Group pairs with property (T), from arithmetic lattices

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To the memory of Armand Borel

Abstract

Let Γ be an arithmetic lattice in an absolutely simple Lie group G with trivial centre. We prove that there exists an integer $N \geq 2$, a subgroup Λ of finite index in Γ, and an action of Λ on $\mathbb{Z}^N$ such that the pair $(\Lambda \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has property (T). If G has property (T), then so does $\Lambda \ltimes \mathbb{Z}^N$. If G is the adjoint group of $Sp(n,1)$, then $\Lambda \ltimes \mathbb{Z}^N$ is a property (T) group satisfying the Baum-Connes conjecture. If Λ_n is an arithmetic lattice in $SO(2n,1)$, then the associated von Neumann algebras $(L(\Lambda_n \ltimes \mathbb{Z}^{N_n}))_{n\geq 1}$ are a family of pairwise non-isomorphic group II_1-factors, all with trivial fundamental groups.

1 Introduction and results

Let G be a locally compact group, and let H be a closed subgroup. The pair (G,H) has *property (T)* if every unitary representation of G almost having invariant vectors, has non-zero H-fixed vectors. The group G has Kazhdan's property (T) if and only if the pair (G,G) has property (T).

Suppose that G acts by automorphisms on a locally compact group N, and form the semi-direct product $G \ltimes N$. In this paper we shall be concerned with the property (T) for the pair $(G \ltimes N, N)$.

Property (T) for the pair $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ already plays a big rôle in Kazhdan's original paper [Kaz67], to establish property (T) for $SL_n(\mathbb{R})$, $n \geq 3$. Later, property (T) for the pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ was exploited by Margulis [Mar73] to give the first explicit example of an infinite family of expanding graphs.

Observe that $SL_2(\mathbb{Z})$ is an arithmetic lattice in the simple Lie group $SL_2(\mathbb{R})$. Our main result states that semi-direct product pairs with property (T) can be obtained, at least virtually, from any arithmetic lattice. Before stating it precisely, we recall the relevant definitions; good references about lattices are [Bor69], [Zim84], [Mar91], [WM].

Definition 1 *Let G be a real, semisimple Lie group with finite centre, and let Γ be a discrete subgroup in G.*

a) Γ is a **lattice** *in G if the homogeneous space G/Γ carries a finite, G-invariant measure.*

b) A lattice Γ in G is **uniform** *if G/Γ is compact.*

c) A lattice Γ in G is **arithmetic** *if there exists a semisimple algebraic $\mathbb{Q}$-group H and a surjective continuous homomorphism $\phi : H(\mathbb{R})^0 \to G$, with compact kernel, such that $\phi(H(\mathbb{Z}) \cap H(\mathbb{R})^0)$ is commensurable with Γ (here $H(\mathbb{R})^0$ is the connected component of identity in $H(\mathbb{R})$).*

Definition 2 *A real Lie group is* **absolutely simple** *if its complexified Lie algebra is simple.*

A simple Lie group is absolutely simple if and only if it is not locally isomorphic (as a real Lie group) to a complex Lie group (see (10.10) in [WM]). With this we can formulate our main result.

Theorem 1 *Let G be a non-compact, absolutely simple Lie group with trivial centre. Let Γ be an arithmetic lattice in G. There exists an integer $N \geq 2$, a subgroup Λ of finite index in Γ, and an action of Λ on $\mathbb{Z}^N$ such that:*

i) the pair $(\Lambda \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has property (T);

ii) $\Lambda \ltimes \mathbb{Z}^N$ is torsion-free and has infinite conjugacy classes.

Note that we have no idea whether Theorem 1 holds true for non-arithmetic lattices (which are known to exist in $SO(n,1)$ for every $n \geq 2$ - see [GPS88], and in $SU(n,1)$ for $1 \leq n \leq 3$ - see [DM86]). A partial generalization of Theorem 1 to the case where Γ is an irreducible, arithmetic lattice in a semisimple Lie group G, will be discussed as Theorem 4 in §2. An explicit value of the integer N in Theorem 1, will be given as part of Theorem 4.

It is known (see p. 26 in [dlHV89]) that $SL_n(\mathbb{Z}) \ltimes \mathbb{Z}^n$ has property (T) for $n \geq 3$. The following Corollary generalizes this fact and provides new examples of groups with property (T).

