1}{9}(3 + 2\gamma_1 + \gamma_2) = \left(\frac{1+\gamma_1}{3}\right)^2$

Comment : Except when m is a power of 2, the cyclotomic representation of C_m is not orthogonal, so the inverse of a cyclotomic matrix is not a cyclotomic matrix (it is G^t -invariant, but not G -invariant). But this does not prevent the simultaneous appearance of E_6 and E_6^* in the list.

3.4 C_{15} -perfect forms

Discriminant $3^4 \cdot 5^6$, determinant d^2 , complete list : one form.

No	Δ_8	s	i	E	Nbr	min	d	Name
1	1.000	120	0	8	10 : 1^{10}	2	1	E_8

Matrix of E_8 : (2 0 0 0 1 -1 -1 0)

Corresponding α : $\frac{1}{15}(6 - \gamma_1 + \gamma_3)$

3.5 C_{20} -perfect forms

Discriminant $2^8 \cdot 5^6$, determinant d^2 , complete list : one form.

No	Δ_8	s	i	E	Nbr	min	d	Name
1	1.000	120	0	10	10 : 1^{10}	2	1	E_8

Matrix of E_8 : (2 0 0 1 -1 0 1 -1)

Corresponding α : $\frac{1}{10}(3 - \gamma_2 + \gamma_3) = \frac{1}{10}(3 - \frac{1+\sqrt{5}}{2} + \sqrt{\frac{5+\sqrt{5}}{2}})$

3.6 C_{24} -perfect forms

Discriminant $2^{16} \cdot 3^4$, determinant d^2 , complete list : 2 forms.

No	Δ_8	s	i	E	Nbr	min	d	Name
1	1.000	120	0	9	8 : 2^8	2	1	E_8
2	2.441	60	1	5	4 : 1^4	4	5^2	

Matrix of E_8 : (2 1 0 0 1 1 0 -1)

Corresponding α : $\frac{1}{12}(3 + \gamma_1 - \gamma_3) = \frac{1}{12}(3 + \sqrt{6})$

Comment : The non-perfect form (No 2) corresponds to a 5-modular lattice in the sense of Quebbemann (A lattice Λ is ℓ -modular, ℓ a prime, if it is even, and if $\sqrt{\ell}\Lambda^*$ is isometric to Λ).

3.7 C_{21} -perfect forms

Discriminant $3^6 \cdot 7^{10}$, determinant d^2 , complete list : 8 forms.

No	Δ_{12}	s	i	E	Nbr	min	d	Name
2	0.177	378	0	15	36 : $1^{12}, 3^6, 4^6, 5^6, 6^6$	4	3^3	K_{12}
4	0.399	210	0	9	13 : $1^4, 2, 3^2, 5, 6^2, 7^2, 8$	6	461	
1	0.410	231	0	10	17 : $1^2, 2^2, 3^1, 4^4, 5^3, 6^2, 7^3$	4	41	
6	0.476	189	0	8	9 : $1^2, 2, 3^2, 4^2, 7, 8$	6	503	
3	0.500	168	0	7	7 : $1, 2, 4^2, 6^2, 8$	8	2897	
7	0.635	147	0	7	10 : $1^3, 4^2, 5, 6, 8^3$	6	$7 \cdot 83$	
8	0.672	126	0	6	6 : $3, 4, 6, 7^3$	8	3359	
5	0.759	126	0	6	6 : $1^3, 2, 4, 7$	8	$43 \cdot 83$	

Matrix of K_{12} : $(4 \ 1 \ -1 \ -2 \ 1 \ 2 \ 1 \ -2 \ -2 \ -1 \ 1 \ 1)$

Corresponding α : $\frac{1}{7}(1 + \gamma_1 + \gamma_2 - 2\gamma_3 + 2\gamma_5)$

3.8 C_{28} -perfect forms

Discriminant $2^{12} \cdot 7^{10}$, determinant d^2 , complete list : 13 forms.

No	Δ_{12}	s	i	E	Nbr	min	d	Name
5	0.177	378	0	21	76 : $1^6, 2^3, 4^3, 5^{13}, 6^6, 7^9, 8^6, 9^6, 10^6, 11^6, 12^9, 13^3$	4	3^3	K_{12}
7	0.585	126	1	8	9 : $1^2, 5^3, 8, 9, 12^2$	12	$29 \cdot 1231$	
1	0.648	154	0	9	10 : $2, 3^2, 4, 5^2, 6, 7^2, 8$	6	587	
12	0.724	112	1	7	8 : $5^3, 6, 7^2, 9, 10$	12	39703	
4	0.765	168	0	9	12 : $1^6, 5^6$	4	$2^3 \cdot 7$	
10	0.955	112	1	7	8 : $2, 3, 5^2, 9^2, 11, 12$	8	$2^6 \cdot 4003$	
9	0.968	112	1	7	8 : $2, 5^2, 7, 10^2, 11, 12$	8	$29 \cdot 139$	
6	1.177	98	0	6	6 : $1, 2, 3, 5^2, 12$	6	$7 \cdot 113$	
8	1.136	84	0	6	6 : $1, 5^2, 7, 11, 13$	10	$3^3 \cdot 617$	
3	1.137	98	0	6	6 : $1^2, 2, 6, 10, 11$	10	$7 \cdot 2381$	
11	1.169	84	1	6	6 : $3, 5^2, 8, 9, 10$	16	$3^3 \cdot 10501$	
13	1.552	84	7	6	6 : $5^3, 8^3$	8	$3^6 \cdot 7$	
2	1.681	98	0	6	6 : $1, 3, 5, 6, 9, 10$	4	83	

Matrix of K_{12} : $(4 \ 2 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ -1 \ 0 \ 0 \ 0)$

Corresponding α : $\frac{1}{14}(5 + 2\gamma_1 + \gamma_4)$

3.9 C_{40} -perfect forms

A complete computation would require more computer time (and power) than reasonable. The densest forms found so far are listed, beginning with Λ_{16} , also called BW_{16} (Barnes-Wall). Only three forms in this list are perfect.

Discriminant $2^{32} \cdot 5^{12}$, determinant d^2 .

No	Δ_{16}	s	i	E	Nbr	min	d	Name	
89	0.0039	2160	0	74	?	4	2^4	Λ_{16}	
159	0.0100	1080	1	39	?	8	3^8		
8	0.0134	840	1	29	196	12	$3^4 \cdot 7^4$		
7	0.0146	840	1	29	184	8	89^4		
23	0.0167	720	4	26	?	10	$3^4 \cdot 5^4$		
34	0.0214	720	0	26	?	6	31^2		
58	0.0224	480	4	18	?	30	$5^2 \cdot 3919^2$		
20	0.0256	780	0	29	?	4	41		
83	0.0261	540	1	20	?	12	521^2		
24	0.0312	540	1	20	?	12	$5^2 \cdot 109^2$		
199	0.0372	460	1	20	?	8	12641		
...		

Matrix of Λ_{16} :

$$(4 \quad -1 \quad 0 \quad -1 \quad 0 \quad 2 \quad -2 \quad 1 \quad -2 \quad 1 \quad 0 \quad -1 \quad 2 \quad -1 \quad 2 \quad -2)$$

3.10 C_{60} -perfect forms

Same remarks as above.

