On the Baum-Connes assembly map for discrete groups

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With an Appendix by Dan KUCEROVSKY

Abstract: In these notes, we study the Baum-Connes analytical assembly maps (or index maps) $\mu_i^\Gamma : RK_i^\Gamma(\underline{E}\Gamma) \to K_i(C_r^*\Gamma)$ and $\tilde{\mu}_i^\Gamma : RK_i^\Gamma(\underline{E}\Gamma) \to K_i(C^*\Gamma)$, for a countable group Γ. Here $RK_i^\Gamma(\underline{E}\Gamma)$ denotes the Γ-equivariant K-homology with Γ-compact supports of the universal space $\underline{E}\Gamma$ for proper Γ-actions, while $K_i(C_r^*\Gamma)$ (resp. $K_i(C^*\Gamma)$) denotes the analytical K-theory of the reduced (resp. full) C*-algebra of Γ. As it is simple and direct, we use the definition of μ_i^Γ suggested by Baum, Connes and Higson in section 3 of [BCH94]. The Baum-Connes conjecture asserts that, for any group Γ, the map μ_i^Γ is an isomorphism ($i = 0, 1$). The contents of this paper are as follows:

1. We make the necessary changes for constructing $\tilde{\mu}_i^\Gamma$, and give a detailed proof that μ_i^Γ and $\tilde{\mu}_i^\Gamma$ provide K-theory elements of the corresponding C*-algebras.

2. We carefully describe the behaviour of the left-hand side of the assembly maps under group homomorphisms, and we prove that $\tilde{\mu}_i^\Gamma$ is natural with respect to arbitrary group homomorphisms. As a consequence, we get a new proof of the fact that, if Γ acts freely on the space X, then the equivariant K-homology $K_*^\Gamma(X)$ is isomorphic to the K-homology $K_*(\Gamma\backslash X)$ of the orbit space.

3. To illustrate the non-triviality of the assembly map, we give a direct proof of the Baum-Connes conjecture for the group $\mathbb{Z}$ of integers, not appealing to equivariant KK-theory.

4. Denote by $\tilde{\kappa}_\Gamma : \Gamma \to K_1(C_r^*\Gamma)$ the homomorphism induced by the canonical inclusion of Γ in the unitary group of $C_r^*\Gamma$. We show that there exists a homomorphism $\tilde{\beta}_t : \Gamma \to RK_1^\Gamma(\underline{E}\Gamma)$ such that $\tilde{\kappa}_\Gamma = \mu_i^\Gamma \circ \tilde{\beta}_t$; this extends a result of Natsume [Nat88] for Γ torsion-free.

The Appendix, by Dan Kucerovsky, discusses the assembly map in terms of unbounded K-homology elements.

Contents

Chapter 1

Introduction

1.1 The Baum-Connes conjecture

Let Γ be a countable, discrete group. The *Baum-Connes conjecture* is a tantalizing programme that identifies two objects associated with Γ, one analytical and one geometrical or topological.

The analytical side involves the K-theory of the *reduced C^*-algebra* $C_r^*\Gamma$, which is the C*-algebra generated by Γ in its left regular representation on the Hilbert space $\ell^2(\Gamma)$. The K-theory used here, $K_i(C_r^*\Gamma)$ for $i = 0, 1$, is the usual K-theory for Banach algebras, as described e.g. in [Tay75].

On the opposite side, one finds the K-homology (with compact supports) of a certain classifying space. More precisely, consider the universal space $\underline{E\Gamma}$ for proper Γ-actions (as described in (1.6) of [BCH94], see also Chapter 2 below; such a space is unique up to Γ-equivariant homotopy). A Γ-invariant subset $Y \subset \underline{E\Gamma}$ is Γ-*compact* if the orbit space $\Gamma\backslash Y$ is compact. The geometric group is the Γ -equivariant K-homology with Γ-compact supports $RK_i^\Gamma(\underline{E\Gamma})$ of $\underline{E\Gamma}$, i.e. the inductive limit of the Γ-equivariant K-homology [1] groups $K_i^\Gamma(Y)$, where Y runs along Γ-compact subsets of $\underline{E\Gamma}$.

The link between both sides of the conjecture is provided by the *analytic assembly map*, or *index map*

$$\mu_i^\Gamma : RK_i^\Gamma(\underline{E\Gamma}) \to K_i(C_r^*\Gamma)$$

[1]The groups $K_i^\Gamma(Y)$ can be defined as the equivariant Kasparov groups $KK_i^\Gamma(C_0(Y), \mathbb{C})$, where $C_0(Y)$ denotes the abelian C*-algebra of continuous functions vanishing at infinity on Y; for equivariant Kasparov theory, we refer to [Kas95], [Kas88]. We shall give more details on that definition in section 2.2. As in Chapter 5 of [Roe96], it is also possible to define $K_i^\Gamma(Y)$ as the *K-theory* of the algebra of pseudo-local operators modulo locally compact operators on Y, in a suitable covariant representation of $C_0(Y)$.

($i = 0, 1$). The definition of the assembly map can be traced back to a result of Kasparov [Kas83]: suppose that Z is a proper Γ-compact manifold endowed with a Γ-invariant elliptic differential operator D on some Γ-vector bundle over Z. Then, in spite of the non-compactness of the manifold Z, the *index* of D has a well-defined meaning as an element of the K-theory $K_i(C_r^*\Gamma)$. On the other hand, using the universal property of $\underline{E}\Gamma$, the manifold Z maps continuously Γ-equivariantly to $\underline{E}\Gamma$, and the pair (Z, D) defines an element of the equivariant K-homology with compact supports $RK_i^\Gamma(\underline{E}\Gamma)$. Then, one sets

$$\mu_i^\Gamma(Z, D) = Index(D).$$

Elaborating on this, and using the concept of abstract elliptic operator (or Kasparov triple), one defines the assembly map μ_i^Γ, which is a group homomorphism.

Conjecture 1 *(the Baum-Connes conjecture) For $i = 0, 1$, the assembly map*

$$\mu_i^G : RK_i^\Gamma(\underline{E}\Gamma) \to K_i(C_r^*\Gamma)$$

is an isomorphism.

This conjecture is part of a more general conjecture (discussed in [BCH94]) where discrete groups are replaced by arbitrary locally compact groups [2]. The reason for restricting to discrete groups is that, in a sense, this case is both interesting and difficult. The main difficulty comes from the analytical side: e.g., there is no general structure result for the reduced C*-algebra of a discrete group, so that its K-theory is usually quite hard to compute (recall that, in many important cases, e.g. lattices in semi-simple Lie group, $C_r^*\Gamma$ is actually *simple*, see [BCdlH94]). The interest of Conjecture 1 is that it *implies* several other famous conjectures in topology, geometry, algebra and functional analysis.

Conjecture 2 *(the Novikov conjecture) For closed oriented manifolds with fundamental group Γ, the higher signatures coming from $H^*(\Gamma, \mathbb{Q})$ are oriented homotopy invariants.*

[2] Not to mention an even more general conjecture, that we deliberately ignore here, concerning either locally compact groups acting on locally compact spaces or foliated manifolds, and with coefficients in an arbitrary auxiliary C*-algebra.

