On certain positive integer sequences

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Let $k \geq 2$, $l \geq 1$, $m \geq 2$ be positive integers. We say that a positive integer n satisfies the property (k, l, m) if the sum of digits of n^m in its expansion in base k is l times the sum of digits of the expansion of n in base k. Such numbers will be called (k, l, m)-numbers.

The simplest case is $(k, l, m) = (2, 1, 2)$, which corresponds to the positive integers n for which the numbers of ones in their binary expansion is equal to the number of ones in n^2. The $(2, 2, 2)$-numbers are the integers n such that n^2 has twice as many 1's in its binary expansion as n. Roughly speaking, since the lenght of n^2 is twice the lenght of n if a digit 0 is allowed, the $(2, 2, 2)$ numbers are the integers n such that n and n^2 have the same percentage of 1's.

In a recent paper that will appear on *Journal of Number Theory*, I studied the $(2, 1, 2)$-numbers.

Several questions, concerning both the structure properties and asymptotic behaviour, can be raised. Is there a necessary and sufficient condition to assure that a number is of type $(2, 1, 2)$? What is the asymptotic behaviour of the counting function of the $(2, 1, 2)$-numbers? The irregularity of distribution does not suggest a clear answer to these questions.

Let $p_{(k,l,m)}(n)$ be the number of the (k, l, m)-numbers which does not exceed n.

In cooperation with C. Sàndor of Budapest, I proved some estimates of particular cases of this function. In particular we proved that $p_{(2,2,2)}(x) \gg x^{0.0909}$ and $x^{0.025} \ll p_{(2,1,2)}(x) \ll x^{0.9183}$, providing non-trivial polynomial bounds.

The following conjectures, supported by computations and by an empirical approach based on probability theory, are open:

Conjecture 1 *For each k one has:*

$$p_{(2,k,k)}(x) = \frac{x}{(\log x)^{1/2}} G_k + R(x),$$

where $G_k = \sqrt{\frac{2 \log 2}{\pi(k^2+k)}}$ *and* $R(x) = o(x/(\log x)^{1/2})$.

Conjecture 2

$$p_{(2,1,2)}(x) = x^{\alpha+o(1)}$$

where $\alpha = \log 1.6875/\log 2 \simeq 0.7548875$.