Discriminant $2^{16} \cdot 3^8 \cdot 5^{12}$, determinant d^2 .

No	Δ_{16}	s	i	E	Nbr	min	d	Name	
36	0.0039	2160	0	52	?	4	2^4	Λ_{16}	
23	0.0095	1320	0	34	?	4	5^2		
17	0.0164	840	0	22	?	6	29^2		
22	0.0180	660	1	19	?	16	$19^2 \cdot 79^2$		
69	0.0204	570	1	16	?	16	$421 \cdot 5701$		
1	0.0242	660	1	19	48	8	101^2		
18	0.0214	570	1	18	?	8	$59 \cdot 179$		
21	0.0376	510	1	16	?	8	12721		
..		

Matrix of Λ_{16} :

$$(4 \quad -1 \quad 0 \quad 2 \quad -1 \quad 2 \quad 0 \quad 0 \quad 2 \quad -2 \quad 2 \quad 1 \quad -2 \quad 2 \quad -1 \quad 0)$$

4 The prime cases $C_5, C_7, C_{11}, C_{13}, C_{17}$

The famous Craig lattices $A_n^{(r)}$ are thoroughly investigated in [?] and [?]. Their direct construction as difference lattices gives them matrices (cf.[?] p. 433) which are invariant under the n -dimensional representation of C_{n+1} having characteristic polynomial $1 + x + \dots + x^{n-1} + x^n$. The diagonals are

constant, and the first row is filled from left and from right with binomial coefficients :

$$\left(\binom{2r}{r} - \binom{2r}{r+1} \binom{2r}{r+2} \cdot \cdot \cdot \binom{2r}{r+4} - \binom{2r}{r+3} \binom{2r}{r+2} \right)$$

This description is very convenient, but does not show the value of the minimum, whose determination is a difficult combinatorial problem (cf. [?] p. 139).

When $n + 1$ is a prime p , the Craig lattices $A_{p-1}^{(r)}$ are *cyclotomic*. In our range of investigation, their minimum is $2r$, by [?], and their density is pretty high. We shall evidently meet these forms below. In high dimensions, Craig lattices are sometimes the densest known (e.g. $p=151$, cf [?] p. 43), although it is generally accepted that the C_p -Hermite constant should be lower than the absolute constant.

4.1 C_5 -perfect forms

Discriminant 5^3 , determinant $5d^2$, complete list : one form.

No	Δ_4	s	i	E	Nbr	min	d	Name
1	5.000	10	0	2	$\mathbf{2} : 1^2$	2	1	$A_4^{(1)} = A_4$

Matrix of the densest form : $(2 \quad -1 \quad 0 \quad 0)$

Corresponding $\alpha : \frac{1}{5}(2 - \gamma_1) = \frac{5-\sqrt{5}}{10}$

4.2 C_7 -perfect forms

Discriminant 7^5 , determinant $7d^2$, complete list : 2 forms.

No	Δ_6	s	i	E	Nbr	min	d	Name
2	5.359	21	0	3	$\mathbf{3} : 1^3$	4	7	$A_6^{(2)}$
1	7.00	21	0	3	$\mathbf{3} : 2^3$	2	1	$A_6^{(1)} = A_6$

Matrix of the densest form : $(4 \quad -2 \quad 1 \quad -1 \quad -1 \quad 1)$

Corresponding $\alpha : \frac{1}{7}(5 - \gamma_1 + 2\gamma_2)$

4.3 C_{11} -perfect forms

Discriminant 11^9 , determinant $11d^2$, complete list : 5 forms.

No	Δ_{10}	s	i	E	Nbr	min	d	Name
2	1.299	110	0	10	20 : $1^5, 3^{15}$	4	11	$A_{10}^{(2)}$
4	1.995	66	0	6	5 : 3^5	10	11^3	$A_{10}^{(4)}$
3	2.248	66	0	6	7 : $2^3, 3^2, 4, 5$	8	463	
5	2.727	55	0	5	5 : 3^5	6	11^2	$A_{10}^{(3)}$
1	11.000	55	0	5	5 : 2^5	2	1	$A_{10}^{(1)} = A_{10}$

Matrix of the densest form : $(4 \ -2 \ 0 \ 0 \ 1 \ -1 \ -1 \ 1 \ 0 \ 0)$

Corresponding $\alpha : \frac{1}{11}(5 - \gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4)$

4.4 C_{13} -perfect forms

Discriminant 13^{11} , determinant $13d^2$, complete list : 25 forms.

No	Δ_{12}	s	i	E	Nbr	min	d	Name
2	0.536	195	0	15	84 : $1^6, 3^6, 4^6, 5^6, 6^6, 7^6, 8^6, 9^6, 10^{12}, 11^6, 12^{12}, 13^6$	4	13	$A_{12}^{(2)}$
16	0.698	156	0	10	18 : $5^6, 9^6, 14^6$	6	13^2	$A_{12}^{(3)}$
14	0.941	78	0	6	6 : 3, 4, 10, 11, 15, 16	28	$5^2 \cdot 81043$	
4	0.967	78	0	6	6 : 2, 5, 6, 12^2 , 14	16	71527	
11	1.017	78	0	6	6 : 2, 3, 5, 12, 14, 15	24	$103 \cdot 8111$	
5	1.019	78	0	6	6 : 2, 4, 9, 10, 11, 16	32	4698461	
20	1.031	78	12	6	6 : 7, 9, 13, 18, 19, 23	78	$53 \cdot 79 \cdot 236783$	
12	1.049	91	0	7	10 : $2^2, 4^2, 6, 10, 11, 17, 21, 25$	12	13259	
15	1.093	78	0	6	6 : 3, 6, 7, 11, 14, 18	18	154127	
3	1.106	78	0	6	6 : 7, 9, 11, 13, 14, 15	20	$103 \cdot 2833$	
19	1.121	78	0	6	6 : 7, 13, 17, 18^2 , 20	24	877109	
10	1.126	78	0	6	6 : 2^2 , 5, 12, 13, 14	24	879113	
6	1.160	78	0	6	6 : 2, 4, 7, 12, 15, 17	24	891983	
18	1.178	78	0	6	6 : 7, 15, 17, 19^2 , 20	16	$13^2 \cdot 467$	
13	1.240	78	0	6	6 : 2, 3, 10, 19, 20, 22	42	$467 \cdot 56731$	
7	1.247	78	0	6	6 : 2, 6, 15, 18, 19, 20	14	36451	
23	1.361	78	12	6	6 : 8, 20, 21, 22^2 , 24	26	$1093 \cdot 1429$	
22	1.511	78	0	6	6 : 8, 13, 21, 23^2 , 24	18	181193	
17	1.541	78	0	6	6 : 6, 12, 18, 19, 24, 25	12	16067	
21	1.568	78	0	6	6 : 8, 12, 22, 23, 24^2	24	$443 \cdot 2341$	
24	1.569	78	0	6	6 : 8, 17, 21^2 , 22, 23	16	91079	
9	1.839	78	0	6	6 : 2, 3, 5, 8, 16, 20	12	17551	
8	1.983	91	0	7	12 : $2^2, 9^2, 21^2, 22^2, 23^2, 24^2$	4	5^2	
25	2.313	78	0	6	6 : 12^3 , 17^3	4	3^3	
1	13	78	0	6	6 : 2^6	2	1	$A_{12}^{(1)} = A_{12}$