The Novikov conjecture follows from the rational injectivity of μ_i^Γ (see [BCH94], Theorem 7.11; [FRR95], section 6).

Conjecture 3 *(one direction of the Gromov-Lawson-Rosenberg conjecture) If M is a closed spin manifold with fundamental group Γ, and if M is endowed with a metric of positive scalar curvature, then all higher $\hat{A}$-genera (coming from $H^*(\Gamma, \mathbb{Q})$) do vanish.*

Conjecture 3 is also a consequence of the rational injectivity of μ_i^Γ (see Theorem 7.11 in [BCH94]).

Let us also mention the conjecture of idempotents for $C_r^*\Gamma$; since $C_r^*\Gamma$ is a completion of the complex group algebra $\mathbb{C}\Gamma$, this conjecture is stronger than the classical conjecture of idempotents, discussed e.g. in [Pas85].

Conjecture 4 *(the conjecture of idempotents, or Kaplansky-Kadison conjecture) Let Γ be a torsion-free group. Then $C_r^*\Gamma$ has no idempotent other than 0 or 1.*

This conjecture would follow from the surjectivity of μ_0^Γ (see Proposition 7.16 in [BCH94]; Proposition 3 in [Val89]).

It has to be emphasized that Conjecture 1 makes $K_i(C_r^*\Gamma)$ computable, at least up to torsion. The reason is that $RK_i^\Gamma(\underline{E\Gamma})$ is computable up to torsion. Let us explain this briefly. Let $F\Gamma$ be the space of complex-valued functions on Γ, whose support is finite and contained in the set of torsion elements of Γ. Letting Γ act by conjugation on torsion elements, $F\Gamma$ becomes a Γ-module; denote by $H_j(\Gamma, F\Gamma)$ the j-th homology group of Γ with coefficients in $F\Gamma$. In section 6 of [BC88a], Baum and Connes define a *Chern character*

$$ch_\Gamma : RK_i^\Gamma(\underline{E\Gamma}) \to \bigoplus_{n=0}^{\infty} H_{i+2n}(\Gamma, F\Gamma),$$

and state in Proposition 15.2 of [BC88a] that the Chern character is an isomorphism after tensoring by $\mathbb{C}$, i.e.

$$ch_\Gamma \otimes 1 : RK_i^\Gamma(\underline{E\Gamma}) \otimes_{\mathbb{Z}} \mathbb{C} \to \bigoplus_{n=0}^{\infty} H_{i+2n}(\Gamma, F\Gamma)$$

is an isomorphism (another Chern character, having all the desired properties, has been constructed by Matthey in Theorem 1.4 of [Mat]; he conjectures that his Chern character coincides with Baum-Connes', and proves this for $\Gamma = G \times \mathbb{Z}/n\mathbb{Z}$, with BG a closed manifold).

As an example, consider the case when Γ is torsion-free. The Γ-module $F\Gamma$ is just the trivial module $\mathbb{C}$; on the other hand, let $B\Gamma$ be a classifying space for Γ, i.e. a $K(\Gamma,1)$-space; let $E\Gamma$ be its universal covering space. Since Γ is torsion-free, any proper action is automatically free, so we may take $\underline{E\Gamma} = E\Gamma$. Then there is a canonical isomorphism

$$RK_i^\Gamma(E\Gamma) \simeq RK_i(B\Gamma),$$

where $RK_i(B\Gamma)$ denotes the K-homology with compact supports of $B\Gamma$. This identification is compatible with the usual Chern character in K-homology, i.e. there is a commutative diagram

$$\begin{array}{ccccc} RK_i^\Gamma(E\Gamma) & & \simeq & & RK_i(B\Gamma) \\ & ch_\Gamma \searrow & & \swarrow ch & \\ & & \bigoplus_{n=0}^\infty H_{i+2n}(\Gamma,\mathbb{C}) & & \end{array}$$

(see [BCH94], p. 274; [Mat], Theorem 1.4).

As another example, take for Γ a finite group. Then

$$RK_0^\Gamma(\underline{E\Gamma}) = K_0^\Gamma(pt) \simeq R(\Gamma),$$

where $R(\Gamma)$ is the representation ring of Γ. On the other hand,

$$\bigoplus_{n=0}^{\infty} H_{2n}(\Gamma, F\Gamma) \; = \; H_0(\Gamma, F\Gamma)$$

is the complex vector space on the set of conjugacy classes of Γ. In other words, the fact that the Chern character is an isomorphism (after tensoring with $\mathbb{C}$) incorporates the classical but not quite obvious fact that, for a finite group, the number of irreducible representations is equal to the number of conjugacy classes.

1.2 What these Notes are about

The basic sources for the Baum-Connes conjecture are the original articles [BC00], [BC88a], [BC88b], [BCH94]; in textbooks, see various sections in the books by Connes [Con94] and by Higson-Roe [HR00b]. For expository presentations entirely devoted to the Baum-Connes conjecture, see the Bourbaki seminars by Julg [Jul98] and Skandalis [Ska99], and the pre-book by the author [Val]. In particular, the last three references contain relevant information about the status of the Baum-Connes conjecture, especially for which classes of discrete groups surjectivity and/or injectivity of the assembly map has been proved.

These Notes were begun in 1996-97, and have gone through many states since then. At the origin, they were aimed at backing up some aspects of the proof of the Baum-Connes conjecture for one-relator groups in [BBV99]. Publication was delayed, due to the feeling that the results contained in these Notes were known to every expert. Part of the Notes was used as we were working on [Val], which explains some overlap, for which we apologize. Eventually I yielded to the friendly insistance of some colleagues and some graduate students, who convinced me that these Notes, although not fully original, could be of some interest for beginners. Let us discuss now the content of these Notes.

It has been noticed by many authors (see e.g. [Con94] p.99, [FRR95] section 6, [Jul98]) that the analytical assembly map $\mu_i^\Gamma : RK_i^\Gamma(\underline{E\Gamma}) \to K_i(C_r^*\Gamma)$ factors through the K-theory of the *full* C*-algebra $C^*\Gamma$, which is the universal C*-completion of the group algebra $\mathbb{C}\Gamma$. More precisely, there is a homomorphism

$$\tilde{\mu}_i^\Gamma : RK_i^\Gamma(\underline{E\Gamma}) \to K_i(C^*\Gamma)$$

such that

$$\mu_i^\Gamma = (\lambda_\Gamma)_* \circ \tilde{\mu}_i^\Gamma,$$

where $\lambda_\Gamma : C^*\Gamma \to C_r^*\Gamma$ is the canonical epimorphism corresponding to the left regular representation of Γ. As we shall see, the assembly map $\tilde{\mu}_i^\Gamma$ enjoys better naturality properties than μ_i^Γ.

