Isometric group actions on Banach spaces and representations vanishing at infinity

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Abstract

Our main result is that the simple Lie group $G = Sp(n, 1)$ acts properly isometrically on $L^p(G)$ if $p > 4n+2$. To prove this, we introduce property BP_0^V, for V be a Banach space: a locally compact group G has property BP_0^V if every affine isometric action of G on V, such that the linear part is a C_0-representation of G, either has a fixed point or is metrically proper. We prove that solvable groups, connected Lie groups, and linear algebraic groups over a local field of characteristic zero, have property BP_0^V. As a consequence for unitary representations, we characterize those groups in the latter classes for which the first cohomology with respect to the left regular representation on $L^2(G)$ is non-zero; and we characterize uniform lattices in those groups for which the first L^2-Betti number is non-zero.

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1 Introduction

Proper isometric actions of groups on Banach spaces which are not Hilbert spaces, were put on centre stage recently: on the one hand due to the work of Fisher-Margulis [FM] on local rigidity of actions of property (T) groups by diffeomorphisms; on the other hand due to the result of Kasparov-Yu [KY], that the Novikov conjecture holds for finitely generated groups embedding uniformly in uniformly convex Banach spaces (in particular for those admitting a proper isometric action on such a space).

This new line of research already produced the following outcome:

- (G. Yu [Yu]) Every Gromov hyperbolic group admits a proper isometric action on the uniformly convex space $\ell^p(\Gamma \times \Gamma)$, for $p \gg 0$.

- (U. Haagerup and A. Przybyszewska [HP], generalizing a previous result in [BG]) Every separable locally compact group G admits a proper isometric action on the strictly convex Banach space $\overline{\bigoplus}_{n=1}^{\infty} L^{2n}(G)$, where $\overline{\bigoplus}$ denotes the ℓ^2-direct sum.

- (U. Bader, A. Furman, T. Gelander and N. Monod [BFGM], Theorem B) Let $G = \prod_{i=1}^{m} G_i(k_i)$, where the k_i's are local fields, $G_i(k_i)$ is the group of k_i-points of a simple algebraic k_i-group G_i with k_i-rank ≥ 2. Let (X, μ) be a σ-finite measure space. Let B be either a closed subspace or a quotient space of $L^p(X, \mu)$ $(1 < p < \infty)$. Let H be either G or a lattice in G. Any proper isometric action of H on B has a (globally) fixed point.

Our main result is the following theorem.

Theorem 1.1. *Let k be a local field. Let G be a simple algebraic group of rank 1 over k. Let p_0 be the Hausdorff dimension of the visual boundary of G. Then, for every $p > \max\{1, p_0\}$, there exists a proper affine action of G on $L^p(G)$ with linear part $\lambda_{G,p}$.*

A key ingredient of the proof is a result of Pansu [Pa1], who computes the first L^p-cohomology for semi-simple Lie groups for $1 < p < \infty$. The first L^p-cohomology actually coincides[1] with the first cohomology of the group with values in the right regular representation on $L^p(G)$. The proof of Theorem 1.1 then consists in showing that non-trivial 1-cocycles on such representations are automatically proper. This latter fact is part of a more general phenomenon: the properness of non-trivial 1-cocycles on an isometric L^p-representation π of a group G is actually true under very general assumptions on G and π.

Our approach was initially motivated by the following example. The cyclic group $\mathbf{Z}$ acts naturally on $\ell^2(\mathbf{Z})$; the corresponding operator T, given by the action of the positive generator of $\mathbf{Z}$ is usually called the bilateral shift. Now take $f \in \ell^2(\mathbf{Z})$, and let us consider the affine isometry T_f of $\ell^2(\mathbf{Z})$ given by $T_f(v) = Tv + f$. It is immediately checked that this isometry has a fixed point if and only if $f \in \operatorname{Im}(T-1)$. We show that otherwise the corresponding action is proper, that is, for every/some $v \in \ell^2(\mathbf{Z})$, $\|{T_f}^n(v)\| \to \infty$ when $|n| \to \infty$. Our aim is to make this observation systematic.

One essential feature in the above context is that the representation of $\mathbf{Z}$ on $\ell^2(\mathbf{Z})$ is C_0. In a general context, let V be a Banach space. An isometric linear

[1] See Section 3.2.

representation π of a locally compact group G is C_0 if for every $L \in V^*$ (the topological dual), and every $v \in V$, we have $L(\pi(g)v) \to 0$ when g tends to infinity. In other words, $\pi(g)v$ weakly tends to zero for every $v \in V$.

Definition 1.2. Let V be a Banach space. A locally compact group G has *Property* (BP_0^V) if, for every C_0 isometric linear representation π of G on V, any affine isometric action of G with linear part π either has a bounded orbit or is proper. We say that G has property (BP_0) if it has (BP_0^V) for every Hilbert space V.

The acronym BP_0 stands for "Bounded or Proper with respect to C_0-representations".

Thus the observation above amounts to prove that $\mathbf{Z}$ has Property BP_0. This is part of the following result.

Proposition 1.3. *(see Proposition 2.10) Let G be a locally compact group, and V a Banach space.*

(1) Suppose that G has two non-compact normal subgroups centralizing each other. Then G has Property BP_0^V.

(2) Suppose that G has a non-compact normal subgroup with Property BP_0^V. Then G has Property BP_0^V.

Corollary 1.4. *(see Corollary 2.12 and Proposition 2.14) Let V be a Banach space.*

1) *Every solvable locally compact group has Property BP_0^V.*

2) *Every connected Lie group or linear algebraic group over a p-adic field has property BP_0^V.*

As an application of Corollary 1.4, we classify in Proposition 4.10 those connected Lie groups and linear algebraic groups over a p-adic field, such that the first cohomology of G with coefficients in the left regular representation λ_G on $L^2(G)$ is non-zero; this generalizes a result of Guichardet (Proposition 8.5 in Chapter III of [Gu2]) for simple Lie groups.

Proposition 1.5. *(see Proposition 4.10) Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$, the group of K-points of a linear algebraic group $\mathbf{G}$ over a local field $\mathbf{K}$ of characteristic zero. Assume G non-compact. Then the following are equivalent*

(i) $H^1(G, \lambda_G) \neq 0$.

(ii) Either G is amenable, or there exists a compact normal subgroup $K \subset G$ such that G/K is isomorphic to $\mathrm{PSL}_2(\mathbf{R})$ (case of Lie groups), or a simple algebraic group of rank one (case of an algebraic group over a p-adic field).

We also characterize those uniform lattices Γ in a group as above, whose first L^2-Betti number $\beta^1_{(2)}(\Gamma)$ is non-zero. For uniform lattices in connected Lie groups, this gives a new proof of Theorem 4.1 in [Eck].

Corollary 1.6. *(see Corollary 4.11) Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$ where $\mathbf{K}$ is a local field of characteristic zero; let Γ be a uniform lattice in G. If the first L^2-Betti number $\beta^1_{(2)}(\Gamma)$ is non-zero, then Γ is commensurable either to a non-abelian free group or to a surface group.*

In contrast with property BP_0, we have

Proposition 1.7. *(see Proposition 5.2) There exists an affine isometric action of $\mathbf{Z}$ on a complex Hilbert space, that is neither proper nor bounded, and whose linear part has no finite-dimensional subrepresentation.*

This result can be extended to $\mathbf{R}$ in view of the following result.