Corollary 1 *Let G be a non-compact, absolutely simple Lie group with trivial centre, which is not locally isomorphic to $SO(n,1)$ or $SU(m,1)$. Let Γ be a lattice in G. There exists an integer $N \geq 2$, a subgroup Λ of finite index in Γ, and an action of Λ on $\mathbb{Z}^N$ such that $\Lambda \ltimes \mathbb{Z}^N$ is torsion-free, has infinite conjugacy classes and property (T).*

We conclude the paper by giving two applications of Theorem 1. The first one is about the Baum-Connes conjecture (see [BCH94]). It is known that, until the work of V. Lafforgue [Laf98], property (T) was a major stumbling block for proving the Baum-Connes conjecture (see [Jul98]). So it seems interesting to construct new examples of groups with property (T) which satisfy the Baum-Connes conjecture. Building on results of P. Julg [Jul02], who established the Baum-Connes conjecture for $Sp(n,1)$, we prove:

Theorem 2 *Keep the notations and assumptions of Corollary 1. Assume moreover that G is the adjoint group of $Sp(n,1)$ $(n \geq 2)$. Then the group $\Lambda \ltimes \mathbb{Z}^N$ is a property (T) group for which the Baum-Connes conjecture holds.*

Our second application is about von Neumann factors of type II_1. Let M be a II_1-factor; for $t > 0$, denote by M_t the compression of $M \overline{\otimes} \mathcal{B}(\mathcal{H})$ by any projection with trace t. The fundamental group of M is

$$\mathcal{F}(M) = \{t \in \mathbb{R}_+^\times : M^t \simeq M\},$$

a subgroup of the multiplicative group of positive real numbers. It was a problem asked by R.V. Kadison in 1967, whether there exists a II_1-factor M such that $\mathcal{F}(M) = \{1\}$. This was solved by Popa in [Popa] (see also [Popb] for a shorter proof): building on Gaboriau's theory of L^2-Betti numbers for measurable equivalence relations [Gab02], Popa proved that, for $\Gamma = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, the corresponding factor $L(\Gamma)$ has trivial fundamental group. Using the same techniques, we prove:

Theorem 3 *Set $\Gamma_n = SO(2n,1)(\mathbb{Z})$, a non-uniform arithmetic lattice in the simple Lie group $SO(2n,1)$ $(n \geq 1)$. Set $N_n = n(2n+1) = \dim_{\mathbf{R}} SO(2n,1)$, and let Γ_n act via the adjoint representation on $\mathbb{Z}^{N_n}$, viewed as the integral points in the Lie algebra of $SO(2n,1)$. Set finally $M_n = L(\Gamma_n \ltimes \mathbb{Z}^{N_n})$. Then $(M_n)_{n\geq 1}$ is a sequence of pairwise non-isomorphic* **group** *II_1-factors, all with trivial fundamental group.*

We emphasize here the fact that the M_n's are group factors: indeed, if $\mathcal{F}(M) = \{1\}$, then the M^t's, for $t > 0$, provide uncountably many pairwise non-isomorphic factors, all with trivial fundamental group.

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2 Proofs of Theorem 1 and Corollary 1

We first recall a useful sufficient condition for property (T) of semi-direct product pairs; see Proposition 2.3 in [Val94] for a proof.

Proposition 1 *Let V be a finite-dimensional, real vector space; let $H \subset GL(V)$ be a semisimple subgroup. If the product of the non-compact simple factors of H has no non-zero fixed vector in V, then the pair $(H \ltimes V, V)$ has property (T).* □

We will need some material about algebraic groups. Let k be a number field, i.e a finite extension of $\mathbb{Q}$, and let X be the set of field embeddings of k into $\mathbb{C}$. As usual, we say that two distinct embeddings $\sigma, \tau : k \to \mathbb{C}$ are *equivalent* if $\sigma(x) = \overline{\tau(x)}$ for all $x \in k$; an *archimedean place* of k is an equivalence class of embeddings, and we denote by $\overline{X}$ the set of archimedean places of k.

If G is a linear algebraic group defined over k, set $R_{k/\mathbb{Q}}(G) = \prod_{\tau \in X} G^\tau$, where G^τ is obtained from G by applying τ to the polynomials defining G. This is the *restriction of* G *to* $\mathbb{Q}$, of which we recall the main properties (for all this, see [Zim84], Proposition 6.1.3).