Matrix of the densest form : $(4 \ -2 \ 0 \ 0 \ 1 \ -1 \ -1 \ 1 \ 0 \ 0)$

Corresponding $\alpha : \frac{1}{11}(5 - \gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4)$

Comment : There are two non-perfect forms in this list, both with a default of perfection of 12, and both with a minimum divisible by 13. Similarly, in the list of 1344 C_{17} -perfect forms, one finds 11 non-perfect forms, all with a default of perfection of 16, and a minimum divisible by 17 (These forms will be listed in Appendix B). This suggests

Observation 2. *The default of perfection of a C_p -perfect form is divisible by $(p - 1)$.*

Observation 3. *The minimum of an integral C_p -perfect, non-perfect C_p -form is divisible by p .*

Observation 2 was first proved by J. Martinet (unpublished). Here is a more general result (proof of my own) settling the claim :

Lemma 2. *Let A be the matrix of a positive cyclotomic form. Call $\delta(A)$, resp. $\delta_p(A)$ its default of perfection, resp. default of C_p -perfection. Then $\delta(A) - \delta_p(A)$ is divisible by $(p - 1)$.*

Proof : The group C_p acts on $\mathcal{D}(A)$, fixing $\mathcal{D}_p(A)$. As any fixpoint-free rational representation of C_p has dimension divisible by $(p - 1)$, the codimension of $\mathcal{D}_p(A)$ in $\mathcal{D}(A)$ is divisible by $(p - 1)$. Hence $(p(p - 1)/2 - \delta(A)) - ((p - 1)/2 - \delta_p(A))$ is divisible by $(p - 1)$. The lemma follows.

Observation 3 remains, at present, at the botanical stage. Arguments in favour are the fact that the orthogonal projection of $\mathcal{D}(A)$ on $\mathcal{D}_p(A)$ is the average on orbits, i.e. division by p , and the conjecture that a non-perfect C_p -perfect form is eutactic.

4.5 C_{17} -perfect forms

Theorem. *There are 1344 C_{17} -perfect forms. The densest among them has matrix (first row) :*

$$(6 \quad -3 \quad 2 \quad -1 \quad -2 \quad 2 \quad -3 \quad 2 \quad 0 \quad 0 \quad 2 \quad -3 \quad 2 \quad -2 \quad -1 \quad 2)$$

This form has minimum 6, and determinant $17 \cdot 2^{16}$. Its representative element α in $\mathbb{Q}(\zeta_{17})$ is :

$$\frac{1}{17}(6 - 3\gamma_1 + 2\gamma_2 - \gamma_3 - 2\gamma_4 + 2\gamma_5 - 3\gamma_6 + 2\gamma_7)$$

Proof : As is clear from Voronoï's algorithm, it suffices to screen the complete list, and extract the densest form.

The table begins with the six densest forms, followed by forms selected according to interesting parameters (kissing number, minimum, name, or

non-perfection). The discriminant is 17^{15} , so the determinant is $17d^2$.

No	Δ_{16}	s	i	E	Nbr	min	d	Name
350	0.02588	680	0	28	402	6	2^8	Densest C_{17} -form, new
133	0.03298	544	0	24	396	6	17^2	$A_{16}^{(3)}$
28	0.06640	442	0	22	497	4	2^4	
1086	0.06927	136	0	8	8	580	373·632503·13535087641	
659	0.06967	136	0	8	8	654	10875511·769549496923	Second largest minimum
658	0.07032	136	0	8	8	1066	140761·2976246257560181	Largest minimum
2	0.07496	476	0	28	768	4	17	$A_{16}^{(2)}$
872	0.07496	255	0	15	145	8	$17 \cdot 2^8$	
278	0.07976	323	0	15	129	8	67^2	$A_{16}^{(4)}$
351	0.08774	255	0	13	84	12	$67 \cdot 1801$	
1251	0.09553	238	0	14	96	8	17^3	
1328	0.10551	136	16	8	8	136	$13^2 \cdot 67 \cdot 8737 \cdot 364073$	Densest non-perfect form
1344	0.12087	136	0	8	8	406	$17681 \cdot 2428417 \cdot 5663347$	Last form found
1267	0.47205	136	0	8	8	136	$384473 \cdot 198141799$	Second less dense form
1	17	136	0	8	8	2	1	$A_{16}^{(1)} = A_{16}$

Comments : The Craig forms $A_{16}^{(5)}$ ($\Delta_{16} = 0.04203, s = 408$) and $A_{16}^{(6)}$ ($\Delta_{16} = 0.12175, s = 153$) are not C_{17} -perfect. The C_{17} -Voronoi domain : oï domain of $A_{16}^{(5)}$ is a face of the domain of the densest C_{17} -form. Both forms numbered 28 ($A_{16}^{(2)}$) and 287 have the same density $\Delta_{16} = 17^3/2^{16} = 0.07496$. This is the only example so far.

The complete list of C_{17} -perfect forms can be found in part II.

5 The groups C_{16} , C_{36} and C_{32}

5.1 C_{16} -perfect forms

See above.

5.2 C_{36} -perfect forms

Discriminant $2^{12} \cdot 3^{18}$, determinant d^2 , complete list : 23 forms.

No	Δ_{12}	s	i	E	Nbr	min	d	Name
8	0.536	108	5	6	$\mathbf{6} : 2, 4, 6, 11, 13, 21$	36	4463 · 5581	
11	0.557	108	6	6	$\mathbf{6} : 2, 6, 8, 16, 21, 22$	40	3 · 73 · 218233	
6	0.579	108	0	6	$\mathbf{6} : 2, 7, 8, 11, 13, 18$	32	37 · 109 · 3167	
4	0.599	108	5	6	$\mathbf{6} : 1, 7, 8, 9, 10, 15$	36	71 · 370801	
10	0.620	126	1	7	$\mathbf{6} : 2, 3^2, 4, 14, 16$	20	3 · 262657	
2	0.669	144	0	8	$\mathbf{12} : 1^3, 2^2, 6, 7, 8, 9, 10, 11, 12$	8	3 · 1117	
21	0.681	108	5	6	$\mathbf{6} : 8, 9, 11, 13, 16, 17$	24	2464919	
16	0.693	108	0	6	$\mathbf{6} : 5, 10, 11, 13, 15, 21$	26	4021019	
15	0.705	108	0	6	$\mathbf{6} : 4, 16, 17, 20, 22^2$	14	3 ² · 10979	
18	0.713	108	5	6	$\mathbf{6} : 6, 9, 14, 17, 19^2$	48	73 · 1117 · 1979	
9	0.728	108	3	6	$\mathbf{6} : 2, 4, 5, 18, 20, 21$	12	2 ⁶ · 73 · 34919	
12	0.790	144	0	8	$\mathbf{6} : 2^6$	6	2 ³ · 3 ⁴	
14	0.797	108	2	6	$\mathbf{6} : 3, 5, 7, 10, 18, 19$	54	37 · 9349559	
17	0.791	108	1	6	$\mathbf{6} : 5, 15, 18, 20, 21, 22$	16	3 ² · 25919	
22	0.803	108	0	6	$\mathbf{6} : 11, 15^2, 17, 20, 23$	20	3 ² · 99611	
3	0.833	126	3	7	$\mathbf{6} : 1, 5, 10^2, 13, 14$	12	37 · 1151	
7	0.900	108	0	6	$\mathbf{6} : 2, 4, 6, 14, 19, 20$	22	1681271	
13	0.913	108	0	6	$\mathbf{6} : 3, 6, 8, 16, 19, 21$	16	71 · 3529	
5	0.986	108	2	6	$\mathbf{6} : 1, 3, 9, 14, 16, 17$	18	107 · 4933	
19	1.001	108	3	6	$\mathbf{6} : 7, 13, 14, 18^2, 20$	24	2988143	
20	1.221	108	1	6	$\mathbf{6} : 7, 9, 15, 17, 19, 22$	8	3 ² · 503	
1	1.230	126	1	7	$\mathbf{6} : 2^3, 3, 4, 5$	4	71	
23	1.265	108	6	6	$\mathbf{6} : 22^6$	4	2 ³ · 3 ²	