Proposition 1.8. *(see Proposition 5.3) Every isometric action of $\mathbf{Z}$ on a complex Hilbert space can be extended to a continuous action of $\mathbf{R}$.*

While property BP_0 is a rule for connected Lie groups or linear algebraic groups over local fields of characteristic zero, this is certainly not the case for discrete groups:

Proposition 1.9. *(see Proposition 4.2 and Corollary 4.6) Non-abelian free groups and surface groups do not have property BP_0.*

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2 Actions on Banach spaces

We define a *Banach pair* as a pair $(V, \mathcal{L})$ where V is a Banach space and $\mathcal{L}$ is a linear subspace of V^*. We call it a Banach-Steinhaus pair if it satisfies the Banach-Steinhaus Property: any subset $X \subset V$ is bounded if and only if $L(X)$ is bounded for every $L \in \mathcal{L}$. For instance, the Banach-Steinhaus Theorem states that (V^*, V) is a Banach-Steinhaus pair for every Banach space V, and in particular (V, V^*) is a Banach-Steinhaus pair. If $(V, \mathcal{L})$ is a Banach-Steinhaus pair, and if W is a closed subspace of V, then $(W, \mathcal{L}|_W)$ is clearly a Banach-Steinhaus pair, where $\mathcal{L}|_W$ is the set of restrictions of $L \in \mathcal{L}$ to W.

If $(V, \mathcal{L})$ is a Banach pair, we say that an isometric linear action π of a locally compact group G on V is $C_0^{\mathcal{L}}$ if $L(\pi(g)v)$ tends to zero when $g \to \infty$, for every $v \in V$ and $L \in \mathcal{L}$. Note that being $C_0^{\mathcal{L}}$ definitely depends on $\mathcal{L}$ (see the example below); however when the context is clear we write it C_0.

Example 2.1. Let G be a discrete, infinite group. Consider its regular ℓ^1-representation. Then it is $C_0^{c_0(G)}$ but not $C_0^{\ell^\infty(G)}$. Note that both are Banach-Steinhaus pairs. This example motivates the introduction of Banach-Steinhaus pairs different from (V, V^*).

If π is a C_0 representation as above, and if W is a closed invariant subspace, defining a subrepresentation $\pi|^W$, then $\pi|^W$ is $C_0^{\mathcal{L}|_W}$.

Definition 2.2. Let $(V, \mathcal{L})$ be a Banach pair. A locally compact group G has *Property* $(FH_0^{(V,\mathcal{L})})$ *(respectively* $(BP_0^{(V,\mathcal{L})})$*)* if, for every C_0-representation π of G, any affine isometric action of G on V with linear part π has a bounded orbit (resp. either has a bounded orbit or is proper).

We say that G has *Property* $(FH_0^{([V],\mathcal{L})})$ if it has Property $(\mathrm{FH}_0^{(W,\mathcal{L}|_W)})$ for every closed subspace W of V. We define analogously Property $(\mathrm{BP}_0^{([V],\mathcal{L})})$.

Similarly, we say that G has *Property* (FH_0^V) *(respectively* (BP_0^V)*)* if it has *Property* $(FH_0^{(V,\mathcal{L})})$ *(respectively* $(BP_0^{(V,\mathcal{L})})$*)* for $\mathcal{L} = V^*$.

When the space V is superreflexive, i.e. isomorphic to a uniformly convex space, it is known that every nonempty bounded subset has a unique circumcenter (also called Chebyshev center, see p. 27 in [BL]). As a consequence, every isometric action with a bounded orbit on V has a globally fixed point.

Lemma 2.3. *Let a compact group K act by affine isometries on a Banach space. Then it fixes a point.*

Proof. Let Ω be an orbit. As Ω is compact, its closed convex hull X is also compact (see e.g. Theorem 3.25 in [Rud]). As K is amenable and acts on X by affine transformations, it has a fixed point. □

Lemma 2.4. *Let $(V, \mathcal{L})$ be a Banach pair. Let K be a compact, normal subgroup of G. The following are equivalent.*

(i) G has Property $FH_0^{([V],\mathcal{L})}$ (resp. $BP_0^{([V],\mathcal{L})}$);

(ii) G/K has Property $FH_0^{([V],\mathcal{L})}$ (resp. $BP_0^{([V],\mathcal{L})}$).

Proof. The implication (i)⇒(ii) is clear. Suppose that G/K has Property $\mathrm{BP}_0^{([V],\mathcal{L})}$. By Lemma 2.3, the set W of K-fixed points is a non-empty closed affine subspace; moreover it is G-invariant. As G has Property $\mathrm{BP}_0^{(W,\mathcal{L}|_W)}$, its action on W, and therefore on V, is either bounded or proper. The case of Property $\mathrm{FH}_0^{([V],\mathcal{L})}$ is similar. □

Lemma 2.5. *Let $(V,\mathcal{L})$ be a Banach-Steinhaus pair. Let H, K be closed, non-compact subgroups of the locally compact group G which centralize each other. Let α be an affine isometric action of G on V, whose linear part π is a C_0-representation. Then either $\alpha|_H$ and $\alpha|_K$ are both bounded, or they are both proper.*

Proof. Set $b(g) = \alpha(g)(0)$. We assume that $\alpha|_H$ is not proper, i.e.

$$M =: \liminf_{h\to\infty, h\in H} \|b(h)\| \; < \; \infty.$$

For $k \in K$, $h \in H$, the 1-cocycle relation gives

$$\pi(k)b(h) + b(k) \; = \; b(kh) \; = \; b(hk) \; = \; \pi(h)b(k) + b(h),$$

which we will use in the following form:

$$b(k) \; = \pi(h)b(k) + (1 - \pi(k))b(h).$$

Then, for every $L \in \mathcal{L}$ we have

$$L(b(k)) = L(\pi(h)b(k)) + L((1 - \pi(k))b(h)),$$

and thus

$$|L(b(k))| \le |L(\pi(h)b(k))| + |L((1 - \pi(k))b(h))| \le |L(\pi(h)b(k))| + 2\|L\|\|b(h)\|.$$

Taking the inferior limit when $h \to \infty$ in H, we obtain

$$|L(b(k))| \le 2\|L\|M.$$

Thus $L(b(K))$ is bounded for every L; as $(V,\mathcal{L})$ is a Banach-Steinhaus pair this means that $b(K)$ is bounded.

Inverting the roles of H and K, we can easily conclude. □

The following proposition is an immediate consequence of Lemma 2.5, by taking $H = G$ and $K = Z(G)$.

Proposition 2.6. *Let G be a locally compact group with non-compact centre (e.g. a non-compact, locally compact abelian group). Then G has property $BP_0^{(V,\mathcal{L})}$ for every Banach-Steinhaus pair $(V, \mathcal{L})$.* $\square$

In order to enlarge the class of groups for which we are able to prove Property BP_0, we need the following lemma.

Lemma 2.7. *Let $(V, \mathcal{L})$ be a Banach-Steinhaus pair. Let α be an affine isometric action of G on V, with linear part a C_0-representation π. Set $b(g) = \alpha(g)(0)$. Let H be a closed, non-compact subgroup of G. Assume that there exists a sequence $(g_k)_{k\geq 1}$ in G, going to infinity, such that*

- *the sequence $(b(g_k))$ is bounded in V;*
- *for every $h \in H$, the set $\{g_k^{-1}hg_k | \, k \geq 1\}$ is relatively compact in G.*

Then $\alpha|_H$ is bounded.

Proof. Fix $M > 0$ such that $\|b(g_k)\| \leq M$ for every $k \geq 1$, and, for $h \in H$ define K_h as the closure of the set $\{g_k^{-1}hg_k | \, k \geq 1\}$, which is compact by assumption. Let us show that $\|b(h)\| \leq 2M$ for every $h \in H$. Noting that $hg_k = g_k h_k$, where $h_k = g_k^{-1}hg_k$, we expand $b(hg_k) - b(g_k) - b(h)$ in two ways, and we obtain

$$\pi(g_k)b(h_k) - b(h) = (\pi(h) - 1)b(g_k).$$

So, for every $L \in \mathcal{L}$,

$$|L(b(h))| \leq |L(\pi(g_k)b(h_k))| + |L((\pi(h) - 1)b(g_k))|$$

$$\leq |L(\pi(g_k)b(h_k))| + 2\|L\|\|b(g_k)\|.$$

Using the assumption that $h_k \in K_h$ for every k, and the fact that for a C_0-representation the decay of coefficients to 0 is uniform on compact subsets of the ambient Banach space, we get for $k \to \infty$,

$$|L(b(h))| \leq 2M\|L\|.$$

As $(V, \mathcal{L})$ is a Banach-Steinhaus pair, this implies that $b(H)$ is bounded. $\square$

The following lemma is a kind of a geometric Hahn-Banach statement.

