Crucial weights
Towards an understanding of weightings by partial order theory

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The basic problem

• Posets support decisions?
• Why yes?
• Why no?
Notations and basics

• \((X, \preceq)\) the poset based on \(m\) indicators and \(X\) the set of objects;
  Objects denoted as \(x(i1), x(i2)\) or just by \(i1, i2\)
  Entries of the data matrix: \(x(i,j)\) for the \(i\)th object and the \(j\)th indicator

• \(g(j)\) the weight for the \(j\)th indicator

• DSS-Model:

\[
\text{Cl}(i, \mathbf{g}) = \sum g(j) \cdot x(i,j); \quad x(i,j) \in [0,1],
\]

\[
\mathbf{g} = (g(1), g(2),...,g(j),...,g(m)) \in G
\]
The technical problem is twofold:

1. Select a tuple $g$
2. Let us model the uncertainty by (mathematical) environments around $g$
Need of a control function

• Selection of a starting tuple $g$
• System of environments around $g$: $\text{Env}_1(g) \subseteq \text{Env}_2(g),...$
• A control function is needed to check the effects of uncertainty with respect to the weights selection.
• This control function is $U = |\{(x,y) \in X^2: x||y\}|$
U = s*U0 is a good approximation [Bruggemann, Carlsen, 2017]...

...with \( s \in [0,1] \) a measure for the uncertainty in weights ...and U0 being the number of incomparabilities if \( s=1 \) (i.e. all weights possible, the original poset based on the indicators)
Fictitious example: 14 objects, $m = 3$ indicators
Cont‘d

Understanding of the deviations from the line $U = s^* U_0$, the „fine-structure“. Analysis of the simplest system for posets with some incomparabilities an $m=2$-system
Motivation: Crucial weights

• Consider two objects A and B and two indicators:
  • A = (0.3, 0.6) and B = (0.4, 0.5): A \parallel B
  • Weights g_1, g_2 with g_1 + g_2 = 1
  • CI(g_1=0.3, g_2=0.7; A) = 0.51; CI(…;B) = 0.47:
    \Rightarrow A > B
  • CI(g_1=0.7, g_2=0.3; A) = 0.39; CI(…;B) = 0.43
    \Rightarrow A < B
  • Crucial weights defined by the requirement:
    CI(A) = CI(B)
Crucial weights

Three crucial weights for six objects \((i_1, \ldots, i_6)\), assuming three Incomparabilities \((i_1, i_2); (i_3, i_4); (i_5, i_6)\)
Two indicators, hence weights $g_1, g_2$ ($g_1 + g_2 = 1$), simplified notation

Assume a certain fixed weight $g_1^*$:

$CI(g_1^*)$ may lead to:

- $i_1 < i_2$
- $i_3 > i_4$
- $i_5 < i_6$

$CI(i_1) = CI(i_2)$
$CI(i_3) = CI(i_4)$
$CI(i_5) = CI(i_6)$

Incomparability increases (starting from $g = g^*, s=0$) with increasing uncertainty interval, when a „crucial weight“ (red vertical lines) is included in the $g_{\text{min}}$ — $g_{\text{max}}$ interval
For $m=2$ all the (gc: crucial weight) gc-values can be calculated by a closed formula (Bruggemann et al, 2008)

The distribution of the gc-values is responsible for the deviations from $U = s*U_0$

$$U = N \star \int h(g, gc) * d\ g$$

• $N$ a normalization factor
• $h$ the distribution of the gc-values,
  seen as quasi continuous function of $g$
• Integral from 0 to 1 (!!!)

Examples....
For example: Modelling by $\int h(x) \, dx$

**Idealized:** $h(x) = N(1-x)$, $x$ instead of $gc(1)$ as convenient abbreviation. $x \in [0,1]$

$U = 2 \int (1-x) \, dx = 2x - x^2$, $U_0$ for the sake of simplicity $= 1$

$U_{standard} = s \cdot U_0$
The crucial type

This is the most crucial type for weighting processes, because 

$h(x)$ has its maximum at 0.5 (Nardo Range)

Example: bridge stability

$U_{an}: \sim \text{error-function of normal distribution.}$
Interim summary

• The deviations from the straight line $U = s*U_0$ are a consequence of the distribution of the crucial weights

• The two dimensional system may be a sufficiently good approximation for a more general system (ambiguity graph)

• Up to now: start value for $g_1$: 0, $s$ increases from 0 to 1.
However!!!

- Following Nardo’s recommendation:
  - $g(j) \approx \frac{1}{m}$, i.e. in an $m=2$-system. $g1 \approx 0.5$
  - We have to take care for the starting tuple $g$
  - Starting with weights near 0 is not the standard!!

Not:

But:

Crucial weights
Child well being

Six indicators: wb, hs, fa, ed, br, sub

2-indicator system: fa, ed
gc(1): Child well being

Indicators:
wb, hs, ed, fa, br, sub,

Actual indicators:
ed, fa

Stable decision situation
i.e. only relatively few incomp.

Unstable decision situation
i.e. relatively many incomp.

$g_1 \approx 0$
Increasing s:
• Few... then
• many, finally
• few
• crucial weights

$g_1 \approx 0.5$
Increasing s:
• many then
• few
• crucial weights
PyHasse

File F:/Python Programme/PyHassedatafiles/child_ed_fa_normalized was opened with 21 rows and 2 columns

start-value of weight1  Steps of the evolution  How many MCruns
0.5  0.1  1000

Length of uncertainty interval
1

(4) weight-evolution
(5) Draw HD of 2-system
(6) show crucial weights
(7) exit
Results

a) Start weight: 0.1, \( s = 0 \) until \( s = 1 \):

b) Start weight: 0.5, \( s = 0 \) until \( s = 1 \):
Thank you for your attention

Questions,
Comments?
References


### PyHasse, normalization

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Results

$g_1 = 0.1$, $s = 0.3$, $MC = 1000$

$g_1 = 0.5$, $s = 0.3$, $MC = 1000$
$s(j) = |\Delta g(j)|$,
assumption: $s(1) = s(2) = \ldots = s(m) = s$

$s$: Uncertainty around $g$

$U = f(s)$