Matrix of the densest form :

$$\left(\begin{array}{cccccccccccc} 36 & 18 & 8 & -1 & -6 & 0 & 18 & 18 & 14 & 0 & -14 & -18 \end{array} \right)$$

Corresponding α :

$$\frac{1}{18}(64 + 36\gamma_1 + 22\gamma_2 - \gamma_3 - 20\gamma_4 - 18\gamma_5)$$

Comment : Again, the high amount of non-perfect forms, and the low density, are amazing. None of these forms seem to be known, except possibly (?) forms 12 and 23, having simple determinants.

5.3 C_{32} -perfect forms

The complete computation was achieved July 2001. There are 4624 C_{32} -perfect forms. The first part of the list below gives the densest forms, the second part gives high kissing numbers, and all forms with minimum 2 and 4. Apart from D_{16} , these forms seem to be entirely new.

Discriminant 2^{64} , determinant d^2 .

No	Δ_{16}	s	i	E	Nbr	min	d	Name
657	0.0398	400	1	21	190	12	$2 \cdot 31 \cdot 5407$	Densest form
292	0.0425	320	3	18	91	16	$2^3 \cdot 432833$	
1977	0.0453	160	2	9	?	356	$2 \cdot 191 \cdot 1217 \cdot 461744636863$	
1968	0.0455	176	3	9	?	584	$2^3 \cdot 1035891743 \cdot 1361782847$	
303	0.0462	192	0	10	11	324	$2^3 \cdot 31 \cdot 97 \cdot 4242811210207$	
890	0.0476	352	0	20	108	12	$2 \cdot 183361$	
3200	0.0478	144	1	8	8	720	$2 \cdot 10490593 \cdot 2942663273057$	
1973	0.0478	160	2	9	?	260	$2 \cdot 3391 \cdot 53857 \cdot 48847391$	
1289	0.0480	176	1	9	?	90	$2 \cdot 31 \cdot 35423 \cdot 1677857$	
2230	0.0481	144	2	8	?	140	$2^3 \cdot 11681 \cdot 1353691649$	
2232	0.0484	176	1	9	?	120	$2^3 \cdot 479 \cdot 1471 \cdot 6561857$	
328	0.0496	144	2	8	8	228	$2^3 \cdot 193 \cdot 11807 \cdot 13537 \cdot 25759$	
758	0.0549	432	0	24	240	6	$2 \cdot 769$	
83	0.0586	496	0	24	380	4	$2 \cdot 31$	Highest kissing no
283	0.2346	240	0	13	55	4	$2^2 \cdot 31$	
1555	0.5742	144	1	8	8	4	$2 \cdot 97$	
1	4	240	0	15	128	2	2	D_{16}

Matrix of the densest form :

$$\begin{pmatrix} 12 & 5 & 0 & 3 & 4 & 3 & 0 & -4 & 0 & 4 & 0 & -3 & -4 & -3 & 0 & -5 \end{pmatrix}$$

Corresponding α :

$$\frac{1}{16}(12 + 5\gamma_1 + 3\gamma_3 + 4\gamma_4 + 3\gamma_5 - 4\gamma_7)$$

As was verified by direct computation, the perfect lattice (No 83, determinant $2^2 \cdot 31^2$) contains the perfect lattice (No 283, determinant $2^4 \cdot 31^2$) with the same minimum, and index 2. This situation is common in the non-equivariant case (e.g. E_8 contains A_8 with index 3, and D_8 with index 2), but not here.

The data on this case suggest

Observation 4. *All perfect cyclotomic forms are even.*

Comment : In the case the prime 2 is unramified in the cyclotomic field, then any cyclotomic form, perfect or not, is even¹. The ramified case is unexplained. Related to this is

Observation 5. *The determinant of an integral C_{32} -perfect form is even.*

¹This result is due to J. Martinet. The representative element α , being real, has an even trace. When 2 is unramified, this property implies that the corresponding lattice is even.

6 Appendix A : Reduction of all C_m -perfect cyclotomic forms with $\phi(m) \leq 12$;

Recall that a cyclotomic matrix $A = (a_{i,j})$ has constant diagonals ($a_{i,j} = a_{i+k,j+k}$). Hence, matrices can be reconstructed from their first row (ROW1). All are given in their smallest integral representative, and could be successfully reduced : the minimum of the form is always on the main diagonal of the matrix.

All forms are given in PARI-notation, so that they can be directly incorporated in a PARI-program.

6.1 Perfect cyclotomic forms for the cyclic group of order 8

```
FORMNUMBER = 1;  
\ \ (D4)  
ROW1=[2, 1, 0, -1];  
KISS = 12; DEFAULT = 0;  
EDGES = 3; FACES = 2;  
DET = {2} ^ {2};  
GAMMA4 = 1.4142;
```

6.2 Perfect cyclotomic forms for the cyclic group of order 12

```
FORMNUMBER = 1;  
\ \ (D4)  
ROW1 = [2, 1, 1, 0];  
KISS = 12; DEFAULT = 0;  
EDGES = 2; FACES = 2;  
DET = {2} ^ {2};  
GAMMA4 = 1.4142;
```

6.3 Perfect cyclotomic forms for the cyclic group of order 9

```
FORMNUMBER = 1;  
\ \ (E6)
```

ROW1 = [2, 1, 0, -1, -1, -1];
KISS = 36; DEFAULT = 0;
EDGES = 4; FACES = 3;
DET = {3} \wedge {1};
GAMMA6 = 1.6653;

FORMNUMBER = 2;
\\ (E_6^*)
ROW1 = [4, 1, -2, -2, 1, 1];
KISS = 27; DEFAULT = 0;
EDGES = 3; FACES = 3;
DET = {3} \wedge {5};
GAMMA6 = 1.6012;