In Chapter 2, we give the definition of Baum-Connes-Higson [BCH94] for μ_i^Γ and provide the necessary changes for $\tilde{\mu}_i^\Gamma$; these definitions have the advantage of being direct and avoiding the use of Kasparov products. However, it is not completely apparent from Definition (3.8) in [BCH94] that the map μ_i^Γ is well-defined and actually provides K-theory elements of $C_r^*\Gamma$. There is a number of checks to be made, for which we give the relevant details; in particular, we pay due attention to positivity questions, often overlooked. In the process, we also show that the two definitions of μ_i^Γ, given in sections 3 and 8 of [BCH94], are truly the same: namely, for X a Γ-compact subset of

$\underline{E}\Gamma$ and $x \in KK_i^\Gamma(C_0(X), \mathbb{C})$, then $\mu_i^\Gamma(x)$ can be defined by first applying Kasparov's descent homomorphism j_Γ to x, and then taking the Kasparov product of $j_\Gamma(x)$ by the canonical line bundle $[\mathcal{L}_X]$ over $C_0(X) \rtimes \Gamma$.

Next we will prove that the analytic assembly map $\tilde{\mu}_i^\Gamma$ is natural, i.e.:

Theorem 1 *Let $\alpha : \Gamma_1 \to \Gamma_2$ be a group homomorphism; then there is a commutative diagram:*

$$\begin{array}{ccc} RK_i^{\Gamma_1}(\underline{E}\Gamma_1) & \stackrel{\tilde{\mu}_i^{\Gamma_1}}{\to} & K_i(C^*\Gamma_1) \\ \alpha_* \downarrow & & \downarrow \alpha_* \\ RK_i^{\Gamma_2}(\underline{E}\Gamma_2) & \stackrel{\tilde{\mu}_i^{\Gamma_2}}{\to} & K_i(C^*\Gamma_2) \end{array}$$

This result is of course known to experts (see [Con94], pp.96-97; [FRR95], p.44; [Ros83]; [Lueb]). But a proof never appeared in the literature, as far as we know. Most of Chapter 3 is devoted to the proof of Theorem 1: 3.1 deals with the case of group monomorphisms, 3.2 with group epimorphisms (since any group homomorphism is the product of an epimorphism with a monomorphism, this clearly implies the general case). Actually one difficulty in the proof is to carefully describe the functoriality of the left-hand side, i.e how $RK_i^\Gamma(\underline{E}\Gamma)$ behaves under group homomorphisms. For an epimorphism $\alpha : \Gamma_1 \twoheadrightarrow \Gamma_2$ with kernel N, the construction is inspired from Kasparov's descent homomorphism (Theorem 3.4 in [Kas88]): if Γ_1 acts properly on a space X, this provides a homomorphism $\alpha_* : K_i^{\Gamma_1}(X) \to K_i^{\Gamma_2}(N\backslash X)$. In particular, if Γ acts *freely* on X, by considering the constant homomorphism α from Γ to the trivial group, we get an explicit map $\alpha_* : K_i^\Gamma(X) \to K_i(\Gamma\backslash X)$ which, as shown in Corollary 4, coincides with the isomorphism constructed classically using Morita equivalence (see [Gre77]).

The proof of Theorem 1 uses the fact that the full group C*-algebra is functorial for arbitrary group homomorphisms. By way of contrast, the reduced C*-algebra is functorial only for group monomorphisms. This purely analytical reason is responsible for the limited naturality of μ_i^Γ, that we now state precisely.

Let $\alpha : \Gamma_1 \to \Gamma_2$ be a group monomorphism; then there is a commutative diagram:

$$\begin{array}{ccc} C^*\Gamma_1 & \stackrel{\alpha_*}{\to} & C^*\Gamma_2 \\ \lambda_{\Gamma_1} \downarrow & & \downarrow \lambda_{\Gamma_2} \\ C_r^*\Gamma_1 & \stackrel{\alpha_*}{\to} & C_r^*\Gamma_2 \end{array}$$

As an immediate consequence, we have:

Corollary 1 *For a group monomorphism $\alpha : \Gamma_1 \to \Gamma_2$, there is a commutative diagram:*

$$\begin{array}{ccc} RK_i^{\Gamma_1}(\underline{E}\Gamma_1) & \stackrel{\mu_i^{\Gamma_1}}{\to} & K_i(C_r^*\Gamma_1) \\ \alpha_* \downarrow & & \downarrow \alpha_* \\ RK_i^{\Gamma_2}(\underline{E}\Gamma_2) & \stackrel{\mu_i^{\Gamma_2}}{\to} & K_i(C_r^*\Gamma_2) \end{array}$$

It is however possible to restore full naturality by imposing conditions on the source group. Recall that a group Γ is *amenable* if $\lambda_\Gamma : C^*\Gamma \to C_r^*\Gamma$ is an isomorphism (this is one among lots of equivalent definitions, see [Ped79] 7.3). More generally, Γ is *K-amenable* in the sense of Cuntz [Cun83] if

$$\lambda_\Gamma^* : K^0(C_r^*\Gamma) \to K^0(C^*\Gamma)$$

is an isomorphism in K-homology; this is known to imply that the K-theory map

$$(\lambda_\Gamma)_* : K_i(C^*\Gamma) \to K_i(C_r^*\Gamma)$$

is an isomorphism (see [Cun83], Theorem 2.1). Until Lafforgue's remarkable results [Laf98], all the groups for which Conjecture 1 was proved, belonged to the class of K-amenable groups.

Corollary 2 *Let $\alpha : \Gamma_1 \to \Gamma_2$ be a group homomorphism, where Γ_1 is K-amenable; then there is a commutative diagram:*

$$\begin{array}{ccc} RK_i^{\Gamma_1}(\underline{E}\Gamma_1) & \stackrel{\mu_i^{\Gamma_1}}{\to} & K_i(C_r^*\Gamma_1) \\ \alpha_* \downarrow & & \downarrow \alpha_* \\ RK_i^{\Gamma_2}(\underline{E}\Gamma_2) & \stackrel{\mu_i^{\Gamma_2}}{\to} & K_i(C_r^*\Gamma_2) \end{array}$$

At this juncture, notice that, since the left-hand side of the Baum-Connes conjecture (i.e. the geometric group $RK_i^\Gamma(\underline{E}\Gamma)$) is fully functorial with respect to group homomorphisms, it would follow from the *truth* of the Baum-Connes conjecture that the right-hand side is fully functorial, i.e. for any group homomorphism $\alpha : \Gamma_1 \to \Gamma_2$ there should exist a functorial $\alpha_* : K_i(C_r^*\Gamma_1) \to K_i(C_r^*\Gamma_2)$; this functoriality, on which we elaborate in Example 2 of Chapter 3, can be "explained" by a conjecture of J.-B. Bost (see [Ska99]): the range of the Baum-Connes assembly map μ_*^Γ should be the K-theory of $\ell^1\Gamma$ rather than the one of $C_r^*\Gamma$; and of course $\ell^1\Gamma$ is fully functorial with respect to group homomorphisms.