Lemma 2.8. *Let $(V, \mathcal{L})$ be a Banach-Steinhaus pair. Then there exists $\eta > 0$ with the following property: for every $v \in V$, there exists $L \in \mathcal{L}$ such that $\|L\| \leq 1$ and $L(v) \geq \eta\|v\|$.*

Proof. Suppose the contrary. For every n, there exists $v_n \in V$ of norm one such that for every $L \in \mathcal{L}$, we have $L(v_n) < 2^{-n}\|L\|$. Set $X = \{2^n v_n | n \geq 0\}$. Then $L(X)$ is bounded for every $L \in \mathcal{L}$; by the Banach-Steinhaus Property, X is bounded; this is a contradiction. □

Lemma 2.9. *Let $(V, \mathcal{L})$ be a Banach-Steinhaus pair. Let G be a locally compact group, and N a non-compact, closed normal subgroup. Let α be an affine isometric action of G on V whose linear part is C_0.*

(1) Suppose that $\alpha|_N$ is bounded. Then α is also bounded.

(2) Suppose that $\alpha|_N$ is proper. Then α is also proper.

Proof. (1) For $M \geq 0$, define A_M as the set of all $x \in V$ whose N-orbit has diameter at most M. Clearly A_M is G-invariant. By assumption, for some M (which we fix now), the set A_M is non-empty. We claim that it is bounded, allowing us to conclude.

Consider $x, y \in A_M$, and set $v = x - y$. Then, for $h \in N$

$$\pi(h)v - v = \alpha(h)x - x - \alpha(h)y + y.$$

So

$$\|\pi(h)v - v\| \leq \|\alpha(h)x - x\| + \|y - \alpha(h)y\| \leq 2M.$$

Fix η and L as in Lemma 2.8. Then

$$\eta\|v\| \leq L(v) \leq |L(v) - L(\pi(h)v)| + |L(\pi(h)v)| \leq 2M + |L(\pi(h)v)|.$$

As N is non-compact, letting $h \to \infty$, we obtain $\|v\| \leq 2M/\eta$. Thus the diameter of A_M is bounded by $2M/\eta$.

(2) Suppose by contradiction that $\alpha|_N$ is proper and α is not proper. Then there exists a sequence (g_k) in G, tending to infinity, such that $(b(g_k))$ is bounded. As $\alpha|_N$ is unbounded, there exists, by Lemma 2.7, an element $n \in N$ such that the sequence $(g_k^{-1} n g_k)_{k \geq 1}$ is not relatively compact in N; extracting if necessary we can suppose that it tends to infinity. Therefore, as $\alpha|_N$ is proper, $\|b(g_k^{-1} n g_k)\|$ tends to infinity. But it is bounded by $2\|b(g_k)\| + \|b(n)\|$, which is bounded, a contradiction. □

From this we deduce the following.

Proposition 2.10. *Let $(V, \mathcal{L})$ be a Banach-Steinhaus pair, and let G be a locally compact group. Let N be a non-compact, closed, normal subgroup of G. If either the centralizer $C_G(N)$ of N is non-compact, or N has Property $BP_0^{(V,\mathcal{L})}$, then G also has Property $BP_0^{(V,\mathcal{L})}$.*

Proof. Let α be an affine isometric action of G, with linear part a C_0-representation π. If $\alpha|_N$ is bounded, then α is bounded by Lemma 2.9(1).

Assume then that $\alpha|_N$ is unbounded. Then $\alpha|_N$ is proper (in case $C_G(N)$ is non-compact, this follows from lemma 2.5). Accordingly, by Lemma 2.9(2), α is proper. □

Corollary 2.11. *Let $(V, \mathcal{L})$ be a Banach-Steinhaus pair. Then Properties $BP_0^{([V],\mathcal{L})}$ and $FH_0^{([V],\mathcal{L})}$ are preserved by extensions.*

Proof. Let $1 \to N \to G \to H \to 1$ be an extension of locally compact groups, and suppose that N and H have Property $\mathrm{BP}_0^{([V],\mathcal{L})}$. If N is compact, then, by Lemma 2.4, since H has Property $\mathrm{BP}_0^{([V],\mathcal{L})}$, so does G. If N is not compact, then, since it has Property $\mathrm{BP}_0^{([V],\mathcal{L})}$, by Proposition 2.10, G has Property $\mathrm{BP}_0^{([V],\mathcal{L})}$. The case of Property $\mathrm{FH}_0^{([V],\mathcal{L})}$ is similar (and easier). □

Corollary 2.12. *Locally compact, solvable groups have Property $BP_0^{(V,\mathcal{L})}$ for every Banach-Steinhaus pair $(V, \mathcal{L})$.*

Proof. Since, using Proposition 2.6, locally compact abelian groups have Property $\mathrm{BP}_0^{(V,\mathcal{L})}$, this follows from Corollary 2.11. □

Lemma 2.13. *Let $(V, \mathcal{L})$ be a Banach pair. Property $BP_0^{(V,\mathcal{L})}$ is inherited from cocompact subgroups.*

Proof. The proof is straightforward. □

Proposition 2.14. *Connected Lie groups, and linear algebraic groups over p-adic fields, have Property $BP_0^{(V,\mathcal{L})}$ for every Banach-Steinhaus pair $(V, \mathcal{L})$.*

Proof. This follows from Lemma 2.13 and Corollary 2.12, since G contains a solvable cocompact subgroup H: for linear algebraic groups over local fields of characteristic zero, this follows from [BT, Théorème 8.2]; for Lie groups, taking the quotient by the maximal solvable normal subgroup, we can also use [BT, Théorème 8.2]. □

3 Proper affine actions of rank 1 groups on L^p-spaces

3.1 Spaces with measured walls, and the non-archimedean case

Recall that a locally compact σ-compact group is a-T-menable if it acts properly isometrically on some Hilbert space.

Proposition 3.1. *Let Γ be a countable, discrete group. The following are equivalent:*

i) Γ is a-T-menable;

ii) for every $p \geq 1$, the group Γ acts properly isometrically on some L^p-space.

Proof. We prove the non-trivial implication $(i) \Rightarrow (ii)$. We recall from [CMV] that a space with measured walls is a 4-tuple $(X, \mathcal{W}, \mathcal{B}, \mu)$ where X is a set, $\mathcal{W}$ is a set of partitions of X into 2 classes (called walls), $\mathcal{B}$ is a σ-algebra of sets on $\mathcal{W}$, and μ is a measure on $\mathcal{B}$ such that, for every pair x, y of distinct points in X, the set $\omega(x, y)$ of walls separating x from y belongs to $\mathcal{B}$ and satisfies $w(x, y) =: \mu(\omega(x, y)) < \infty$.

It was proved in Proposition 1 of [CMV] that a countable group is a-T-menable if and only if it admits a proper action on some space with measured walls (by this we means that Γ preserves the measured wall space structure, and that the function $g \mapsto w(gx, x)$ is proper on Γ).