6.4 Perfect cyclotomic forms for the cyclic group of order 15

FORMNUMBER = 1;
\\ (E_8)
ROW1 = [2, 0, 0, 0, 1, -1, -1, 0];
KISS = 120; DEFAULT = 0;
EDGES = 8; FACES = 10;
DET = {1};
GAMMA8 = 2.0000;

6.5 Perfect cyclotomic forms for the cyclic group of order 20

FORMNUMBER = 1;
\\ (E_8)
ROW1 = [2, 0, 0, 1, -1, 0, 1, -1];
KISS = 120; DEFAULT = 0;
EDGES = 10; FACES = 10;
DET = {1};
GAMMA8 = 2.0000;

6.6 Perfect cyclotomic forms for the cyclic group of order 24

FORMNUMBER = 1;

$\backslash \backslash (E_8)$

ROW1 = [2, 1, 0, 0, 1, 1, 0, -1];

KISS = 120; DEFAULT = 0;

EDGES = 9; FACES = 8;

DET = {1};

GAMMA8 = 2.0000;

FORMNUMBER = 2;

ROW1 = [4, 0, -1, -2, 2, 2, 0, -2];

KISS = 60; DEFAULT = 1;

EDGES = 5; FACES = 4;

DET = {5} \wedge {4};

GAMMA8 = 1.7888;

6.7 Perfect cyclotomic forms for the cyclic group of order 21

FORMNUMBER = 1;

ROW1 = [4, 2, 1, 0, 1, 1, 0, -2, -2, -2, -1, -1];

KISS = 231; DEFAULT = 0;

EDGES = 10; FACES = 17;

DET = {41} \wedge {2};

GAMMA12 = 2.1540;

FORMNUMBER = 2;

$\backslash \backslash (K_{12})$

ROW1 = [4, 1, 1, -1, -1, -2, -2, -2, 1, 1, 2, 2];

KISS = 378; DEFAULT = 0;

EDGES = 15; FACES = 36;

DET = {3} \wedge {6};

GAMMA12 = 2.3094;

FORMNUMBER = 3;

ROW1 = [8, -2, 0, 1, -4, 4, -1, -4, 3, -4, 3, 3];

KISS = 168; DEFAULT = 0;

EDGES = 7; FACES = 16;

DET = {2897} \wedge {2};

GAMMA12 = 2.1188;

FORMNUMBER = 4;

ROW1 = [6, -1, 3, 0, 0, 0, 0, -3, 1, -3, 0, 0];

KISS = 210; DEFAULT = 0;
EDGES = 9; FACES = 13;
DET = $\{461\} \wedge \{2\}$;
GAMMA12 = 2.1587;

FORMNUMBER = 5;
ROW1 = [8, -1, 0, 2, -2, 4, -2, -4, 3, -4, 0, 0];
KISS = 126; DEFAULT = 0;
EDGES = 6; FACES = 6;
DET = $\{43\} \wedge \{2\} * \{83\} \wedge \{2\}$;
GAMMA12 = 2.0464;

FORMNUMBER = 6;
ROW1 = [6, 2, 2, -1, -2, -3, -3, -3, 1, 1, 3, 3];
KISS = 189; DEFAULT = 0;
EDGES = 8; FACES = 9;
DET = $\{503\} \wedge \{2\}$;
GAMMA12 = 2.1276;

FORMNUMBER = 7;
ROW1 = [6, 3, 1, 0, 2, 1, -1, -3, -2, -2, -2, -2];
KISS = 147; DEFAULT = 0;
EDGES = 7; FACES = 16;
DET = $\{7\} \wedge \{2\} * \{83\} \wedge \{2\}$;
GAMMA12 = 2.0770;

FORMNUMBER = 8;
ROW1 = [8, 4, 2, 1, 3, 1, -2, -4, -2, -3, -4, -4];
KISS = 126; DEFAULT = 0;
EDGES = 6; FACES = 6;
DET = $\{3359\} \wedge \{2\}$;
GAMMA12 = 2.0672;

6.8 Perfect cyclotomic forms for the cyclic group of order 28

FORMNUMBER = 1;
ROW1 = [6, 2, 0, -1, 0, 1, 3, 0, -3, -1, 0, 1];
KISS = 154; DEFAULT = 0;
EDGES = 9; FACES = 10;
DET = $\{587\} \wedge \{2\}$;

GAMMA12 = 2.0735;
 FORMNUMBER = 2;
ROW1 = [4, 2, 0, -1, -1, 0, 1, 0, -1, 0, 1, 1];
 KISS = 98; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {83} \wedge {2};
 GAMMA12 = 1.9152;
 FORMNUMBER = 3;
ROW1 = [10, 2, -2, -1, -2, -1, 5, 0, -5, 1, 2, 1];
 KISS = 98; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {7} \wedge {2} * {2381} \wedge {2};
 GAMMA12 = 1.9785;
 FORMNUMBER = 4;
ROW1 = [4, 2, 1, -1, -1, -1, 0, 0, 0, 1, 1, 1];
 KISS = 168; DEFAULT = 0;
 EDGES = 9; FACES = 12;
 DET = {2} \wedge {6} * {7} \wedge {2};
 GAMMA12 = 2.0450;
 FORMNUMBER = 5;
 \ \ (K_{12})
ROW1 = [4, 2, 1, 0, 0, 0, 1, 0, -1, 0, 0, 0];
 KISS = 378; DEFAULT = 0;
 EDGES = 21; FACES = 76;
 DET = {3} \wedge {6};
 GAMMA12 = 2.3094;
 FORMNUMBER = 6;
ROW1 = [6, 3, 3, 1, 1, 0, 1, 0, -1, 0, -1, -1];
 KISS = 98; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {7} \wedge {2} * {113} \wedge {2};
 GAMMA12 = 1.9729;
 FORMNUMBER = 7;
ROW1 = [12, 4, 2, 2, 1, 0, 5, 0, -5, 0, -1, -2];
 KISS = 126; DEFAULT = 1;
 EDGES = 8; FACES = 9;

DET = {29} \wedge {2} * {1231} \wedge {2};
GAMMA12 = 2.0912;

FORMNUMBER = 8;
ROW1 = [10, 2, 0, 2, 0, -1, 5, 0, -5, 1, 0, -2];
KISS = 84; DEFAULT = 0;
EDGES = 6; FACES = 6;
DET = {3} \wedge {6} * {617} \wedge {2};
GAMMA12 = 1.9787;

FORMNUMBER = 9;
ROW1 = [8, 4, 1, 0, -2, -2, 1, 0, -1, 2, 2, 0];
KISS = 112; DEFAULT = 1;
EDGES = 7; FACES = 16;
DET = {29} \wedge {2} * {139} \wedge {2};
GAMMA12 = 2.0053;

FORMNUMBER = 10;
ROW1 = [8, 0, 3, 2, 1, -2, 2, 0, -2, 2, -1, -2];
KISS = 112; DEFAULT = 1;
EDGES = 7; FACES = 16;
DET = {4003} \wedge {2};
GAMMA12 = 2.0076;

FORMNUMBER = 11;
ROW1 = [16, 2, -3, 2, -3, -4, 8, 0, -8, 4, 3, -2];
KISS = 84; DEFAULT = 1;
EDGES = 6; FACES = 6;
DET = {3} \wedge {6} * {10501} \wedge {2};
GAMMA12 = 1.9740;