To illustrate the construction of the analytical assembly map, we give in Chapter 4 a direct proof of the fact that

$$\mu_1^{\mathbb{Z}} : RK_1^{\mathbb{Z}}(\underline{E}\mathbb{Z}) \to K_1(C^*\mathbb{Z})$$

is an isomorphism. Of course, it is well-known that the group $\mathbb{Z}$ satisfies the Baum-Connes conjecture, and it might even be tempting to believe that this result is obvious (this is essentially the opinion expressed in lemma 3.5 of [BC88b]). I think that, although both groups involved are isomorphic to $\mathbb{Z}$, the definition of the assembly map is intricate enough, so that one really has to check that $\mu_1^{\mathbb{Z}}$ maps generator to generator. It is informative here to look at Kasparov's dual map

$$\alpha : K^i(C^*\Gamma) \to RK^i(B\Gamma),$$

defined in [Kas75], section 8; [Kas95], section 9; [Kas88], section 6, and which was considered prior to the Baum-Connes assembly map. Then

$$\alpha : K^1(C^*\mathbb{Z}) \to K^1(B\mathbb{Z})$$

is an isomorphism: the non-triviality of this fact is apparent from the proofs in [Kas75], Theorem 1 of section 8; [Ros84], lemma 3.2. Coming back to "$\mu_1^{\mathbb{Z}}$ is an isomorphism", it is clear that this result is contained in Kasparov's conspectus [Kas95], but hidden in the wide generality of Theorem 1 of section 7. What is explicit there (and non-trivial) is the fact the Connes-Kasparov conjecture holds for the 1-dimensional Lie group $\mathbb{R}$ ([Kas95], lemma 4 in section 5). Then one appeals to the machinery of equivariant KK-theory, whose powerful functorialities allow to descend from a Lie group to a discrete subgroup. I thought that it was worthwhile to give a direct proof. Another direct proof can be found in Example 12.5.9 of the recent book [HR00b].

In the final Chapter 5, we consider the canonical homomorphism

$$\tilde{\kappa}_\Gamma : \Gamma \to K_1(C_r^*\Gamma)$$

obtained from the canonical embedding of Γ into the unitary group of $C_r^*\Gamma$. Since $K_1(C_r^*\Gamma)$ is abelian, $\tilde{\kappa}_\Gamma$ factors through a homomorphism

$$\kappa_\Gamma : \Gamma^{ab} \to K_1(C_r^*\Gamma)$$

where Γ^{ab} denotes the abelianization of Γ. It is known (see [EN87], [BV96]) that κ_Γ is always rationally injective.

Theorem 2 *There exists a homomorphism*

$$\beta_t : \Gamma^{ab} \to RK_1^\Gamma(\underline{E}\Gamma)$$

such that, as homomorphisms $\Gamma^{ab} \to K_1(C_r^*\Gamma)$*, one has*

$$\kappa_\Gamma = \mu_1^\Gamma \circ \beta_t.$$

For Γ torsion-free, such a map β_t was previously constructed by Natsume [Nat88] who, however, does not give the proof that β_t is a group homomorphism. The proof of Theorem 2 appeals to Theorem 1 together with the fact (proved in Chapter 4) that conjecture 1 holds for the group of integers. Recall that $\Gamma^{ab} = H_1(\Gamma, \mathbb{Z})$, and that the inclusion $\mathbb{C} \hookrightarrow F\Gamma$ associated with the trivial conjugacy class in Γ, induces an inclusion

$$\Gamma^{ab} \otimes_{\mathbb{Z}} \mathbb{C} = H_1(\Gamma, \mathbb{C}) \hookrightarrow \bigoplus_{n=0}^{\infty} H_{2n+1}(\Gamma, F\Gamma) \simeq RK_1^\Gamma(\underline{E}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Theorem 2 together with the rational injectivity of κ_Γ then imply that μ_1^Γ is rationally injective on the image of β_t, i.e. on the lowest dimensional part of $RK_1^\Gamma(\underline{E}\Gamma)$.

Deep generalizations of Theorem 2 have been proposed in Matthey's PhD thesis [Mat00], in the form of a "delocalized" version allowing him to treat other conjugacy classes in Γ than the trivial one. More precisely, for $0 \leq j \leq 2$, he constructs maps $\beta_t : H_j(\Gamma, F\Gamma) \to RK_j^\Gamma(\underline{E}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C}$ and $\beta_a : H_j(\Gamma, F\Gamma) \to K_j(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{C}$, commuting with $\mu_j^\Gamma \otimes 1$.

The Appendix, by Dan Kucerovsky, presents the construction of the Baum-Connes assembly map "in the unbounded picture", i.e. when the K-homology elements in $RK_*^\Gamma(\underline{E}\Gamma)$ are given by unbounded Kasparov elements (as in [BJ83], [Kuc94]). The interest of this approach is that most K-homology elements of geometric origin are given by unbounded operators: e.g. first order, elliptic, differential operators (like de Rham, Dirac, or signature operators) define unbounded operators on the Hilbert space of L^2-sections of the corresponding vector bundles over the underlying manifold.

1.3 Other descriptions of assembly maps

Other approaches to the Baum-Connes assembly map have been proposed.

- For an arbitrary metric space X, J. Roe and N. Higson (see [HR00a], [Roe96]) define the coarse K-homology of X, denoted by $KX_*(X)$, the C*-algebra C^*X of operators with finite propagation on X, and the coarse assembly map

$$A_\infty : KX_*(X) \to K_*(C^*X).$$

Note that $KX_*(X) = RK_*(X)$ for X uniformly contractible with bounded geometry ([HR00a], Proposition 3.8). The *coarse Baum-Connes conjecture* is the statement that, for X a complete path metric space with bounded geometry, the map A_∞ is an isomorphism. For Γ a finitely generated group, view a Cayley graph $|\Gamma|$ as a complete path metric space. At least when Γ admits a finite complex as a classifying space, there is a "descent principle" allowing to deduce, from the conjectured isomorphism $A_\infty : KX_*(|\Gamma|) \to K_*(C^*|\Gamma|)$, the injectivity of the Baum-Connes assembly map $\mu_*^\Gamma : RK_*(B\Gamma) \to K_*(C_r^*\Gamma)$ (see [Roe96], Theorem 8.4). A comparison between this approach and the "classical" one can be found in [Roe].

- J.F. Davis and W. Lueck [DL98] give a categorical definition of assembly maps in algebraic K-theory, topological K-theory and L-theory, by means of spectra over the orbit category of the group Γ. The source of the Baum-Connes assembly map is defined there by considering the "orbit category" $Or(\Gamma, \mathcal{F}in)$ of quotients of Γ by finite subgroups, applying the functor $K^{top} : Or(\Gamma, \mathcal{F}in) \to SPECTRA$ constructed in section 2 of [DL98], considering the classifying space $E(\Gamma, \mathcal{F}in)$ of $Or(\Gamma, \mathcal{F}in)$, forming the "tensor product" spectrum $E(\Gamma, \mathcal{F}in) \otimes_{Or(\Gamma, \mathcal{F}in)} K^{top}$, as in section 1 of [DL98], and finally applying homotopy groups. W. Lueck has communicated to us a simple proof [Luea] of the naturality of the source of the Baum-Connes assembly map under arbitrary group homomorphisms: when expressed in that language, it basically boils down to the fact that the orbit category is natural, i.e. that group homomorphisms map finite subgroups to finite subgroups! In section 5 of [DL98], Davis and Lueck construct an "assembly map" to $K_*(C_r^*\Gamma)$: it was recently proved by I. Hambleton and E.K. Pedersen (see [HP], Corollary 7.4), that this construction is equivalent to the one in [BCH94].