A half-space in a space with measured walls X is a class of the partition defined by some wall in $\mathcal{W}$. Let Ω be the set of half-spaces, $p : \Omega \to \mathcal{W}$ the canonical map (associating to any half-space the corresponding wall), $\mathcal{A} =: p^{-1}(\mathcal{B})$ the pulled-back σ-algebra, and ν the pulled-back measure defined by

$$\nu(A) = \frac{1}{2} \int_{\mathcal{W}} card(A \cap p^{-1}(x)) \, d\mu(x)$$

for $A \in \mathcal{A}$. Let χ_x be the characteristic function of the set of half-spaces through x. For $x, y \in X$, we define a function $c(x, y) \in L^p(\Omega, \nu)$ by:

$$c(x, y) = \chi_x - \chi_y.$$

Suppose that Γ acts properly on $(X, \mathcal{W}, \mathcal{B}, \mu)$. For $p \geq 1$, let π_p denote the quasi-regular representation of Γ on $L^p(\Omega, \nu)$. Observe that:

- $c(x, y) + c(y, z) = c(x, z)$;
- $c(gx, gy) = \pi_p(g) c(x, y)$;
- $\|c(x, y)\|_p^p = w(x, y)$

for every $x, y, z \in X$, $g \in \Gamma$. Fixing a base-point $x_0 \in X$, the map

$$b : \Gamma \to L^p(\Omega, \nu) : g \mapsto c(gx_0, x_0)$$

defines a 1-cocycle in $Z^1(\Gamma, \pi_p)$. Since $\|b(g)\|_p = w(gx_0, x_0)^{1/p}$, this cocycle is proper, so that the corresponding affine isometric action is proper. $\square$

Remark 3.2. : What the above proof really shows is that every locally compact group acting properly on a space with measured walls, admits a proper action on some L^p-space, for every $p \geq 1$. Several non-discrete examples appear in [CMV].

A tree $X = (V, E)$ is an example of a space with measured walls (with $\mathcal{W} = E$, $\mu =$ counting measure). The set Ω of half-spaces identifies with the set $\mathbb{E}$ of oriented edges. Suppose that a locally compact group G acts properly on a tree X, transitively on edges (so that G has one or two orbits on $\mathbb{E}$). We choose a reference edge $e_0 \in \mathbb{E}$ and use it to lift the cocycle $b \in Z^1(G, \ell^p(\mathbb{E}))$ from the previous proof to a cocycle $\tilde{b} \in Z^1(G, L^p(G))$, by the formula $(\tilde{b}(g))(h) = (b(g))(he_0)$. Then $\|\tilde{b}(g)\|_p^p = \frac{m_0 d(gx_0, x_0)}{k}$, where m_0 is the Haar measure of the stabilizer of e_0 in G, and $k \in \{1, 2\}$ is the number of orbits of G in $\mathbb{E}$. This shows that $\tilde{b}$ is a proper cocycle. We have proved:

Proposition 3.3. *Let G be a locally compact group. If G acts properly a tree, transitively on edges (e.g. if G is a rank 1 simple algebraic group over a non-archimedean local field), then for every $p \geq 1$, the group G admits a proper isometric action on $L^p(G)$, with linear part the left regular representation $\lambda_{G,p}$.* $\square$

3.2 The Lie group case

Let M be a Riemannian manifold equipped with its Riemannian measure μ. Fix $p > 1$. Denote by $D_p(M)$ the vector space of differentiable functions whose gradient is in $L^p(TM)$. Equip $D_p(M)$ with a pseudo-norm $\|f\|_{D_p} = \|\nabla f\|_p$, which induces a norm on $D_p(M)$ modulo the constants. Denote by $\mathbf{D}_p(M)$ the completion of this normed vector space. We have $W^{1,p}(M) = L^p(M) \cap D_p(M)$. Hence, $W^{1,p}(M)$ canonically embeds in $\mathbf{D}_p(M)$ as a subspace.

The first L^p-cohomology of M is the quotient space

$$H_p^1(M) = \mathbf{D}_p(M)/W^{1,p}(M).$$

The first *reduced* L^p-cohomology of M is the quotient space

$$\overline{H_p}^1(M) = \mathbf{D}_p(M)/\overline{W^{1,p}(M)},$$

where $\overline{W^{1,p}(M)}$ is the closure of $W^{1,p}(M)$ in the Banach space $\mathbf{D}_p(M)$. Note that the two latter spaces coincide if and only if the norm $\|\cdot\|_{D_p}$ on the Sobolev space $W^{1,p}(M)$ is equivalent to the usual Sobolev norm $\|\cdot\|_p + \|\cdot\|_{D_p}$, that is, if M satisfies the strong Sobolev inequality in L^p: $\|f\|_p \leq C\|\nabla f\|_p$. If the group of isometries G of M acts co-compactly on M, the strong Sobolev inequality in L^p is satisfied if and only if G is either non-amenable or non-unimodular [Pit].

Assume now that $M = G$ is a connected, unimodular Lie group, endowed with a left invariant Riemannian metric. Denote by $\rho_{G,p}$ the right regular representation on $D_p(G)$. Let $g \in G$ and $\gamma : [0, d(1,g)] \to G$ be a geodesic between 1 and g. For any $f \in D_p(G)$ and $x \in G$, we have

$$(f - \rho_{G,p}(g)f)(x) = f(x) - f(xg) = \int_0^{d(1,g)} \nabla f(\gamma_x(t)) \cdot \gamma_x'(t) dt,$$

where $\gamma_x(t) = x\gamma(t)$. Using Hölder's inequality, we deduce that

$$\|f - \rho_{G,p}(g)f\|_p \leq d(1,g)\|\nabla f\|_p.$$

Therefore, there is a well defined map from $D_p(G)$ to $Z^1(G, \rho_{G,p})$

$$J : f \mapsto (b_f : g \mapsto f - \rho_{G,p}(g)f).$$

The map J induces an injective map from $\mathbf{D}_p(G)$ to $Z^1(G, \rho_{G,p})$. Moreover, b_f is a co-boundary if and only if f is in $L^p(G) + \{\text{constants}\}$, i.e. if and only if the class of f is zero in $H_p^1(G)$. Hence, J induces an injective[2] linear map from $H_p^1(G)$ to $H^1(G, \rho_{G,p})$.

Let G be a simple Lie group of rank 1 equipped with a left-invariant Riemannian metric. Up to taking the quotient by a normal compact subgroup, G is $PO(n,1)$, $PU(n,1)$, $PSp(n,1)$ or $F_{4(-20)}$. Let ∂G be the sphere at infinity of G, and let p_0 be its Hausdorff dimension, so that

$$p_0 = \begin{cases} n-1 & if \quad G = PO(n,1) \\ 2n & if \quad G = PU(n,1), n \geq 2 \\ 4n+2 & if \quad G = PSp(n,1) \\ 22 & if \quad G = F_{4(-20)} \end{cases}$$

By a result of P. Pansu [Pa1], $H_{(p)}^1(G) \neq 0$ if and only if $p > \max\{1, p_0\}$. From the above discussion, we deduce that $H^1(G, \rho_{G,p}) \neq 0$ for those groups as soon as $p > \max\{1, p_0\}$. Together with the fact that connected Lie groups have property $(BP_0^{L^p})$ for $1 < p < \infty$, this yields the following result.

Theorem 3.4. *Let G be a simple Lie group of rank 1 over k. Let p_0 be the Hausdorff dimension of the visual boundary of G. Then, for every $p > \max\{1, p_0\}$, there exists a proper affine action of G on $L^p(G)$ with linear[3] part $\lambda_{G,p}$.* □

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[2]Actually, J induces an isomorphism of topological vector spaces [T] but this is much more delicate and not needed here.

[3]Since G is unimodular, the representations $\lambda_{G,p}$ and $\rho_{G,p}$ are isomorphic.

4 Action on Hilbert spaces

4.1 Property BP_0

Recall that a locally compact group G has Property (FH) if every affine isometric action of G on a Hilbert space has a fixed point. For G σ-compact, this is known to be equivalent to the celebrated Kazhdan's Property (T) (see [HV]).

When V is a Hilbert space (sufficiently large in comparison to G), we write BP_0 and (FH_0) for (BP_0^V) and (FH_0^V).