FORMNUMBER = 12;
ROW1 = [12, -2, -2, 3, -4, -6, 4, 0, -4, 6, 4, -3];
KISS = 112; DEFAULT = 1;
EDGES = 7; FACES = 16;
DET = {39703} \wedge {2};
GAMMA12 = 2.0545;

FORMNUMBER = 13;
ROW1 = [12, 4, 6, 3, 0, 0, 0, -4, -6, 0, 2, 1];
KISS = 84; DEFAULT = 7;
EDGES = 6; FACES = 6;

DET = $\{3\} \wedge \{12\} * \{7\} \wedge \{2\}$;
GAMMA12 = 1.9280;

6.9 Perfect cyclotomic forms for the cyclic group of order 5

FORMNUMBER = 1;
\\ (A₄)
ROW1 = [2, 0, -1, -1];
KISS = 10; DEFAULT = 0;
EDGES = 2; FACES = 2;
DET = $\{5\}$;
GAMMA4 = 1.3374;

6.10 Perfect cyclotomic forms for the cyclic group of order 7

FORMNUMBER = 1;
\\ (A₆)
ROW1 = [2, 0, 0, -1, -1, 0];
KISS = 21; DEFAULT = 0;
EDGES = 3; FACES = 3;
DET = $\{7\} \wedge \{1\}$;
GAMMA6 = 1.4460;

FORMNUMBER = 2;
\\ (A₆⁽²⁾)
ROW1 = [4, -2, 1, -1, -1, 1];
KISS = 21; DEFAULT = 0;
EDGES = 3; FACES = 3;
DET = $\{7\} \wedge \{3\}$;
GAMMA6 = 1.5118;

6.11 Perfect cyclotomic forms for the cyclic group of order 11

FORMNUMBER = 1;
\\ (A₁₀)
ROW1 = [2, 0, 0, 0, 0, -1, -1, 0, 0, 0];

KISS = 55; DEFAULT = 0;
 EDGES = 5; FACES = 5;
 DET = {11};
 GAMMA10 = 1.5735;

FORMNUMBER = 2;
 \ \ ($A_{10}^{(2)}$)
ROW1 = [4, -2, 0, 0, 1, -1, -1, 1, 0, 0];
 KISS = 110; DEFAULT = 0;
 EDGES = 10; FACES = 20;
 DET = {11} \wedge {3};
 GAMMA10 = 1.9482;

FORMNUMBER = 3;
ROW1 = [8, -4, 0, 3, -4, 1, 1, -4, 3, 0];
 KISS = 66; DEFAULT = 0;
 EDGES = 6; FACES = 11;
 DET = {11} * {463} \wedge {2};
 GAMMA10 = 1.8443;

FORMNUMBER = 4;
 \ \ ($A_{10}^{(4)}$)
ROW1 = [10, -4, -2, 5, -5, 1, 1, -5, 5, -2];
 KISS = 66; DEFAULT = 0;
 EDGES = 6; FACES = 5;
 DET = {11} \wedge {7};
 GAMMA10 = 1.8664;

FORMNUMBER = 5;
 \ \ ($A_{10}^{(3)}$)
ROW1 = [6, -3, 1, 1, -2, 0, 0, -2, 1, 1];
 KISS = 55; DEFAULT = 0;
 EDGES = 5; FACES = 5;
 DET = {11} \wedge {5};
 GAMMA10 = 1.8090;

6.12 Perfect cyclotomic forms for the cyclic group of order 13

FORMNUMBER = 1;
 \ \ (A_{12})

ROW1 = [2, 0, 0, 0, 0, 0, -1, -1, 0, 0, 0, 0];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13};
 GAMMA12 = 1.6151;

FORMNUMBER = 2;
 $\setminus \setminus (A_{12}^{(2)})$
ROW1 = [4, -2, 0, 0, 0, 1, -1, -1, 1, 0, 0, 0];
 KISS = 195; DEFAULT = 0;
 EDGES = 15; FACES = 84;
 DET = {13} \wedge {3};
 GAMMA12 = 2.1065;

FORMNUMBER = 3;
ROW1 = [20, -7, -3, 0, 6, -4, -2, -2, -4, 6, 0, -3];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13} * {103} \wedge {2} * {2833} \wedge {2};
 GAMMA12 = 1.9831;

FORMNUMBER = 4;
ROW1 = [16, -7, -1, -1, 5, -3, -1, -1, -3, 5, -1, -1];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13} * {71527} \wedge {2};
 GAMMA12 = 2.0054;

FORMNUMBER = 5;
ROW1 = [32, -16, 2, -4, 10, -7, -1, -1, -7, 10, -4, 2];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13} * {4698461} \wedge {2};
 GAMMA12 = 1.9967;

FORMNUMBER = 6;
ROW1 = [24, -8, -6, 0, 8, -3, -3, -3, -3, 8, 0, -6];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13} * {891983} \wedge {2};
 GAMMA12 = 1.9754;

FORMNUMBER = 7;
ROW1 = [14, -7, 0, 3, 0, -4, 1, 1, -4, 0, 3, 0];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{36451} \wedge {2};
 GAMMA12 = 1.9634;

FORMNUMBER = 8;
ROW1 = [4, -2, 0, 1, -2, 2, -1, -1, 2, -2, 1, 0];
 KISS = 91; DEFAULT = 0;
 EDGES = 7; FACES = 12;
 DET = {13}*{5} \wedge {4};
 GAMMA12 = 1.8890;

FORMNUMBER = 9;
ROW1 = [12, -6, 0, 4, -6, 5, -3, -3, 5, -6, 4, 0];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{17551} \wedge {2};
 GAMMA12 = 1.9009;

FORMNUMBER = 10;
ROW1 = [24, -12, 5, 5, -10, 6, -6, -6, 6, -10, 5, 5];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{879113} \wedge {2};
 GAMMA12 = 1.9802;

FORMNUMBER = 11;
ROW1 = [24, -6, -10, 2, 2, 9, -9, -9, 9, 2, 2, -10];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{103} \wedge {2}*{8111} \wedge {2};
 GAMMA12 = 1.9970;

FORMNUMBER = 12;
ROW1 = [12, -6, 1, -2, 1, 3, -3, -3, 3, 1, -2, 1];
 KISS = 91; DEFAULT = 0;
 EDGES = 7; FACES = 15;
 DET = {13}*{13259} \wedge {2};
 GAMMA12 = 1.9919;

FORMNUMBER = 13;
ROW1 = [42, -21, 10, 5, -10, 3, -8, -8, 3, -10, 5, 10];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{467} \wedge {2}*{56731} \wedge {2};
 GAMMA12 = 1.9644;

FORMNUMBER = 14;
ROW1 = [28, -11, -3, -1, 9, -6, -2, -2, -6, 9, -1, -3];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{5} \wedge {4}*{81043} \wedge {2};
 GAMMA12 = 2.0101;

FORMNUMBER = 15;
ROW1 = [18, -6, -4, 0, 6, -3, -2, -2, -3, 6, 0, -4];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{154127} \wedge {2};
 GAMMA12 = 1.9851;