1.4 Remarks on background

The reader of this paper is advised to have some background in C*-algebras. The reason is not that the author has been educated in the C*-faith, but rather that the difficulties encountered are analytical in nature - even to describe the functoriality of the left-hand side of the Baum-Connes conjecture! I shall use freely those parts of C*-algebra theory relevant for Kasparov's KK-theory, namely positivity, representations, and multipliers; they can be found either in Arveson's book [Arv76] or in Pedersen's book [Ped79]; for group C*-algebras, Dixmier's book [Dix77] is compulsory; for Hilbert C*-modules, I recommand Lance's lovely little book [Lan95]; for KK-theory itself, I suggest the book [JT91] by Knudsen Jensen and Thomsen.

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Chapter 2

The analytical assembly map

2.1 Proper actions

Let X be a metrizable space on which the group Γ acts by homeomorphisms.

Definition 1 *The Γ-space X is proper if every point in X admits a Γ-invariant open neighbourhood which maps Γ-equivariantly continuously to an homogeneous space Γ/H, where H is a finite subgroup of Γ.*

This definition is *stronger* than the usual definition of a proper action, which requires that, for any compact subsets K, L of X, the set

$$\{\gamma \in \Gamma : \gamma K \cap L \neq \emptyset\}$$

is finite (or, equivalently, the action map $\Gamma \times X \to X \times X : (\gamma, x) \mapsto (\gamma.x, x)$ is proper in the usual sense that the inverse image of a compact subset is compact). For locally compact spaces, Definition 1 is actually equivalent to the classical one (see [Pal61]).

According to our definition, a proper Γ-space is locally of the form $\Gamma \times_H Y$, a space induced from the action of a finite subgroup H on a space Y. We say that a proper Γ-space X is Γ-*compact* if $\Gamma\backslash X$ is compact; note that a proper, Γ-compact space has to be locally compact.

Definition 2 *A proper Γ-space $\underline{E}\Gamma$ is universal if it satisfies the following universal property: for every proper Γ-space X, there exists a Γ-equivariant continuous map $X \to \underline{E}\Gamma$, and any two such maps are Γ-equivariantly homotopic.*

The Γ-equivariant map $X \to \underline{E}\Gamma$ is not proper in general, but it is proper as soon as X is Γ-compact (see Lemma 10 below). It is clear from Definition 2 that a universal proper Γ-space is unique, up to Γ-equivariant homotopy. If Γ is torsion-free, any proper Γ-action is free, so we may take for $\underline{E}\Gamma$ the universal covering space $E\Gamma$ of the classifying space $B\Gamma$. If Γ is finite, we may take $\underline{E}\Gamma = pt$, the one-point space. The non-classical definition of properness in Definition 1 is required because there are natural examples where $\underline{E}\Gamma$ is definitely not a locally compact space. For example, the following fairly simple description of $\underline{E}\Gamma$ (see section 2 in [BCH94]), valid for an arbitrary Γ, is not locally compact as soon as Γ is infinite. The lemma below appears in section 2 of [BCH94]; our proof is slightly more direct than the original one.

Lemma 1 *Let $\underline{E}\Gamma$ be the space of finitely supported probability measures on Γ, endowed with the metric*

$$\|\mu - \nu\|_\infty = \sup\{|\mu(\gamma) - \nu(\gamma)| : \gamma \in \Gamma\},$$

and with the action of Γ given by left multiplication. Then $\underline{E}\Gamma$ is a universal proper Γ-space.

Proof: We first check that the action of Γ on $\underline{E}\Gamma$ is proper. Fix $\mu \in \underline{E}\Gamma$, and let Γ_μ be its stabilizer, a finite subgroup in Γ. Set

$$R = \inf\{\|\mu - \gamma(\mu)\|_\infty : \gamma \in \Gamma - \Gamma_\mu\};$$

one sees easily that $R > 0$. For $\epsilon > 0$, define

$$U = \{\nu \in \underline{E}\Gamma : \exists \gamma \in \Gamma : \|\nu - \gamma(\mu)\|_\infty < \epsilon\};$$

it is an open, Γ-invariant subset of $\underline{E}\Gamma$. Moreover, for $\epsilon < \frac{R}{2}$, the open set U is such that, for $\nu \in U$, the element $\gamma \in \Gamma$ with $\|\nu - \gamma(\mu)\|_\infty < \epsilon$ is unique modulo Γ_μ. Sending ν to the coset $\gamma\Gamma_\mu$ then defines a Γ-equivariant map $U \to \Gamma/\Gamma_\mu$. So $\underline{E}\Gamma$ is a proper Γ-space.

Let X be a proper Γ-space. We have to show that there exists a continuous Γ-equivariant map, which is unique up to Γ-equivariant homotopy. Uniqueness is clear, since $\underline{E}\Gamma$ is a convex set on which Γ acts affinely. For the existence, denote by W the disjoint union of the Γ/H's, where H runs along finite subgroups of Γ. Define a Γ-equivariant map $\phi : W \to \underline{E}\Gamma$ by sending the coset γH to the uniform probability measure on γH. By the lemma in

Appendix 1 of [BCH94], there exists a countable partition of unity $(\alpha_k)_{k\geq 1}$ on X, consisting of Γ-invariant functions and such that, for every $k \geq 1$, there is a Γ-equivariant continuous map $\psi_k : \alpha_k^{-1}]0,1] \to W$. Then the map

$$\Psi : X \to \underline{E}\Gamma : x \mapsto \sum_{k=1}^{\infty} \alpha_k(x)(\phi \circ \psi_k)(x)$$

is continuous and Γ-equivariant. $\square$

2.2 Equivariant K-homology

We now recall, following [BCH94], the definition of the geometric group $RK_i^\Gamma(\underline{E}\Gamma)$ that appears in the left hand side of the Baum-Connes conjecture.

Definition 3 *The Baum-Connes geometric group is*

$$RK_i^\Gamma(\underline{E}\Gamma) = \varinjlim_{\substack{X \subset \underline{E}\Gamma \\ X \text{ is } \Gamma-\text{compact}}} K_i^\Gamma(X),$$

where $K_i^\Gamma(X)$ is the Γ-equivariant K-homology of X.

More generally, for Y a proper Γ-space, we define $RK_i^\Gamma(Y)$ as the inductive limit of the $K_i^\Gamma(X)$'s, where X runs along Γ-compact subsets of Y. If Γ is the trivial group, we drop the superscript Γ.

An element of $K_i^\Gamma(X)$ is represented by a *Kasparov triple* or *abstract elliptic operator* $(\mathcal{H}, \pi, F)$, where $\mathcal{H}$ is a Hilbert space endowed with a unitary representation of Γ, where π is a covariant representation of $C_0(X)$ on $\mathcal{H}$, and where F is a bounded self-adjoint operator on $\mathcal{H}$ such that $[F, \pi(f)]$, $\pi(f)(F^2 - 1)$ and $\pi(f)[\gamma, F]$ are compact operators on $\mathcal{H}$ for any $f \in C_0(X)$ and any $\gamma \in \Gamma$. For $i = 0$, we require moreover $\mathcal{H}$ to be a $\mathbb{Z}/2$-graded Hilbert space, the representations of Γ and $C_0(X)$ to be by degree 0 operators (i.e. they preserve the grading), and F to be a degree 1 operator (i.e. it reverses the grading).