- For $g \in G(k)$, set $\Delta(g) = (\tau(g))_{\tau \in X}$. Then $R_{k/\mathbb{Q}}(G)$ is an algebraic group over $\mathbb{Q}$, such that
$$(R_{k/\mathbb{Q}}(G))(\mathbb{Q}) = \Delta(G(k)).$$
- Let $\mathcal{O}$ be the ring of integers of k. Then
$$(R_{k/\mathbb{Q}}(G))(\mathbb{Z}) = \Delta(G(\mathcal{O})).$$
- Let τ_0 be the identity of k. The projection $p : R_{k/\mathbb{Q}}(G) \to G^{\tau_0} = G$ is defined over k, and yields bijections $(R_{k/\mathbb{Q}}(G))(\mathbb{Q}) \to G(k)$ and $(R_{k/\mathbb{Q}}(G))(\mathbb{Z}) \to G(\mathcal{O})$.
- For every subfield F of $\mathbb{C}$ such that $\tau(k) \subset F$ for every $\tau \in X$, each G^τ is defined over F and
$$(R_{k/\mathbb{Q}}(G))(F) = \prod_{\tau \in X} G^\tau(F).$$

Let K be the normal closure of k, and let $Gal(K/\mathbb{Q})$ be its Galois group over $\mathbb{Q}$. Let $W = \bigotimes_{\tau \in X} K^n$ be the tensor product over K of $|X|$ copies of K^n (so that $\dim_K W = n^{|X|}$). Let $GL_n(k)$ act on W by

$$\rho(g) = \bigotimes_{\tau \in X} \tau(g)$$

($g \in GL_n(k)$). If $H = R_{k/\mathbb{Q}}(GL_n)$, then ρ is a representation of H defined over K. Since $H(K) = \prod_{\tau \in X} GL_n(K)$, we have $\rho((g_\tau)_{\tau \in X}) = \bigotimes_{\tau \in X} g_\tau$ for $(g_\tau)_{\tau \in X} \in H(K)$. The following lemma was kindly provided by Y. Benoist.

Lemma 1 *With notations as above, the representation ρ is defined over $\mathbb{Q}$, and there exists a $GL_n(k)$-invariant $\mathbb{Q}$-subspace U of W which is a $\mathbb{Q}$-form of ρ, i.e. the map $K \otimes_{\mathbb{Q}} U \to W$ is a $GL_n(k)$-equivariant isomorphism.*

Proof: Let I be the set of maps $X \to \{1, \dots, n\}$, so that we may denote by $(e_i)_{i \in I}$ the standard basis of W associated with the standard basis of K^n. Let $Gal(K/\mathbb{Q})$ act on I by $\gamma \cdot i(\tau) = i(\gamma^{-1} \circ \tau)$, for $\tau \in X$, $i \in I$. Consider the semi-linear representation of $Gal(K/\mathbb{Q})$ on W given by

$$\gamma(\sum_{i \in I} \lambda_i e_i) \ = \ \sum_{i \in I} \gamma(\lambda_i) e_{\gamma \cdot i}$$

($\lambda_i \in K$). Observe that, for $v_\tau \in K^n$ ($\tau \in X$):

$$\gamma(\bigotimes_{\tau \in X} v_\tau) \ = \ \bigotimes_{\tau \in X} \gamma(v_{\gamma^{-1} \circ \tau}).$$

This shows that the action of $Gal(k/\mathbb{Q})$ on W commutes with the representation ρ of $GL_n(k)$.

Recall that the group $Gal(K/\mathbb{Q})$ acts on representations $\pi : H \to GL(W)$ defined over K, by $\gamma \cdot \pi = \gamma \circ \pi \circ \gamma^{-1}$ (where $\gamma \in Gal(K/\mathbb{Q})$). Here, since $\gamma \cdot \rho = \rho$ for every $\gamma \in Gal(K/\mathbb{Q})$, the representation ρ is defined over $\mathbb{Q}$, by [Bor91], AG 14.3.

Finally, let U be the space of points in W which are fixed under $Gal(K/\mathbb{Q})$: by [Bor91], AG 14.2, this is a $\mathbb{Q}$-form for W. Since ρ commutes with the action of $Gal(K/\mathbb{Q})$ on W, the space U is $GL_n(k)$-invariant. □

Let G be a real, semisimple Lie group with trivial centre and no compact factor. Recall that a lattice Γ in G is *irreducible* if, for any non-central, closed, normal subgroup N in G, the projection of Γ in G/N is dense.

Assume that Γ is an irreducible, arithmetic lattice in G. By definition 1(c), there is a semisimple algebraic $\mathbb{Q}$-group H and $\phi : H(\mathbb{R})^0 \to G$ a surjective homomorphism, with compact kernel, such that $\phi(H(\mathbb{Z}) \cap H(\mathbb{R})^0)$ is commensurable with Γ.

Such an H is obtained as follows: by Corollary 6.54 in [WM], there exists a number field k and a *simple* algebraic k-group L such that $H = \prod_{\tau \in \overline{X}} L^\tau$ and ϕ can be identified with the projection of $H(\mathbb{R})^0$ onto the product of its non-compact simple factors. If Γ is not uniform in G, then by Corollary 6.1.10 in [Zim84] we may assume that $H(\mathbb{R})^0$ has no compact factor.