There is a simple characterization of groups with Property (FH_0) among groups with Property BP_0.

Proposition 4.1. *Let G be a locally compact group.*

1) *Suppose that G has Property BP_0. Then either G is a-T-menable or has Property (FH_0).*

2) *If G is both a-T-menable and has Property (FH_0), then it is compact.*

Proof. The first statement is clear. Suppose that G is a-T-menable and is not compact. Then G is σ-compact (take a proper action α and consider $K_n = \{g \in G : \|\alpha(g)(0)\| \leq n\}$). Since G is a-T-menable, it is Haagerup, so that it has a C_0-representation π with almost invariant vectors; since G is not compact, π has no invariant vector. By Proposition 2.5.3 in [BHV], $\infty\pi$ has nontrivial 1-cohomology, while it is C_0. Hence G does not have Property (FH_0). □

4.2 Discrete groups without BP_0

Proposition 2.14 provides a wealth of groups with Property BP_0. We now provide examples of groups *without* Property BP_0 (in particular the free group F_n on n generators, $n \geq 2$).

Proposition 4.2. *Let H be an infinite group, K a non-trivial group, and F a common finite subgroup of H and K, which is distinct from K. Let $G = H *_F K$ be the amalgamated product. Then there exists a 1-cocycle with respect to the regular representation λ_G which is neither bounded nor proper. In particular, G does not have Property BP_0.*

Proof. Let w be a ℓ^2 function on G which is left F-invariant, but not left K-invariant (in particular $w \neq 0$). Define $\alpha(k) = \lambda_G(k)$ for $k \in K$, and $\alpha(h) = t_w \circ \lambda_G(h) \circ t_{-w}$ for $h \in H$, where t_w denotes the translation by w on $\ell^2(G)$. Then α is well-defined on $H *_F K$ (by the F-invariance assumption on w). The fixed point set of K is the set of all left K-invariant functions. The set of fixed points of H is reduced to $\{w\}$ (since H is infinite). Accordingly, the action has no fixed point. On the other hand, since H is infinite and has a fixed point, the action is not proper. □

To produce more examples of groups without (BP_0), we first establish a connection with a classical conjecture on discrete groups. For a group Γ, we denote by $\mathbf{C}\Gamma$ the group algebra over $\mathbf{C}$, and by denote again by λ_Γ the left regular representation of $\mathbf{C}\Gamma$ on $\ell^2(\Gamma)$:

$$\lambda_\Gamma(f)\xi = f * \xi$$

($f \in \mathbf{C}\Gamma$, $\xi \in \ell^2(\Gamma)$). Here is the *analytical zero-divisor conjecture*:

Conjecture 1. If Γ is a torsion-free group, then $\lambda_\Gamma(f)$ is injective, for every non-zero $f \in \mathbf{C}\Gamma$.

The main result on Conjecture 1 is due to P. Linnell [Lin]: it holds for groups which are extensions of a right-orderable group by an elementary amenable group; in particular, we will use the fact that it holds for non-abelian free groups.

Lemma 4.3. *Let Γ be a group satisfying Conjecture 1. Let $f_1, f_2 \in \mathbf{C}\Gamma$ be non-zero elements. There exists non-zero functions $\xi_1, \xi_2 \in \ell^2(\Gamma)$ such that $\lambda_\Gamma(f_1)\xi_1 + \lambda_\Gamma(f_2)\xi_2 = 0$.*

Proof. We start with a

Claim: If $f \in \mathbf{C}\Gamma$ is a non-zero element, then $\lambda_\Gamma(f)$ has dense image. To see that, observe that the orthogonal of the image of $\lambda_\Gamma(f)$ is the kernel of $\lambda_\Gamma(f^*)$, which is $\{0\}$ as Γ satisfies Conjecture 1.

Let $L(\Gamma)$ be the von Neumann algebra of Γ, i.e. the bi-commutant of $\lambda_\Gamma(\mathbf{C}\Gamma)$ in $\mathcal{B}(\ell^2(\Gamma))$. A (non-necessarily closed) subspace of $\ell^2(\Gamma)$ is *affiliated with* $L(\Gamma)$ if it is invariant under the commutant $\lambda_\Gamma(\mathbf{C}\Gamma)'$ of $\lambda_\Gamma(\mathbf{C}\Gamma)$. E.g., if $f \in \mathbf{C}\Gamma$, the image of $\lambda_\Gamma(f)$ is an affiliated subspace. A result of L. Aagaard [Aag] states that the intersection of two dense, affiliated subspaces is still dense. We apply this with the images of $\lambda_\Gamma(f_1)$ and of $\lambda_\Gamma(-f_2)$, so that there exist non-zero ξ_1, ξ_2 such that $\lambda_\Gamma(f_1)\xi_1 = \lambda_\Gamma(-f_2)\xi_2$. □

Proposition 4.4. *Fix $k \geq 2$. Let w be a non-trivial reduced word in the free group F_k. There exists an unbounded 1-cocycle $b_w \in Z^1(F_k, \lambda_{F_k})$, such that $b_w(w) = 0$. In particular, b is not proper.*

Proof. We start with $k = 2$. Write F_2 as the free group on 2 generators s, t. Write w as a reduced word in $s^{\pm 1}, t^{\pm 1}$:

$$w = x_1^{\epsilon_1} x_2^{\epsilon_2} ... x_n^{\epsilon_n}$$

($x_j \in \{s, t\}; \epsilon_j \in \{-1, 1\}$). If either s or t does not appear in w, then the existence of the desired cocycle follows from the proof of Proposition 4.2 (with $H = K = \mathbf{Z}$). So may assume that both s and t appear in w. Set $\delta_j = \frac{\epsilon_j - 1}{2}$ and define two elements $f_{w,s},\ f_{w,t} \in \mathbf{C}F_2$ by:

$$f_{w,s} = \sum_{j:x_j = s} \epsilon_j x_1^{\epsilon_1} ... x_{j-1}^{\epsilon_{j-1}} x_j^{\delta_j};$$

$$f_{w,t} = \sum_{j:x_j = t} \epsilon_j x_1^{\epsilon_1} ... x_{j-1}^{\epsilon_{j-1}} x_j^{\delta_j}.$$

Note that $f_{w,s}$ and $f_{w,t}$ are non-zero, as s and t appear in w. Since F_2 satisfies Conjecture 1 (by Linnell's result already quoted [Lin]), we may appeal to lemma 4.3 and find non-zero functions $\xi_s, \xi_t \in \ell^2(F_2)$ such that $\lambda_{F_2}(f_{w,s})\xi_s + \lambda_{F_2}(f_{w,t})\xi_t = 0$.

Set then $b_w(s) = \xi_s$, $b_w(t) = \xi_t$ and, using freeness of F_2, extend uniquely to a 1-cocycle $b_w \in Z^1(F_2, \lambda_{F_2})$. Using the relations $b(g_1 g_2 ... g_m) = \sum_{j=1}^m \lambda_{F_2}(g_1 ... g_{j-1}) b(g_j)$ and $b(g^{-1}) = -\lambda_{F_2}(g^{-1}) b(g)$ (for a cocycle b and $g_1, ..., g_m, g \in F_2$), one checks that

$$b_w(w) = \lambda_{F_2}(f_{w,s}) b_w(s) + \lambda_{F_2}(f_{w,t}) b_w(t) = 0.$$

It remains to show that b_w is unbounded, i.e. that the corresponding affine action α_w has no fixed point. Let $H =< w >$ be the cyclic subgroup generated by w. As the linear action is C_0, the only fixed point of $\alpha_w|_H$ is 0. But 0 is clearly not fixed under $\alpha_w(s)$ or $\alpha_w(t)$, which completes the proof in case $k = 2$.