FORMNUMBER = 16;
 $\setminus \setminus (A_{12}^{(3)})$
ROW1 = [6, -3, 1, -1, 2, -2, 0, 0, -2, 2, -1, 1];
 KISS = 156; DEFAULT = 0;
 EDGES = 10; FACES = 18;
 DET = {13} \wedge {5};
 GAMMA12 = 2.0606;

FORMNUMBER = 17;
ROW1 = [12, -2, -6, 0, 5, 0, -3, -3, 0, 5, 0, -6];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{16067} \wedge {2};
 GAMMA12 = 1.9291;

FORMNUMBER = 18;
ROW1 = [16, -8, 0, 4, -1, -4, 1, 1, -4, -1, 4, 0];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13} \wedge {5}*{467} \wedge {2};
 GAMMA12 = 1.9728;

FORMNUMBER = 19;
ROW1 = [24, -12, 1, 5, -1, -6, 1, 1, -6, -1, 5, 1];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{877109} \wedge {2};
 GAMMA12 = 1.9809;

FORMNUMBER = 20;
ROW1 = [78, -7, -39, -18, 28, 5, -8, -8, 5, 28, -18, -39];
 KISS = 78; DEFAULT = 12;
 EDGES = 6; FACES = 6;
 DET = {13}*{53} \wedge {2}*{79} \wedge {2}*{236783} \wedge {2};
 GAMMA12 = 1.9947;

FORMNUMBER = 21;
ROW1 = [24, -12, 1, 4, -11, 12, -6, -6, 12, -11, 4, 1];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{443} \wedge {2}*{2341} \wedge {2};
 GAMMA12 = 1.9264;

FORMNUMBER = 22;
ROW1 = [18, -9, 1, 3, -9, 9, -4, -4, 9, -9, 3, 1];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{181193} \wedge {2};
 GAMMA12 = 1.9323;

FORMNUMBER = 23;
ROW1 = [26, -12, -2, 8, -13, 13, -7, -7, 13, -13, 8, -2];
 KISS = 78; DEFAULT = 12;
 EDGES = 6; FACES = 6;
 DET = {13}*{1093} \wedge {2}*{1429} \wedge {2};
 GAMMA12 = 1.9492;

FORMNUMBER = 24;
ROW1 = [16, -8, 1, 3, -8, 8, -4, -4, 8, -8, 3, 1];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{91079} \wedge {2};
 GAMMA12 = 1.9262;

FORMNUMBER = 25;
ROW1 = [4, -2, 1, 0, -1, 1, -1, -1, 1, -1, 0, 1];
 KISS = 78; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {13}*{3} \wedge {6};
 GAMMA12 = 1.8649;

6.13 Perfect cyclotomic forms for the cyclic group of order 16

FORMNUMBER = 1;
 $\backslash \backslash (D_8)$
ROW1 = [2, 1, 0, 0, 0, 0, 0, -1];
 KISS = 56; DEFAULT = 0;
 EDGES = 7; FACES = 8;
 DET = {2} \wedge {2};
 GAMMA8 = 1.6817;

FORMNUMBER = 2;
ROW1 = [8, 2, -4, -3, 0, 3, 4, -2];
 KISS = 40; DEFAULT = 3;
 EDGES = 5; FACES = 4;
 DET = {2} \wedge {2}*{241} \wedge {2};
 GAMMA8 = 1.7073;

FORMNUMBER = 3;
ROW1 = [4, 0, -2, 1, 0, -1, 2, 0];
 KISS = 40; DEFAULT = 1;
 EDGES = 5; FACES = 4;
 DET = {2} \wedge {2}*{17} \wedge {2};
 GAMMA8 = 1.6564;

6.14 Perfect cyclotomic forms for the cyclic group of order 36

FORMNUMBER = 1;
ROW1 = [4, 2, 1, 0, 0, 0, 2, 2, 1, 0, -1, -2];
 KISS = 126; DEFAULT = 1;
 EDGES = 7; FACES = 6;
 DET = {71} \wedge {2};

GAMMA12 = 1.9657;
 FORMNUMBER = 2;
ROW1 = [8, 4, 2, 1, 0, 0, 4, 4, 2, 0, -2, -4];
 KISS = 144; DEFAULT = 0;
 EDGES = 8; FACES = 12;
 DET = {3} \wedge {2} * {1117} \wedge {2};
 GAMMA12 = 2.0680;

FORMNUMBER = 3;
ROW1 = [12, 6, 3, -2, -1, 0, 6, 6, 4, 0, -4, -6];
 KISS = 126; DEFAULT = 3;
 EDGES = 7; FACES = 6;
 DET = {37} \wedge {2} * {1151} \wedge {2};
 GAMMA12 = 2.0306;

FORMNUMBER = 4;
ROW1 = [36, 18, 3, 4, 0, 0, 18, 18, 3, 0, -3, -18];
 KISS = 108; DEFAULT = 5;
 EDGES = 6; FACES = 6;
 DET = {71} \wedge {2} * {370801} \wedge {2};
 GAMMA12 = 2.0872;

FORMNUMBER = 5;
ROW1 = [18, 9, 9, 4, 6, 3, 9, 6, 3, 0, -3, -6];
 KISS = 108; DEFAULT = 2;
 EDGES = 6; FACES = 6;
 DET = {107} \wedge {2} * {4933} \wedge {2};
 GAMMA12 = 2.0022;

FORMNUMBER = 6;
ROW1 = [32, 16, 8, -1, -4, 0, 16, 16, 12, 0, -12, -16];
 KISS = 108; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {37} \wedge {2} * {109} \wedge {2} * {3167} \wedge {2};
 GAMMA12 = 2.0930;

FORMNUMBER = 7;
ROW1 = [22, 11, 3, 4, 0, 0, 11, 11, 3, 0, -3, -11];
 KISS = 108; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {1681271} \wedge {2};

GAMMA12 = 2.0175;
 FORMNUMBER = 8;
ROW1 = [36, 18, 8, -1, -6, 0, 18, 18, 14, 0, -14, -18];
 KISS = 108; DEFAULT = 5;
 EDGES = 6; FACES = 6;
 DET = {4463} \wedge {2} * {5581} \wedge {2};
 GAMMA12 = 2.1065;
 FORMNUMBER = 9;
ROW1 = [24, 12, 12, 9, 8, 4, 12, 8, 4, 0, -4, -8];
 KISS = 108; DEFAULT = 3;
 EDGES = 6; FACES = 6;
 DET = {73} \wedge {2} * {34919} \wedge {2};
 GAMMA12 = 2.0534;
 FORMNUMBER = 10;
ROW1 = [20, 10, 8, 10, 6, 3, 10, 7, 2, 0, -2, -7];
 KISS = 126; DEFAULT = 1;
 EDGES = 7; FACES = 6;
 DET = {3} \wedge {2} * {262657} \wedge {2};
 GAMMA12 = 2.0810;
 FORMNUMBER = 11;
ROW1 = [40, 5, -18, -20, -4, 20, 20, -15, -14, 0, 14, 15];
 KISS = 108; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {3} \wedge {2} * {73} \wedge {2} * {218233} \wedge {2};
 GAMMA12 = 2.0997;
 FORMNUMBER = 12;
ROW1 = [6, -1, -3, 3, 0, -2, 3, 1, -3, 0, 3, -1];
 KISS = 144; DEFAULT = 6;
 EDGES = 8; FACES = 6;
 DET = {2} \wedge {6} * {3} \wedge {8};
 GAMMA12 = 2.0396;
 FORMNUMBER = 13;
ROW1 = [16, 8, 4, -3, -2, 0, 8, 8, 6, 0, -6, -8];
 KISS = 108; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {71} \wedge {2} * {3529} \wedge {2};