Theorem 4 *Let Γ be an arithmetic, irreducible lattice in a real, semisimple Lie group G with trivial centre and no compact factor. Let H, ϕ, k, L be as above, and let X be the set of embeddings of k into $\mathbb{C}$. Assume that k is totally real (so that $X = \overline{X}$). Set $N = (\dim_{\mathbf{R}} L(\mathbb{R}))^{|X|}$. The exists a subgroup Λ of finite index in Γ, and an action of Λ on $\mathbb{Z}^N$ such that:*

i) the pair $(\Lambda \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has property (T);

ii) $\Lambda \ltimes \mathbb{Z}^N$ is torsion-free and has infinite conjugacy classes.

If G is absolutely simple, then k is totally real, by [Mar91], (1.5) in Chapter 9; this shows that Theorem 4 implies Theorem 1.

Proof of Theorem 4: The idea of the proof is to construct a representation of $H(\mathbb{R})^0$ on a finite-dimensional space V, satisfying *simultaneously* the following two conditions:

- the pair $(H(\mathbb{R})^0 \ltimes V, V)$ has property (T);
- some finite index subgroup in $H(\mathbb{Z}) \cap H(\mathbb{R})^0$ stabilizes some lattice in V.

The proof is in 3 steps.

1. *Construction of a rational representation of H on $\mathbb{Q}^N$, such that the pair $(H(\mathbb{R})^0 \ltimes \mathbb{R}^N, \mathbb{R}^N)$ has property (T):*

   Since k is totally real, we have $H = R_{k/\mathbb{Q}}(L)$. Let K be the normal closure of k. Let $\mathfrak{l}^\tau$ be the Lie algebra of L^τ, and Ad^τ be the adjoint representation of L^τ on $\mathfrak{l}^\tau$. Both L^τ and the representation Ad^τ of L^τ are defined over K ([Bor91], I.3.13). Set $n = \dim_k \mathfrak{l}$. Choosing a basis in $\mathfrak{l}$, we get an identification of $K \otimes_k \mathfrak{l}^\tau$ with K^n, for every $\tau \in X$; set $W = \bigotimes_{\tau \in X} K^n$, and let ρ be the representation of $GL_n(k)$ on W defined before lemma 1. Define a representation of L on W by

$\pi = \rho \circ Ad$, so that $\pi = \bigotimes_{\tau\in X} Ad^\tau$ is a representation of H, which is defined over $\mathbb{Q}$ by lemma 1. Let U be the $\mathbb{Q}$-form of W given by lemma 1 and its proof. Since π is defined over $\mathbb{Q}$, it defines a rational representation of $H(\mathbb{R})$ over $\mathbb{R}\otimes_{\mathbb{Q}} U \simeq \mathbb{R}^N$.

Note that, as a representation of $H(\mathbb{R})$, this is exactly the external tensor product of the adjoint representations of the $L^\tau(\mathbb{R})$'s ($\tau \in X$). Take τ such that $L^\tau(\mathbb{R})$ is non-compact; since $L^\tau(\mathbb{R})^0$ is simple, it has no non-zero fixed vector in its adjoint representation. So the product of these $L^\tau(\mathbb{R})^0$'s, i.e. the product of the non-compact simple factors of $H(\mathbb{R})^0$, has no non-zero fixed vector in $\bigotimes_{\tau\in X} \mathfrak{l}^\tau(\mathbb{R}) \simeq \mathbb{R}^N$. By Proposition 1, the pair $(H(\mathbb{R})^0 \ltimes \mathbb{R}^N, \mathbb{R}^N)$ has property (T).

2. *Construction of the semi-direct product $\Lambda \ltimes \mathbb{Z}^N$:* By lemma 1, the space U is invariant under $H(\mathbb{Q}) = \{\Delta(g) : g \in L(k)\}$; in particular, it is invariant under $H(\mathbb{Z}) = \{\Delta(g) : g \in L(\mathcal{O})\}$. Choose a $\mathbb{Q}$-basis of U, and let M be the $\mathbb{Z}$-module generated by that basis (so that $M \simeq \mathbb{Z}^N$). By Proposition 7.12 in [Bor69], there exists a congruence subgroup Λ_1 in $H(\mathbb{Z}) \cap H(\mathbb{R})^0$ which leaves M invariant.