Suppose now $k \geq 2$. View F_k as a subgroup of index $k - 1$ in F_2. The restriction of λ_{F_2} to F_k is the direct sum of $k - 1$ copies of λ_{F_k}. Project the cocycle b_w given by the first part of the proof, to each of these $k - 1$ summands. This way, get $k - 1$ cocycles in $Z^1(F_k, \lambda_{F_k})$, each of them vanishing on w. At least one of them is unbounded, because $b|_{F_k}$ is unbounded. □

Corollary 4.5. *For $k \geq 2$, let $\Gamma = F_k *_{\mathbf{Z}} G$ be an amalgamated product over $\mathbf{Z}$ an infinite cyclic subgroup. Then Γ does not have property (BP_0).*

Proof. Let $w \in F_k$ and $g \in G$ correspond to the positive generators of the copies of $\mathbf{Z}$ that are amalgamated. Choosing representatives for the left cosets of F_k

in Γ, identify $\lambda_\Gamma|_{F_k}$ with $\infty\lambda_{F_k} =: \lambda_{F_k} \oplus \lambda_{F_k} \oplus ....$ Let $b_w \in Z^1(F_k, \lambda_{F_k})$, as in Proposition 4.4. Define an affine action α of F_k, with linear part $\lambda_\Gamma|_{F_k}$, by:

$$\alpha(x)(v_1, v_2, v_3, ...) \; = \; (\lambda_{F_k}(x)v_1 + b_w(x), \lambda_{F_k}(x)v_2, \lambda_{F_k}(x)v_3, ...)$$

$(x \in F_k)$. On the other hand, view $\lambda_\Gamma|_G$ as an affine action of G. Since

$$\alpha(w) = \lambda_\Gamma(w) = \lambda_\Gamma(g),$$

these two affine actions can be "glued together", i.e. extend to an affine action $\tilde{\alpha}$ of Γ, with linear part λ_Γ. By the very construction, $\tilde{\alpha}$ has unbounded orbits and is not proper. □

Corollary 4.6. *The surface groups Γ_g $(g \geq 2)$ do not have Property (BP$_0$).*

Proof. The presentation

$$\Gamma_g \; = \; \langle a_1, ..., a_g, b_1, ..., b_g | [a_1, b_1]^{-1} = \prod_{j=2}^{g} [a_j, b_j] \rangle$$

exhibits Γ_g as an amalgamated product $F_2 *_{\mathbf{Z}} F_{2g-2}$ so Corollary 4.5 applies. □

Here is an intriguing question, in view of the fact that $PSL_2(\mathbf{Z})$ contains a free group of finite index:

Question 1. Does $PSL_2(\mathbf{Z}) \simeq C_2 * C_3$ have Property BP$_0$?

4.3 Application to the regular representation

Let us recall that Guichardet [Gu1, Théorème 1] proved that, if π is a representation without non-zero fixed vector of a locally compact, σ-compact group, the space $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$ (endowed with the topology of uniform convergence on compact subsets) if and only if π does not almost have invariant vectors. In particular $H^1(G, \pi) \neq 0$ if π almost has invariant vectors. This rests on a clever use of the open mapping theorem for Fréchet spaces. Using this, we can reprove the following result, first proved in [AW] (see also [BCV]).

Proposition 4.7. *Let G be a σ-compact, locally compact group. If G is Haagerup, then it is a-T-menable.*

Proof. Set $H = G \times \mathbf{Z}$; then H is σ-compact, locally compact, is Haagerup, and has noncompact center. Hence, by Proposition 2.6, it has Property BP$_0$. Take a C_0-representation π of H, almost having invariant vectors. By Guichardet's result recalled above, there exists an affine action α of H, with linear part π, and without fixed point. By property BP$_0$, the action α is proper. So the restriction $\alpha|_G$ is proper too. □

If G is σ-compact and amenable, the representation π in the above proof can be taken as the left regular representation of $G \times \mathbf{Z}$ on $L^2(G \times \mathbf{Z})$. (By way of contrast, if Γ is a discrete, non-amenable group, then $H^1(\Gamma \times \mathbf{Z}, \lambda_{\Gamma\times\mathbf{Z}}) = 0$ by Corollary 10 in [BV]).

Concerning affine actions on $L^2(G)$, we have the following

Conjecture 2. For an amenable group G, every affine action with linear part λ_G is either bounded or proper.

Evidence for this conjecture comes from the fact that Proposition 2.10, Corollary 2.12 and Proposition 2.14 establish it in numerous cases: amenable groups with infinite center, solvable groups, amenable Lie groups, etc... More evidence comes from a result proved in [MV]: if Γ is a countable amenable group, and A is any infinite subgroup, then the restriction map $H^1(\Gamma, \lambda_\Gamma) \to H^1(A, \lambda_\Gamma|_A)$ is injective. If true, our conjecture would provide a conceptual explanation of this fact.

Being more ambitious, one may even ask

Question 2. Does every amenable group have Property BP_0?

A test-case for this question and the conjecture is provided by locally finite groups.

We now turn to the study of some groups G for which $H^1(G, \lambda_G) \neq 0$.

Lemma 4.8. *Let G be a locally compact, second countable group. Suppose that, for some $k \geq 2$, the group G has closed normal subgroups $N_1, \ldots, N_k$ such that $[N_i, N_j] = 1$ whenever $i \neq j$ and $G = N_1 \cdot \cdots \cdot N_k$. Let π be a unitary representation such that $\overline{H^1}(G, \pi) \neq 0$. Then at least one of the N_i has an invariant vector by π.*

Proof. There is an obvious map p of $N = \prod_{i=1}^{k} N_i$ onto G. Then $\overline{H^1}(N, \pi \circ p) \neq 0$. This uses the standard fact that every compact subset of G is the image of a compact subset of N (note that we use here σ-compactness).

Suppose that for some i, the group N_i has no invariant vector by $\pi \circ p$. Write $N = N_i \times \prod_{j \neq i} N_j$; by [Sha, Proposition 3.2] (which uses second countability), $\prod_{j \neq i} N_j$ has an invariant vector by $\pi \circ p$, so that for every $j \neq i$, N_j has an invariant vector by $\pi \circ p$. □

Proposition 4.9. *Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$, the group of $\mathbf{K}$-points of a linear algebraic group $\mathbf{G}$ over a local field $\mathbf{K}$ of characteristic zero. Suppose that G has a C_0-representation π such that $\overline{H^1}(G, \pi) \neq 0$. Then either G is amenable, or has a compact subgroup K such that G/K is a simple Lie group (resp. a simple linear algebraic group) with trivial centre.*

Proof. By Property BP_0, G has the Haagerup Property. If G is a connected Lie group, by [CCJJV, Chap. 4], $G = RS_1 \dots S_k$ where $R, S_1, \dots, S_k$ centralize each other, R is a connected amenable Lie group, and each S_i is a simple, noncompact, connected Lie group with the Haagerup Property (with possibly infinite centre). In the case of an algebraic group, the same conclusion holds [Cor], except that the S_i's are simple linear algebraic groups.

If G is not amenable, then $k \geq 1$, and in this case by Lemma 4.8 it follows that $k = 1$ and R is compact. By [Sha, Corollary 3.6], the centre $Z(G)$ has an invariant vector by π and thus is compact since π is C_0; since in our situation $Z(S_1) \subset Z(G)$, we see that S_1 has finite centre, so that $K = RZ(S_1)$ is compact and G/K is a simple group with trivial centre. □

Proposition 4.10. *Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$ where $\mathbf{K}$ is a local field of characteristic zero. Assume G non-compact. Then the following are equivalent*

(i) $H^1(G, \lambda_G) \neq 0$.