GAMMA12 = 2.0151;
 FORMNUMBER = 14;
ROW1 = [54, 27, 27, 8, 16, 9, 27, 18, 11, 0, -11, -18];
 KISS = 108; DEFAULT = 2;
 EDGES = 6; FACES = 6;
 DET = {37} \wedge {2} * {9349559} \wedge {2};
 GAMMA12 = 2.0381;
 FORMNUMBER = 15;
ROW1 = [14, 6, 0, 2, 0, -1, 7, 7, 0, 0, 0, -7];
 KISS = 108; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {3} \wedge {4} * {10979} \wedge {2};
 GAMMA12 = 2.0590;
 FORMNUMBER = 16;
ROW1 = [26, 8, 13, 8, 12, 1, 13, 7, 1, 0, -1, -7];
 KISS = 108; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {4021019, 2};
 GAMMA12 = 2.0618;
 FORMNUMBER = 17;
ROW1 = [16, 4, -4, -1, 4, 7, 8, -3, -8, 0, 8, 3];
 KISS = 108; DEFAULT = 1;
 EDGES = 6; FACES = 6;
 DET = {3} \wedge {4} * {25919} \wedge {2};
 GAMMA12 = 2.0392;
 FORMNUMBER = 18;
ROW1 = [48, 24, 24, 13, 12, 8, 24, 16, 12, 0, -12, -16];
 KISS = 108; DEFAULT = 5;
 EDGES = 6; FACES = 6;
 DET = {73} \wedge {2} * {1117} \wedge {2} * {1979} \wedge {2};
 GAMMA12 = 2.0571;
 FORMNUMBER = 19;
ROW1 = [24, 12, 2, -2, -7, 0, 12, 12, 9, 0, -9, -12];
 KISS = 108; DEFAULT = 3;
 EDGES = 6; FACES = 6;
 DET = {2988143} \wedge {2};

GAMMA12 = 1.9997;
 FORMNUMBER = 20;
ROW1 = [8, 3, 0, 2, 0, -1, 4, 4, 0, 0, 0, -4];
 KISS = 108; DEFAULT = 1;
 EDGES = 6; FACES = 6;
 DET = {3} \wedge {4} * {503} \wedge {2};
 GAMMA12 = 1.9669;

FORMNUMBER = 21;
ROW1 = [24, 12, 8, 5, 0, 3, 12, 9, 8, 0, -8, -9];
 KISS = 108; DEFAULT = 5;
 EDGES = 6; FACES = 6;
 DET = {2464919} \wedge {2};
 GAMMA12 = 2.0649;

FORMNUMBER = 22;
ROW1 = [20, 3, -8, -10, -4, 8, 10, -5, -4, 0, 4, 5];
 KISS = 108; DEFAULT = 0;
 EDGES = 6; FACES = 6;
 DET = {3} \wedge {4} * {99611} \wedge {2};
 GAMMA12 = 2.0367;

FORMNUMBER = 23;
ROW1 = [4, -1, 2, -2, 1, -1, 2, 0, 1, 0, -1, 0];
 KISS = 108; DEFAULT = 6;
 EDGES = 6; FACES = 6;
 DET = {2} \wedge {6} * {3} \wedge {4};
 GAMMA12 = 1.9611;

7 Appendix B : The non-perfect C_{17} -perfect forms

FORMNUMBER = 196;
ROW1 = [68, -22, -4, -8, -4, 8, 14, -3, -15, -15, -3, 14, 8, -4, -8, -4];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = 17 * {188114204107} \wedge {2};
 GAMMA16 = 2.2197;

FORMNUMBER = 340;

ROW1 = [34, 12, 2, -8, 0, -10, -12, -8, 7, 7, -8, -12, -10, 0, -8, 2];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{811454711\} \wedge \{2\}$;
 GAMMA16 = 2.1923;

FORMNUMBER = 605;
 ROW1 = [68, 5, -34, -32, 17, 25, 5, -18, -2, -2, -18, 5, 25, 17, -32, -34];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{2\} \wedge \{8\} * \{2687\} \wedge \{2\} * \{3443113\} \wedge \{2\}$;
 GAMMA16 = 2.2872;

FORMNUMBER = 831;
 ROW1 = [136, -33, -68, 31, 36, -51, 0, 68, -51, -51, 68, 0, -51, 36, 31, -68];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{137\} \wedge \{2\} * \{372281721463\} \wedge \{2\}$;
 GAMMA16 = 2.2038;

FORMNUMBER = 838;
 ROW1 = [68, -30, -7, 5, -5, 16, -26, 32, -19, -19, 32, -26, 16, -5, 5, -7];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{170035693777\} \wedge \{2\}$;
 GAMMA16 = 2.2479;

FORMNUMBER = 839;
 ROW1 = [68, -30, -8, 6, -3, 13, -25, 32, -19, -19, 32, -25, 13, -3, 6, -8];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{137\} \wedge \{2\} * \{1184168047\} \wedge \{2\}$;
 GAMMA16 = 2.2612;

FORMNUMBER = 846;
 ROW1 = [408, -162, -84, 187, -134, 49, 73, -194, 61, 61, -194, 73, 49, -134, 187, -84];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{145043\} \wedge \{2\} * \{1766024463083\} \wedge \{2\}$;
 GAMMA16 = 2.2787;

FORMNUMBER = 1238;

ROW1 = [68, -5, -34, 8, 20, 13, -23, -34, 21, 21, -34, -23, 13, 20, 8, -34];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{101\} \wedge \{2\} * \{1441826677\} \wedge \{2\}$;
 GAMMA16 = 2.2919;

FORMNUMBER = 1307;
 ROW1 = [204, -17, -102, -51, 75, 76, -102, -52, 71, 71, -52, -102, 76, 75, -51, -102];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{1080088119949789\} \wedge \{2\}$;
 GAMMA16 = 2.2570;

FORMNUMBER = 1318;
 ROW1 = [136, -68, 22, -21, 4, 53, -68, 15, -5, -5, 15, -68, 53, 4, -21, 22];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{13\} \wedge \{4\} * \{67\} \wedge \{2\} * \{8737\} \wedge \{2\} * \{364073\} \wedge \{2\}$;
 GAMMA16 = 2.3018;

FORMNUMBER = 1321;
 ROW1 = [170, -85, 27, -25, 43, -7, -9, -47, 18, 18, -47, -9, -7, 43, -25, 27];
 KISS = 136; DEFAULT = 16;
 EDGES = 8; FACES = 8;
 DET = $17 * \{2551\} \wedge \{2\} * \{86294560643\} \wedge \{2\}$;
 GAMMA16 = 2.2946;

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