   By Selberg's lemma (see [Alp87]), Λ_1 admits a torsion-free, finite index subgroup Λ_2; of course $\phi(\Lambda_2)$ is commensurable with Γ. Replacing Λ_2 by a finite-index subgroup if necessary, we may assume that $\phi(\Lambda_2) \subset \Gamma$. Notice that, since $\ker\phi$ is compact and Λ_2 is torsion-free, Λ_2 intersects $\ker\phi$ trivially. We then set $\Lambda = \phi(\Lambda_2)$, which acts on $\mathbb{Z}^N$ via $(\pi\circ\phi^{-1})_{|\Lambda}$. The desired semi-direct product is then $\Lambda \ltimes \mathbb{Z}^N$. It is torsion-free since Λ and $\mathbb{Z}^N$ both are. Since $\Lambda \ltimes \mathbb{Z}^N$ is a lattice in $H(\mathbb{R})^0 \ltimes \mathbb{R}^N$, the pair $(\Lambda \ltimes \mathbb{Z}^N, Z^N)$ has property (T).

3. $\Lambda \ltimes \mathbb{Z}^N$ *has infinite conjugacy classes:* Indeed, it is a well-known fact that lattices in semisimple Lie groups with trivial centre have infinite conjugacy classes. This already shows that every element in $\Lambda \ltimes \mathbb{Z}^N$ which projects non-trivially to Λ, has infinite conjugacy class. It remains to prove the same for a non-zero element x in the normal subgroup $\mathbb{Z}^N$. Equivalently, we must show that the Λ-orbit of x in $\mathbb{Z}^N$, is infinite. So we take $x \in \mathbb{Z}^N$ with finite Λ-orbit, and show that $x = 0$. Let Λ_x be the stabilizer of x in Λ: it is a finite-index subgroup of Λ. Set then $C = \{h \in H(\mathbb{R}) : \pi(h)(x) = x\}$. Since π is a rational representation, C is a Zariski closed subgroup of $H(\mathbb{R})$, containing $\phi^{-1}(\Lambda_x)$. The latter is a lattice in $H(\mathbb{R})$, so it is Zariski dense, by the Borel density theorem [Bor60]. This means that x is fixed under $H(\mathbb{R})$. By our choice of π, this implies $x = 0$.

$\square$

Remark: If $G = H(\mathbb{R})^0$ and $\Gamma = H(\mathbb{Z}) \cap H(\mathbb{R})^0$, then we may take $\Lambda = \Gamma$ in Theorem 4 (provided we don't insist that Λ be torsion-free). In other words, in this situation there is no need to pass to a finite-index subgroup. To see it, let $M \subset U$ be the $\mathbb{Z}$-module appearing at the beginning of step 2 in the proof of Theorem 4. As M is invariant under the finite-index subgroup Λ_1, the orbit of M under Γ is finite. Then the sum of all the $\mathbb{Z}$-modules in the orbit, is a Γ-invariant free $\mathbb{Z}$-module of rank N.

We now re-visit some examples of arithmetic lattices, taken from [Mar91], 1.7(vi) in Chapter IX. Since Theorem 4 applies to each of them, we will in each case identify k, N and H.

Example 1 *Let Φ be a quadratic form in $n+1$ variables, with signature $(n,1)$, and coefficients in a number field $k \subset \mathbb{R}$. We denote by SO_Φ the special orthogonal group of Φ: this is a simple algebraic group defined over k. Set $\Gamma = SO_\Phi(\mathcal{O})$, where as usual $\mathcal{O}$ is the ring of integers of k.*

a) $\Phi = x_1^2 + \ldots + x_n^2 - x_{n+1}^2$*; here $k = \mathbb{Q}$ and $H = SO_\Phi$, so that $\Gamma = SO(n,1)(\mathbb{Z})$ is a non-uniform arithmetic lattice in $SO_\Phi(\mathbb{R}) = SO(n,1)$, to which the previous remark applies. Here $N = \dim_{\mathbb{R}} SO(n,1) = \frac{n(n+1)}{2}$. Let J be the $(n+1)\times(n+1)$, diagonal matrix with diagonal values $(1,\ldots,1,-1)$; the Lie algebra of $SO(n,1)$ is*

$$\mathfrak{so}(n,1) = \{X \in M_{n+1}(\mathbb{R}) : X^t J + JX = 0\}.$$

The adjoint representation of $SO(n,1)$ on $\mathfrak{so}(n,1)$ is given by $Ad(g)(X) = gXg^{-1}$ $(g \in SO(n,1), X \in \mathfrak{so}(n,1))$. So the restriction of Ad to Γ leaves invariant $\mathfrak{so}(n,1) \cap M_{n+1}(\mathbb{Z}) \simeq \mathbb{Z}^N$. This example will be used below in the proof of Theorem 3.