(ii) Either G is amenable, or there exists a compact normal subgroup $K \subset G$ such that G/K is isomorphic to $\mathrm{PSL}_2(\mathbf{R})$ *(case of Lie groups), or a simple algebraic group of rank one (case of an algebraic group over a p-adic field).*

Proof. Suppose (i). If G is not amenable, then, by the result of Guichardet already mentioned [Gu1, Théorème 1], one has $\overline{H^1}(G, \lambda_G) = H^1(G, \lambda_G) \neq 0$. By Proposition 4.9, G has a compact normal subgroup K such that $S = G/K$ is simple with trivial centre. Moreover, G does not have Property (T), hence has rank one [DK]. This settles the non-Archimedean case. If G is a Lie group, then by [Mar, Theorem 6.4], λ_G contains an irreducible subrepresentation σ factoring through S, such that $\overline{H^1}(G, \sigma) = \overline{H^1}(S, \sigma) \neq 0$. Then $\sigma \leq \lambda_S$, as S is cocompact in G, so that $H^1(S, \lambda_S) \neq 0$. By a result of Guichardet (Proposition 8.5 in Chapter III of [Gu2]), this implies that $S \simeq \mathrm{PSL}_2(\mathbf{R})$.

Conversely suppose (ii). If G is amenable, then $H^1(G, \lambda_G)$ is not Hausdorff, hence is nonzero. Otherwise, suppose G non-amenable, and consider K as in (ii). By Proposition 3.3 and Theorem 3.4 (noticing that $p_0 = 1$ for $PSL_2(\mathbf{R})$), we have $H^1(G/K, \lambda_{G/K}) \neq 0$. Then $H^1(G, \lambda_G) \neq 0$ by the same elementary argument as used in the proof of Proposition 3.3. □

Corollary 4.11. *Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$ where $\mathbf{K}$ is a local field of characteristic zero; let Γ be a uniform lattice in G. If the first L^2-Betti number $\beta^1_{(2)}(\Gamma)$ is non-zero, then Γ is commensurable either to a non-abelian free group or to a surface group (more precisely: Γ has a finite index subgroup Γ_0 with* *a finite normal subgroup N such that Γ_0/N is either a non-abelian free group or a surface group).*

Proof. From $\beta^1_{(2)}(\Gamma) > 0$, it follows that Γ (and also G) is non-amenable: see Theorem 0.2 in [CG]. On the other hand, it was proved in [BV] that, for Γ a finitely generated non-amenable group: $\beta^1_{(2)}(\Gamma) > 0$ if and only if $H^1(\Gamma, \lambda_\Gamma) \neq 0$. Since Γ is uniform in G, we have by Shapiro's lemma (Proposition 4.6 in Chapter III of [Gu2]):

$$0 \neq H^1(\Gamma, \lambda_\Gamma) = H^1(G, Ind_\Gamma^G \lambda_\Gamma) \simeq H^1(G, \lambda_G).$$

By Proposition 4.10, the group G admets a compact normal subgroup K such that G/K is isomorphic either to $\mathrm{PSL}_2(\mathbf{R})$ or to a simple algebraic group of rank 1 over a p-adic group. Let $p : G \to G/K$ be the quotient map. Then $p(\Gamma)$ is a uniform lattice in G/K. By Selberg's lemma, find a finite-index torsion-free subgroup $\tilde{\Gamma}_0$ of $p(\Gamma)$: then $\tilde{\Gamma}_0$ is either a surface group (case of $\mathrm{PSL}_2(\mathbf{R})$) or a non-abelian free group (non-archimedean case). Set $\Gamma_0 = p^{-1}(\tilde{\Gamma}_0)$, a finite-index subgroup of Γ. Conclude by observing that the kernel $\mathrm{Ker} p|_{\Gamma_0}$ is contained in $\Gamma \cap K$, so is finite. □

The preceding result overlaps a result of B. Eckmann (Theorem 4.1 in [Eck]), who classified lattices Γ (not necessarily uniform) with $\beta^1_{(2)}(\Gamma) > 0$ in a connected Lie group.

4.4 Some non-σ-compact groups

Here is a curiosity. Start with the observation from the proof of Proposition 4.1 that a locally compact, non-σ-compact group cannot be a-T-menable. Accordingly, if it has Property BP_0, then it also has Property (FH_0).

The above observation shows that the σ-compactness assumption is necessary in Guichardet's result mentioned above. It also provides, in the non-Fréchet case, some explicit counterexamples to the statement of the open mapping theorem.

Proposition 4.12. *Let G be a non-σ-compact locally compact amenable group with Property BP_0. Endow $Z^1(G, \lambda_G)$ with the topology of uniform convergence on compact subsets. Then the map*

$$\partial : \begin{cases} L^2(G) & \to & Z^1(G, \lambda_G) \\ \xi & \mapsto & (g \mapsto \lambda_G(g)\xi - \xi) \end{cases}$$

is a continuous linear bijective homomorphism, whose inverse is not continuous.

Proof. The map ∂ is linear, injective (as G is not compact) and surjective (since G has property (FH_0)). It is clearly continuous. By amenability of G, for every $\varepsilon > 0$ and every compact subset $K \subset G$, there exists a unit vector $\xi \in L^2(G)$ such that

$$\max_{g \in K} \|(\partial\xi)(g)\| < \varepsilon.$$

This clearly shows that ∂^{-1} is not continuous. □

Example 4.13. Examples of non-σ-compact amenable groups with Property BP_0 include

- Uncountable solvable groups (by Corollary 2.12)
- Discrete groups of the form $G = F^I$, where F is a non-trivial finite group and I is any infinite set. Indeed G is amenable, as it is locally finite, and since G is isomorphic to $G \times G$ (as I is infinite), G contains an infinite normal subgroup with infinite centralizer, so Proposition 2.10 applies.

5 Actions of Z and R

5.1 Actions of Z

We have shown that every action of $\mathbf{Z}$ on a Hilbert space with C_0 linear part is either bounded or proper.

An example of Edelstein [Ede] shows that the C_0 assumption cannot be dropped. Let us briefly recall his example. On $\mathbf{C}$, consider the rotation r_n with centre 1 and angle $2\pi/n!$. Consider the abstract product $\mathbf{C}^{\mathbf{N}}$, and, for $(z_n)_{n\in\mathbf{N}} \in \mathbf{C}^{\mathbf{N}}$, set $r((z_n)_{n\in\mathbf{N}}) = (r_n(z_n))_{n\in\mathbf{N}}$. This self-map is bijective and has the constant sequence 1 as unique fixed point. Moreover, it can be shown that $r(\ell^2(\mathbf{N})) = \ell^2(\mathbf{N})$. Thus r induces an affine isometry of $\ell^2(\mathbf{N})$, which is fixed-point free since the constant 1 is not in $\ell^2(\mathbf{N})$. However, the action is not proper; actually 0 is a recurrent point: an easy computation gives $\|r^{n!}(0)\|^2 \leq \sum_{k>n}(2\pi n!/k!)^2$, and this sum clearly tends to zero.

Observe that this isometry has diagonalizable linear part. Let us now provide another counter-example with further assumptions on the linear part.

Definition 5.1. A unitary or orthogonal representation of a group is *weakly C_0* if it has no nonzero finite dimensional subrepresentation[4].

[4] C_0 (resp. weakly C_0) representations are often called mixing (resp. weakly mixing).

Proposition 5.2. *There exists an affine isometric action of* $\mathbf{Z}$ *on a complex Hilbert space, which is neither bounded nor proper, and has weakly* C_0 *linear part.*

Proof. Write σ for the affine action of $\mathbf{Z}$, and π for its linear part. Let μ be a probability measure on $[0,1]$ and write $H = L^2([0,1],\mu)$. Let $\pi(1)$ be the multiplication by the function $e(x) = \exp(2i\pi x)$. Write $\sigma(1) = \tau_1 \circ \pi(1)$ where τ_1 is the translation by the constant function 1. Note that π is weakly C^0 if and only if the spectrum of $\pi(1)$ has no atom, i.e. μ is nonatomic.