Clearly, by compressing to the orthogonal of the null space of $\pi(C_0(X))$, we may assume that π is a non-degenerate representation. Because the action of Γ on X is proper and X is Γ-compact, we shall see that we may actually assume, in the definition of a Kasparov element $(\mathcal{H}, \pi, F) \in K_i^\Gamma(X)$, that F

is Γ-equivariant (i.e. $[\gamma, F] = 0$ for any $\gamma \in \Gamma$) and *properly supported*, i.e. for any $f \in C_c(X)$ there exists $g \in C_c(X)$ with $(\pi(g) - 1)F\pi(f) = 0$. The latter condition has to be understood as a locality condition.

It turns out that properness of the action and Γ-compactness of X allow for the possibility of averaging over the group Γ, in order to make F equivariant. This is explained in the next lemma.

Lemma 2 *Fix a real-valued $f \in C_c(X)$. For $T \in \mathcal{L}(\mathcal{H})$, set*

$$A_f(T) = \sum_{\gamma\in\Gamma} \gamma\pi(f)T\pi(f)\gamma^{-1}.$$

Then

1. *The sum $\sum_{\gamma\in\Gamma} \gamma\pi(f)T\pi(f)\gamma^{-1}$ converges in the strong operator topology, and $A_f(T)$ is a bounded operator on $\mathcal{H}$; more precisely, there exists a constant $C > 0$, only depending on f, such that $\|A_f(T)\| \leq C\|T\|$.*

2. *$A_f(T)$ is properly supported (and, in particular, it maps $\pi(C_c(X))\mathcal{H}$ into itself).*

3. *$A_f(T)$ is Γ-equivariant, i.e. it commutes with Γ.*

Proof:

1. (inspired by the proof of Lemma 3.2 in [Kas88]). Separating the real and imaginary parts of T, we may assume that T is self-adjoint. Then one has the operator inequalities

   $$-\gamma\pi(f^2)\gamma^{-1}\|T\| \leq \gamma\pi(f)T\pi(f)\gamma^{-1} \leq \gamma\pi(f^2)\gamma^{-1}\|T\|$$

   (for $\gamma \in \Gamma$). Summing over γ, one gets

   $$-(\sum_{\gamma\in\Gamma} \pi(\gamma(f^2)))\|T\| \leq A_f(T) \leq \sum_{\gamma\in\Gamma} \pi(\gamma(f^2))\|T\|.$$

   Thus $\|A_f(T)\| \leq \| \sum_{\gamma\in\Gamma} \gamma(f^2)\|_\infty \cdot \|T\|$.

2. For $g \in C_c(X)$, we have

   $$A_f(T)\pi(g) = \sum_{\gamma\in\Gamma} \pi(\gamma(f))\gamma F\pi(f.\gamma^{-1}(g))\gamma^{-1}.$$

   Consider the finite set $F = \gamma \in \Gamma : f.\gamma^{-1}(g) \neq 0$, and choose $h \in C_c(X)$ equal to 1 on $\bigcup_{\gamma\in F} \gamma(supp(f))$. Then $\pi(h)A_f(T)\pi(g) = A_f(T)\pi(g)$, which shows that $A_f(T)$ is properly supported.

3. Obvious.

$\square$

Proposition 1 *Let X be a proper and Γ-compact space. Any Kasparov triple $(\mathcal{H}, \pi, F)$ in $K_i^\Gamma(X)$ is operatorially homotopic to a Kasparov triple $(\mathcal{H}, \pi, G)$, where G is properly supported and Γ-equivariant.*

Proof: Since the space X is proper and Γ-compact, there exists a non-negative function $c \in C_c(X)$ such that $\sum_{\gamma\in\Gamma} c(\gamma x) = 1$ for every $x \in X$. Taking

$$h = \sqrt{c}, \tag{2.1}$$

we form $G = A_h(F)$.

It already follows from lemma 2 that G is properly supported and Γ-equivariant. Let us check that, for any $f \in C_c(X)$, the operator $\pi(f)(F-G)$ is compact. But since

$$\sum_{\gamma\in\Gamma} \gamma\pi(h^2)\gamma^{-1} = 1 \tag{2.2}$$

(as a consequence of the fact that π is non-degenerate), we have

$$\begin{array}{rcl} F - G & = & \sum_{\gamma\in\Gamma}(\gamma\pi(h^2)\gamma^{-1}F \;-\; \gamma\pi(h)F\pi(h)\gamma^{-1}) \\ = & & \sum_{\gamma\in\Gamma}\gamma\pi(h)[\pi(h)\gamma^{-1}, F] \\ = & & \sum_{\gamma\in\Gamma}\gamma\pi(h)(\pi(h)[\gamma^{-1}, F] + [\pi(h), F]\gamma^{-1}). \end{array}$$

We remark that, in the last summation, all terms are compact, thanks to the assumption on F. Then

$$\pi(f)(F-G) \;=\; \sum_{\gamma\in\Gamma}\gamma\pi(\gamma^{-1}(f).h)(\pi(h)[\gamma^{-1}, F] + [\pi(h), F]\gamma^{-1}).$$

Since there are finitely many γ's such that $\gamma^{-1}(f).h$ is non-zero, we see that $\pi(f)(F-G)$ is compact, as a finite sum of compact operators. Therefore G defines a Kasparov triple which is operatorially homotopic to F, via the homotopy $F_t = (1-t)G + tF$ $(t \in [0,1])$. $\square$

From now on, we shall assume that all operators defining Kasparov triples are Γ-equivariant and properly supported.

It is clear that the direct sum of two Kasparov triples over X is again a Kasparov triple over X. The equivalence relation which turns the set of Kasparov triples over X into the group $K_i^\Gamma(X)$ is the one described in [Kas95]; actually $K_i^\Gamma(X)$ is the Kasparov group $KK_i^\Gamma(C_0(X), \mathbb{C})$.

2.3 Definition of the assembly maps

We proceed to define the *analytic assembly map*

$$\tilde{\mu}_i^\Gamma : RK_i^\Gamma(\underline{E}\Gamma) \to K_i(C^*\Gamma),$$

by suitably modifying the construction given in [BCH94], (3.8), for the map $\mu_i^\Gamma : RK_i^\Gamma(\underline{E}\Gamma) \to K_i(C_r^*\Gamma)$.

We begin with a locally compact, proper Γ-space X and a Hilbert space $\mathcal{H}$ endowed with a covariant representation π of $C_0(X)$. Consider the Γ-module $\pi(C_c(X))\mathcal{H}$ (since π is non-degenerate, this is a dense subspace of H); view it as a right $\mathbb{C}\Gamma$-module by

$$\xi \cdot \gamma = \gamma^{-1}\xi; \tag{2.3}$$

define a $\mathbb{C}\Gamma$-valued scalar product on that module by

$$\langle \xi_1 | \xi_2 \rangle(\gamma) = \langle \xi_1 \cdot \gamma | \xi_2 \rangle = \langle \xi_1 | \gamma\xi_2 \rangle \tag{2.4}$$

(that $\gamma \mapsto \langle \xi_1 | \xi_2 \rangle(\gamma)$ has finite support in Γ already uses properness of the Γ-action) [1].