b) $\Phi = x_1^2 + \ldots + x_n^2 - \sqrt{2}x_{n+1}^2$*; here $k = \mathbb{Q}(\sqrt{2})$ and $H = SO_\Phi \times SO_{\sigma(\Phi)}$, where σ is the non-trivial element of $Gal(k/\mathbb{Q})$. Then $\Gamma = SO_\Phi(\mathbb{Z}[\sqrt{2}])$ is a uniform arithmetic lattice in $SO_\Phi(\mathbb{R}) \simeq SO(n,1)$. Here $N = (\frac{n(n+1)}{2})^2$.*

c) $\Phi = x_1^2 + \ldots + x_n^2 - \delta x_{n+1}^2$ *where $\delta > 0$ is a root of a cubic irreducible polynomial over $\mathbb{Q}$, having two positive roots δ, δ' and one negative root δ''. Here $k = \mathbb{Q}(\delta)$; let σ, τ be the embeddings of k into $\mathbb{R}$ defined by $\sigma(\delta) = \delta'$ and $\tau(\delta) = \delta''$. Then $H = SO_\Phi \times SO_{\sigma(\Phi)} \times SO_{\tau(\Phi)}$ and Γ is an irreducible, uniform, arithmetic lattice in $SO_\Phi(\mathbb{R}) \times SO_{\sigma(\Phi)}(\mathbb{R}) \simeq SO(n,1) \times SO(n,1)$. Here $N = (\frac{n(n+1)}{2})^3$.*

Remark: Let k be a number field, with normal closure K. If k is not totally real, we do not know whether Theorem 4 is still valid. The reason is that, while the Galois group $Gal(K/\mathbb{Q})$ acts on the set X of embeddings $k \to \mathbb{C}$ (by $\sigma\cdot\tau = \sigma\circ\tau$), it does *not* act on the set $\overline{X}$ of archimedean places, because of pairs of complex places. So if we start from $H = \prod_{\tau\in\overline{X}} L^\tau$ (this group is indeed defined over $\mathbb{Q}$, see [WM] ex. (6:30)), we will be unable to appeal to lemma 1, which appeals to the Galois-fixed point criterion for rationality over $\mathbb{Q}$. A concrete example to which this remark applies, is the following (see [Mar91], 1.7(vi)(6) in Chapter IX): set $\Phi = x_1^2 + \ldots + x_n^2 - x_{n+1}^2$ and $k = \mathbb{Q}(\sqrt[3]{2})$. Here k has one real place and one complex place; $\Gamma = SO_\Phi(\mathcal{O})$ is an irreducible, non-uniform arithmetic lattice in $SO(n,1)\times SO(n+1,\mathbb{C})$, for which we do not know whether Theorem 4 holds.

Proof of Corollary 1: Under the assumptions of the Corollary, Γ is an arithmetic lattice in G: if $rank_{\mathbf{R}}G \geq 2$, this is Margulis' famous arithmeticity theorem (see [Mar91], Thm (A) in Chapter IX; [Zim84], 6.1.2); if $rank_{\mathbf{R}}G = 1$ (i.e. G is locally isomorphic either to $Sp(n,1)\,(n\geq 2)$ or to $F_{4(-20)}$), this follows from the work of Corlette [Cor92] and Gromov-Schoen [GS92]. Theorem 1 then applies and provides $N \geq 2$ and a torsion-free Λ acting on $\mathbb{Z}^N$, in such a way that $\Lambda\ltimes\mathbb{Z}^N$ has infinite conjugacy classes and the pair $(\Lambda\ltimes\mathbb{Z}^N, \mathbb{Z}^N)$ has property (T). On the other hand, the assumptions also imply that G has property (T), and hence Λ too (see [dlHV89]). We conclude by using the following fact: let

$$1 \to N \to H \to H/N \to 1$$

be a short exact sequence of locally compact groups; if the pair (H,N) has property (T) and the group H/N has property (T), then the group H has property (T) (the easy proof can be left as an exercice). $\square$

3 Proof of Theorem 2

We recall that a locally compact group G *satisfies the Baum-Connes conjecture* if, for every C^*-algebra A endowed with an action of G, the Baum-Connes assembly map

$$\mu_{A,G} : RKK_*^G(\underline{E}G, A) \to K_*(A \rtimes_r G)$$

is an isomorphism. Here $\underline{E}G$ is the universal space for G-proper actions, $RKK_*^G(\underline{E}G, A)$ denotes the G-equivariant KK-theory with compact supports of $\underline{E}G$ and A, and $K_*(A\rtimes_r G)$ denotes the equivariant K-theory of the reduced crossed product $A\rtimes_r G$; see [BCH94] for details.