Let b be the corresponding cocycle and write $c(n) = \|b(n)\|^2$. An immediate computation shows that

$$c(n) = \int \phi_n(x) d\mu(x)$$

where $\phi_n(x) = |\sin(\pi n x)/\sin(\pi x)|^2$.

Let N_n a increasing sequence of integers and let ε_n be a decreasing sequence in $]0,1[$, such that $\varepsilon_n \to 0$. Moreover, let us assume that $N_n/N_{n+1} = o(\varepsilon_n)$.

For all positive integer n, write

$$I_n(k) = \left[\frac{k-\varepsilon_n}{N_n}, \frac{k+\varepsilon_n}{N_n}\right] \cap [0,1]$$

and

$$K_n = K_{n-1} \cap \bigcup_{k=0}^{N_n} I_n(k).$$

Finally, write

$$K = \bigcap_n K_n.$$

One can check easily that K is homeomorphic to a Cantor space.

Let μ be a probability measure on $[0,1]$ such that

- its support is contained in $K \cap [0,1/2]$;
- There exists a subsequence ε_{k_n} such that

$$\mu\left([0,\sqrt{\varepsilon_{k_n}}]\right) = \sqrt{\varepsilon_{k_n}}. \tag{1}$$

We choose the sequence k_n such that each interval $I_n = [\sqrt{\varepsilon_{k_{n+1}}}, \sqrt{\varepsilon_{k_n}}]$ intersects K nontrivially. Take for μ_n any nonatomic measure supported by $K \cap I_n$ such that $\mu_n(I_n) = \sqrt{\varepsilon_{k_n}} - \sqrt{\varepsilon_{k_{n+1}}}$ and define $\mu = \sum_n \mu_n$: clearly, μ is nonatomic.

Claim 1. The action σ has no fixed point (so is not bounded).

If σ has a fixed point f, then $f(x) = (1 - \exp(2i\pi x))^{-1}$ μ-a.e. Let us show that f does not belong to $L^2([0,1])$. Indeed, note that $|f|^2 = (1/\sin(\pi x))^2$. For all $x \in [0, \sqrt{\varepsilon_{k_n}}]$, we have

$$\sin(\pi x)^2 \leq \pi^2 x^2 \leq \pi^2 \varepsilon_{k_n}$$

and by (1)

$$\mu([0, \sqrt{\varepsilon_{k_n}}]) = \sqrt{\varepsilon_{k_n}}.$$

It follows that

$$\int |f|^2 d\mu \geq \frac{1}{\pi^2 \sqrt{\varepsilon_{k_n}}}$$

which proves claim 1.

Claim 2. If moreover $\varepsilon_{k_n} = o(N_{k_n}^{-4})$ (for instance, $N_n = 2^{n!}$ and $\varepsilon_n = (N_n)^{-5}$), then $c(N_{k_n})$ tends to 0, so that the action is not proper.

Indeed, let us show that $c(N_{k_n}) = o(1)$.

First, note that for all $x \in K$, the fractional part of $N_{k_n}.x$ is less than ε_{k_n}. Thus, for every $x \geq \sqrt{\varepsilon_{k_n}}$ and every $x \in K$, it comes

$$\phi_{N_{k_n}}(x) \leq \left(\frac{\sin(2\pi\varepsilon_{k_n})}{\sin(2\pi\sqrt{\varepsilon_{k_n}})}\right)^2 \leq \pi^2 \varepsilon_{k_n}/4.$$

On the other hand, we have

$$\frac{\sin(2\pi N_{k_n} x)}{\sin(2\pi x)} \leq N_{k_n}$$

and by (1)

$$\mu([0, \sqrt{\varepsilon_{k_n}}]) = \sqrt{\varepsilon_{k_n}}.$$

It follows that

$$c(N_{k_n}) \leq \sqrt{\varepsilon_{k_n}}.N_{k_n}^2 + \pi^2 \varepsilon_{k_n}/4.$$

So we get $c(N_{k_n}) = o(1)$. □

5.2 Actions of R

Let us now show that the "pathological" actions of **Z** described above can be extended to **R**.

Recall that a group G is said to be *exponential* if, for every $g \in G$, there is a one-parameter subgroup through g (i.e. a continuous homomorphism $\beta : \mathbf{R} \to G$ such that $\beta(1) = g$). Clearly, an exponential group has to be arc-connected.

Endow the group of affine isometries of a complex Hilbert space, $\mathcal{H} \rtimes \mathcal{U}(\mathcal{H})$, with the product topology, for the natural topology on $\mathcal{H}$ and the norm operator topology on the unitary group $\mathcal{U}(\mathcal{H})$.

Proposition 5.3. *The group of affine isometries of a complex Hilbert space $\mathcal{H}$, is exponential.*

Proof. Let $\alpha(v) = Uv + b$ be an affine isometry of $\mathcal{H}$.

By the spectral theorem, we find a projection-valued measure $\mathbf{P}$ on $[-\pi, \pi[$ such that $U = \int_{-\pi}^{\pi} e^{ix}\, d\mathbf{P}(x)$, in the sense that, for every $\xi, \eta \in \mathcal{H}$

$$\langle U\xi|\eta\rangle \;=\; \int_{-\pi}^{\pi} e^{ix}\, d\mu_{\xi,\eta}(x)$$

where $\mu_{\xi,\eta}(A) \;=\; \langle \mathbf{P}(A)\xi|\eta\rangle$ for any Borel subset $A \subset [-\pi, \pi[$. Consider the one-parameter group of unitary operators

$$\upsilon(s) = \int_{[-\pi,\pi[} e^{isx}\, d\mathbf{P}(x).$$

For every $\xi \in \mathcal{H}$ and $t \in \mathbf{R}$, define

$$b_\xi(t) = \int_0^t \upsilon(s)\xi\, ds.$$

It is straightforward that $b_\xi \in Z^1(\mathbf{R}, \upsilon)$. Let us consider the operator $A = \int_0^1 \upsilon(s)\, ds$. Then $b_\xi(1) = A\xi$ for every $\xi \in \mathcal{H}$. Thus, to show that there exists $\xi \in \mathcal{H}$ such that $b = b_\xi(1)$, it suffices to establish that A is invertible.

By Fubini's Theorem

$$\int_0^1 \upsilon(s)\, ds \;=\; \int_0^1 \Big(\int_{[-\pi,\pi[} e^{isx}\, d\mathbf{P}(x)\Big)\, ds \;=\; \int_{[-\pi,\pi[} \frac{e^{ix}-1}{ix}\, d\mathbf{P}(x).$$

Since the function $x \mapsto \frac{ix}{e^{ix}-1}$ is bounded on $[-\pi, \pi[$, we obtain that

$$\int_{[-\pi,\pi[} \frac{ix}{e^{ix}-1}\, d\mathbf{P}(x)$$

is a bounded operator on $\mathcal{H}$, and is the inverse of A; so we may take $\xi = A^{-1}(b)$. ∎

In view of Proposition 5.2, we obtain

Corollary 5.4. *There exists an affine isometric action of* $\mathbf{R}$ *on a complex Hilbert space that is neither bounded nor proper. Moreover, it can be chosen weakly* C_0*.* ∎

Remark 5.5. Proposition 5.3 is *false* for *real* Hilbert spaces. This follows from the fact that the orthogonal group of a real Hilbert space is *not* exponential. This is clear in finite dimension (the group $O(n)$ is not connected), and was observed by Putnam and Wintner [PW2] in infinite dimension (although the orthogonal group is then connected [PW1]). An example of an orthogonal transformation which is not in the image of the exponential map is

$$S = \mathrm{diag}(-1, 1, 1, 1, \ldots);$$

this can be seen by noticing that S is not a square in the orthogonal group: indeed if $S = R^2$, since R commutes with S it stabilizes the -1-eigenspace of S, which leads to a contradiction.

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