Since our aim is to complete $\pi(C_c(X))\mathcal{H}$ as a Hilbert C*-module, we have to discuss positivity of the functions $\gamma \mapsto \langle \xi | \gamma\xi \rangle$ (with $\xi \in \pi(C_c(X))\mathcal{H}$) as elements either in the reduced or the full C*-algebra of Γ. Recall that a function ϕ on Γ is *positive-definite* if it is of the form $\phi(g) = \langle \eta | \rho(g)\eta \rangle$ for some unitary representation ρ of Γ and some vector η in the Hilbert space of ρ.

For the reduced C*-algebra $C_r^*\Gamma$, an element $\gamma \mapsto \langle \xi | \gamma\xi \rangle$ as above, is positive for an abstract reason: any finitely supported, positive definite function on Γ defines a positive element in $C_r^*\Gamma$, by [Dix77], 13.7.8. If Γ is amenable, then $C_r^*\Gamma = C^*\Gamma$, so that trivially the same elements are positive in $C^*\Gamma$. But this abstract argument fails in general for the full C*-algebra: it turns out that, if every finitely supported, positive definite function on Γ is a positive element in $C^*\Gamma$, then Γ is amenable (see [Val98]).

[1] Our convention for scalar products: linear on the right, anti-linear on the left

Lemma 3 *Let X be a locally compact, proper Γ-space; let π be a Γ-covariant representation of $C_0(X)$ on a Hilbert space $\mathcal{H}$. For any $\xi \in \pi(C_c(X))\mathcal{H}$, the function $\gamma \mapsto \langle \xi | \gamma\xi \rangle$ defines a positive element in $C^*\Gamma$.*

Before proving Lemma 3, we introduce some preliminaries that will be important for the sequel.

Consider the space $C_c(\Gamma, \mathcal{H})$ of $\mathcal{H}$-valued, finitely supported functions on Γ; it is viewed as a right $\mathbb{C}\Gamma$-module via ordinary convolution:

$$\xi a(\sigma) = \sum_{\gamma \in \Gamma} \xi(\gamma) a(\gamma^{-1}\sigma)$$

($\xi \in C_c(\Gamma, \mathcal{H})$, $a \in \mathbb{C}\Gamma$, $\sigma \in \Gamma$; notice that this does not involve the left action of Γ on $\mathcal{H}$). The module $C_c(\Gamma, \mathcal{H})$ carries a $\mathbb{C}\Gamma$-valued scalar product:

$$\langle \xi | \eta \rangle(\sigma) = \sum_{\gamma \in \Gamma} \langle \xi(\gamma) | \eta(\gamma\sigma) \rangle \tag{2.5}$$

(ξ, $\eta \in C_c(\Gamma, \mathcal{H})$, $\sigma \in \Gamma$) which is positive, i.e. $\langle \xi | \xi \rangle$ is a positive element in $C^*\Gamma$. To see the latter, choose a basis $(e_i)_{i \in I}$ of $\mathcal{H}$, and write $\xi(\sigma) = \sum_{i \in I} \xi_i(\sigma) e_i$ (so that ξ_i is in $\mathbb{C}\Gamma$). Then

$$\langle \xi | \xi \rangle(\sigma) \;=\; \sum_{\gamma \in \Gamma} \sum_{i \in I} \overline{\xi_i(\gamma)} \xi_i(\gamma\sigma) \;=\; \sum_{i \in I} \sum_{\gamma \in \Gamma} \xi_i^*(\gamma) \xi_i(\gamma^{-1}\sigma),$$

i.e. $\langle \xi | \xi \rangle \;=\; \sum_{i \in I} \xi_i^* \xi_i$, which is a positive element in $C^*\Gamma$. This also shows that the completion of $C_c(\Gamma, \mathcal{H})$ as a $C^*\Gamma$-module is the standard module

$$\mathcal{H} \otimes C^*\Gamma \;=\; \{(\xi_i)_{i \in I} : \xi_i \in C^*\Gamma,\ \sum_{i \in I} \xi_i^* \xi_i \textit{ converges in } C^*\Gamma\}.$$

Proof of Lemma 3: The proofs of this lemma and the following one, use ideas from [Pie00], Proposition 2.3.2.

Let $h \in C_c(X)$ be as in formula (2.1), i.e. $h \geq 0$ and $\sum_{\gamma \in \Gamma} \gamma h^2 \;=\; 1$. Let $\ell^2(\Gamma, \mathcal{H})$ be the Hilbert space of $\mathcal{H}$-valued square-summable functions on Γ. Define

$$S : \left\{ \begin{array}{lcl} \mathcal{H} & \to & \ell^2(\Gamma, \mathcal{H}) \\ \xi & \mapsto & (\gamma \mapsto \gamma\pi(\gamma^{-1}h)\xi) \end{array} \right.$$

Note that $S\xi$ really belongs to $\ell^2(\Gamma, \mathcal{H})$, since

$$\|S\xi\|^2 \;=\; \sum_{\gamma \in \Gamma} \|\gamma\pi(\gamma^{-1}h)\xi\|^2 \;=\; \sum_{\gamma \in \Gamma} \|\pi(\gamma^{-1}h)\xi\|^2$$

$$= \sum_{\gamma\in\Gamma} \langle \pi(\gamma^{-1}h)^2\xi|\xi\rangle \; = \; \|\xi\|^2.$$

It is clear that S maps $\pi(C_c(X))\mathcal{H}$ into $C_c(\Gamma,\mathcal{H})$, and it is easy to check that S commutes with the right Γ-actions on these two spaces.

Now define an operator

$$S^* : \begin{cases} C_c(\Gamma,\mathcal{H}) & \to \quad \pi(C_c(X))\mathcal{H} \\ \eta & \mapsto \quad \sum_{t\in\Gamma} t^{-1}\pi(h)(\eta(t)) \end{cases}$$

(the notation S^* will be justified in a minute). For the moment, notice that S^*S is the identity on $\pi(C_c(X))\mathcal{H}$. Now for $\xi \in \pi(C_c(X))\mathcal{H}$, $\eta \in C_c(\Gamma,\mathcal{H})$, $\gamma\in\Gamma$:

$$\langle \xi|S^*\eta\rangle(\gamma) \; = \; \langle \xi|\gamma S^*\eta\rangle \; = \; \sum_{t\in\Gamma} \langle \xi|\gamma t^{-1}\pi(h)(\eta(t))\rangle$$

$$= \sum_{s\in\Gamma} \langle \xi|s^{-1}\pi(h)(\eta(s\gamma))\rangle \; = \; \sum_{s\in\Gamma} \langle \pi(h)s\xi|\eta(s\gamma)\rangle$$

$$= \sum_{s\in\Gamma} \langle s\pi(s^{-1}h)\xi|\eta(s\gamma)\rangle \; = \; \sum_{s\in\Gamma} \langle S\xi(s)|\eta(s\gamma)\rangle \; = \; \langle S\xi|\eta\rangle(\gamma).$$

This proves that S and S^* are really adjoint to each other, with respect to the $\mathbb{C}\Gamma$-valued scalar products. In particular, for $\xi \in \pi(C_c(X))\mathcal{H}$, we have:

$$\langle \xi|\xi\rangle(\cdot) \; = \; \langle \xi|S^*S\xi\rangle(\cdot) \; = \; \langle S\xi|S\xi\rangle(\cdot).$$

But we have seen, just before this proof, that $\langle S\xi|S\xi\rangle(\cdot)$ is a positive element in $C^*\Gamma$. Therefore, so is $\langle \xi|\xi\rangle(\cdot)$. ☐

By Lemma 3, we may form the completion of $\pi(C_c(X))\mathcal{H}$ with respect to the scalar product given by formula (2.4), and get a Hilbert C*-module $\mathcal{E}$ over $C^*\Gamma$.