Let Γ be a lattice in the adjoint group of $Sp(n,1)$, $(n \geq 2)$. Corollary 1 provides N, Λ and an action of Λ on $\mathbb{Z}^N$ such that $\Lambda \ltimes \mathbb{Z}^N$ has property (T). The Baum-Connes conjecture for $\Lambda \ltimes \mathbb{Z}^N$ follows by combining the following facts:

- The Baum-Connes conjecture holds for $\mathbb{Z}^N$ (see [Kas95]).
- The Baum-Connes conjecture holds for $Sp(n,1)$: this is a remarkable result of P. Julg [Jul02].
- The Baum-Connes conjecture is inherited by closed subgroups, as was proved by Chabert and Echterhoff [CE01]; in particular it is satisfied by the lattice Λ.
- Let $1 \to \Gamma_0 \to \Gamma_1 \to \Gamma_2 \to 1$ be a short exact sequence of countable groups. If Γ_0 and Γ_2 satisfy the Baum-Connes conjecture, and Γ_2 is torsion-free, then Γ_1 satisfies the Baum-Connes conjecture. This is a result of Oyono-Oyono (Theorem 7.1 in [OO01]). We apply it to the short exact sequence $1 \to \mathbb{Z}^N \to \Lambda \ltimes \mathbb{Z}^N \to \Lambda \to 1$: since Λ is torsion-free, and $\mathbb{Z}^N$ and Λ satisfy the Baum-Connes conjecture, then so does $\Lambda \ltimes \mathbb{Z}^N$. □

4 Proof of Theorem 3

Recall that a locally compact group H is *a-T-menable*, or has *the Haagerup property*, if H admits a unitary representation almost having invariant vectors, whose coefficient functions vanish at infinity on H. We refer to [CCJ+01] for an extensive study of this class of groups. We will use the fact that closed subgroups of $SO(k,1)$ and $SU(m,1)$ are a-T-menable.

We now recall the portions of Popa's theory [Popa] which are relevant for our proof. Let N be a finite von Neumann algebra and let B be a von Neumann subalgebra. In Definition 2.1 of [Popa], Popa defines *property (H)* for the inclusion $B \subset N$; in Proposition 3.1 of [Popa], he proves that, if a countable group Γ acts on the finite von Neumann algebra B (preserving some normal, faithful, tracial state), and $N = B \rtimes \Gamma$, the inclusion $B \subset N$ has property (H) if and only if Γ is a-T-menable.

In Definition 4.2 of [Popa], Popa also defines property (T) for the inclusion $B \subset N$. In Proposition 5.1 of [Popa], he proves that, if H is a subgroup of Γ, the inclusion $L(H) \subset L(\Gamma)$ has property (T) if and only if the pair (Γ, H) has property (T).

Now let M be a II_1-factor, and let A be a Cartan subalgebra in M. Following Definitions 6.1 and 6.4 of [Popa], we say that A is a *HT_s-Cartan* *subalgebra* if the inclusion $A \subset M$ satisfies both property (H) and property (T). We denote by $\mathcal{HT}_s$ the class of II_1 factors with HT_s-Cartan subalgebras.

Example 2 *Let Γ be an arithmetic group in the adjoint group of $SO(k,1)$ or $SU(m,1)$. Let N, Λ be provided by Theorem 1. Set $M = L(\Lambda \ltimes \mathbb{Z}^N)$. Let $\mathbb{T}^N$ be the N-dimensional torus, viewed as the Pontryagin dual of $\mathbb{Z}^N$. Since $L(\mathbb{Z}^N) \simeq L^\infty(\mathbb{T}^N)$ in a Λ-equivariant way, we have*

$$M = L(\mathbb{Z}^N) \rtimes \Lambda \simeq L^\infty(\mathbb{T}^N) \rtimes \Lambda.$$

Since $\Lambda \ltimes \mathbb{Z}^N$ has infinite conjugacy classes, M is a II_1-factor; equivalently, the action of Λ on $\mathbb{T}^N$ is ergodic.

Set $A = L(\mathbb{Z}^N)$; since Λ is a-T-menable and the pair $(\Lambda \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has property (T), we see that A is an HT_s-Cartan subalgebra in M.

Popa's fundamental result (Theorem 6.2 in [Popa]) is that a factor M in the class $\mathcal{HT}_s$ has a unique HT_s-Cartan subalgebra, up to conjugation by unitaries in M. In particular, there exists a unique (up to isomorphism) standard equivalence relation $\mathcal{R}_M$ on the standard probability space, implemented by the normalizer of any HT_s-Cartan subalgebra of M. This means that any invariant of the equivalence relation $\mathcal{R}_M$ becomes an isomorphism invariant of the factor M.

This brings us to Gaboriau's L^2-Betti numbers for measurable equivalence relations [Gab02]. If $\mathcal{R}$ is a standard equivalence relation on the standard probability space (X, μ), Gaboriau defines, for $n = 0, 1, 2\ldots$ the L^2-Betti number $b_n^{(2)}(\mathcal{R}) \in [0, +\infty[$. We will use two properties of these numbers.

1) If B is a Borel subset of X, with $0 < \mu(B)$, denote by $\mathcal{R}^B$ the restriction of $\mathcal{R}$ to B. Then $b_n^{(2)}(\mathcal{R}^B) = \frac{b_n^{(2)}(\mathcal{R})}{\mu(B)}$ for every $n \geq 0$.

2) If $\mathcal{R}$ is induced by a measure preserving, essentially free action of a countable group Γ, then $b_n^{(2)}(\mathcal{R}) = b_n^{(2)}(\Gamma)$ for every $n \geq 0$; here $b_n^{(2)}(\Gamma)$ denotes the n-th L^2-Betti number of Γ, as defined by Cheeger and Gromov [CG86].

Coming back to a factor M in the class $\mathcal{HT}_s$, we define -after Popa- the n-th L^2-Betti number of M as $b_n^{(2)}(M) = b_n^{(2)}(\mathcal{R}_M)$. From property 1) above, if $0 < b_n^{(2)}(M) < \infty$ for some $n \geq 0$, then $b_n^{(2)}(M^t) = \frac{b_n^{(2)}(M)}{t}$ which implies immediately that $\mathcal{F}(M) = \{1\}$.

Proof of Theorem 3: We know by example 1(a) that $\Gamma_n = SO(2n,1)(\mathbb{Z})$ is an arithmetic lattice in $SO(2n,1)$, and that the pair $(\Gamma_n \ltimes \mathbb{Z}^{N_n}, \mathbb{Z}^{N_n})$ has property (T). By example 2, the von Neumann algebra $M_n = L(\Gamma_n \ltimes \mathbb{Z}^{N_n})$ is a II_1-factor in the class $\mathcal{HT}_s$. Since the equivalence relation $\mathcal{R}_{M_n}$ is induced by the action of Γ_n on $\mathbb{T}^{N_n}$, by property 2) above we have for every $k \geq 0$

$$b_k^{(2)}(M_n) = b_k^{(2)}(\Gamma_n).$$

Now, for any lattice Λ in $SO(2n,1)$, the L^2-Betti number $b_k^{(2)}(\Lambda)$ was estimated by Borel [Bor85]: the result is

$$b_k^{(2)}(\Lambda) = \begin{cases} 0 & if \quad k \neq n \\ > 0 & if \quad k = n \end{cases}$$

So M_n has exactly one non-zero L^2-Betti number, namely the n-th one. This proves simultaneously that $\mathcal{F}(M_n) = \{1\}$ and that the M_n's are pairwise non-isomorphic. □

Remarks:

i) Theorem 3 also holds with $SO(2n,1)$ replaced by $SU(n,1)$, N_n being replaced by $\dim_{\mathbf{R}} SU(n,1) = n(n+2)$ and $SO(2n,1)(\mathbb{Z})$ being replaced by $SU(n,1)(\mathbb{Z}[i])$ (the latter being a non-uniform, arithmetic lattice in $SU(n,1)$. The reason is that, for any lattice Λ in $SU(n,1)$, one has by [Bor85]:

$$b_k^{(2)}(\Lambda) = \begin{cases} 0 & if \quad k \neq n \\ > 0 & if \quad k = n \end{cases}$$

as in the case of $SO(2n,1)$. On the other hand, if Λ is a lattice in $SO(2n+1,1)$, all its L^2-Betti numbers are zero, so the same holds for the corresponding II_1-factors constructed in Example 2.

ii) For $n \geq 2$, let Γ_n be a lattice in the adjoint group of $Sp(n,1)$. Let N_n, Λ_n be provided by Corollary 1; set $M_n = L(\Lambda_n \ltimes \mathbb{Z}^N)$. Then M_n is a II_1-factor with property (T) in the sense of Connes and Jones [CJ85]; so that $\mathcal{F}(M_n)$ is countable, by a result of Connes [Con80]. To the best of our knowledge, it is unknown whether the M_n's are pairwise non-isomorphic. However, it is a result of Cowling and Zimmer [CZ89] that the inclusions $L(\mathbb{Z}^{N_n}) \subset M_n$ are pairwise non-isomorphic.

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