Lemma 4 *Let $T \in \mathcal{L}(\mathcal{H})$ be properly supported and such that $T\gamma = \gamma T$ for any $\gamma \in \Gamma$. Then T extends continuously to an operator $\mathcal{T} \in \mathcal{L}_{C^*\Gamma}(\mathcal{E})$.*

Proof: Let S, S^* be the operators appearing in the proof of Lemma 3, such that $S^*S = 1$. For $\eta \in C_c(\Gamma,\mathcal{H})$, $\gamma\in\Gamma$, an easy computation shows

$$STS^*\eta(\gamma) \; = \; \sum_{s\in\Gamma} \pi(h)T\pi(sh)s\eta(s^{-1}\gamma).$$

If we identify $C_c(\Gamma, \mathcal{H})$ with $\mathcal{H} \otimes \mathbb{C}\Gamma$, this can be re-written

$$STS^* = \sum_{s\in\Gamma} \pi(h)T\pi(sh)s \otimes \lambda_\Gamma(s)$$

where λ_Γ denotes the left regular representation of Γ. Since T is properly supported, this sum is finite. It is then clear that STS^* extends to a continuous $C^*\Gamma$-module map on $\mathcal{H} \otimes C^*\Gamma$. So $T = S^*(STS^*)S$ extends to a continuous operator $\mathcal{T} \in \mathcal{L}_{C^*\Gamma}(\mathcal{E})$. □

Lemmas 3 and 4 are much easier to prove when $C^*\Gamma$ is replaced by $C^*_r\Gamma$ (see lemma 6.1.3 in [Val]).

Let then X be a Γ-compact subset of $\underline{E}\Gamma$, and let $(\mathcal{H}, \pi, F)$ be an element of $KK_i^\Gamma(C_0(X), \mathbb{C})$, where F is Γ-equivariant and properly supported. As above, let $\mathcal{E}$ be the completion of $\pi(C_c(X))\mathcal{H}$ as a Hilbert C*-module over $C^*\Gamma$. The operator F satisfies the assumptions of Lemma 4, so it extends to an operator $\mathcal{F} \in \mathcal{L}_{C^*\Gamma}(\mathcal{E})$. Assume for a moment that the following result (proved in section 2.4) is true.

Proposition 2 *$\mathcal{F}^2 - 1$ is a compact operator on the C^*-module $\mathcal{E}$.*

This proposition says that the pair $(\mathcal{E}, \mathcal{F})$ defines an element in

$$KK_i(\mathbb{C}, C^*\Gamma) = K_i(C^*\Gamma).$$

In (3.10) of [BCH94], this element is called the Γ-*index* of F and denoted by $Index_\Gamma(F)$. It is immediate that the homomorphism

$$KK_i^\Gamma(C_0(X), \mathbb{C}) \to K_i(C^*\Gamma)$$

extends to the direct limit $RK_i^\Gamma(\underline{E}\Gamma)$ in Definition 3.

Definition 4 *The homomorphism*

$$\tilde{\mu}_i^\Gamma : \begin{cases} RK_i^\Gamma(\underline{E}\Gamma) \to K_i(C^*\Gamma) \\ (\mathcal{H}, \pi, F) \mapsto Index_\Gamma(F) \end{cases}$$

is the analytical assembly map.

We also *define* the analytical assembly map $\mu_i^\Gamma : RK_i^\Gamma(\underline{E}\Gamma) \to K_i(C^*_r\Gamma)$ by $\mu_i^\Gamma = (\lambda_\Gamma)_* \circ \tilde{\mu}_i^\Gamma$. We remark that the map $\tilde{\mu}_i^\Gamma$ is denoted by β in [Kas75], [Kas95], [Kas88], [Ros83]; and by A in [FRR95]. The map μ_i^Γ is denoted by A' in [FRR95]. The reader may also enjoy Gromov's point of view in [Gro93], where the analytical assembly map is denoted by K.

Example 1

Let H be a finite subgroup of Γ. Fix a finite-dimensional unitary representation ρ of H on a complex vector space V_ρ. Let us describe an element $\beta_{H,\rho}$ of $RK_0^\Gamma(\underline{E}\Gamma)$ as follows: take $X_H = \Gamma/H$, with action of Γ by left translations; X_H is a proper, Γ-compact space. In the picture of $\underline{E}\Gamma$ by probability measures on Γ, the space X_H identifies with the set of uniform probability measures on left cosets of H (in particular, for $H = 1$, the space X_H identifies with the set of Dirac measures). Take now the induced vector bundle $E_\rho = \Gamma \times_H V_\rho$ over X_H; denote by $\mathcal{H}$ the space of ℓ^2-sections of E_ρ; this is nothing but the space of the representation obtained by inducing up ρ from H to Γ. Consider the Γ-covariant representation π of $C_0(X_H)$ by pointwise multiplication of sections; since X_H is discrete and the fibers of E_ρ are of finite dimension, the representation π acts by compact operators on $\mathcal{H}$, so the triple $(\mathcal{H}, \pi, 0)$ defines an element $\beta_{H,\rho}$ of $RK_0^\Gamma(\underline{E}\Gamma)$. To describe the image $\mu_0^\Gamma(\beta_{H,\rho})$, we may clearly assume that ρ is irreducible. Then

$$\pi(C_c(X_H))\mathcal{H} = \mathbb{C}\Gamma \bigotimes_{\mathbb{C}H} V_\rho,$$

the space of finitely supported sections of E_ρ. Since ρ is irreducible, there exists a minimal projection $p_{H,\rho} \in \mathbb{C}H$ such that the $\mathbb{C}H$-modules V_ρ and $\mathbb{C}H \star \overline{p_{H,\rho}}$ are isomorphic. It is worthwhile to spell this out explicitly. Let ξ be a vector of norm 1 in V_ρ. Then

$$p_{H,\rho}(s) \;=\; \frac{\deg \rho}{|H|}\langle \rho(s)\xi|\xi\rangle$$

for $s \in H$ (it follows from the Schur orthogonality relations, see [Dix77] 14.3.3, that $p_{H,\rho}$ is indeed a projection in $\mathbb{C}H$). This projection $p_{H,\rho}$ is characterized by the fact that $\sigma(p_{H,\rho}) \;=\; 0$ for every irreducible representation σ of H which is not equivalent to $\overline{\rho}$, the contragredient representation of ρ; while $\overline{\rho}(p_{H,\rho})$ is the orthogonal projection on the one-dimensional subspace $\mathbb{C}\xi$ of $V_{\overline{\rho}}$. The map

$$\mathbb{C}H \star \overline{p_{H,\rho}} \to V_\rho : f \mapsto \rho(f)\xi$$

is an isomorphism intertwining the left regular representation λ_H (restricted to $\mathbb{C}H \star \overline{p_{H,\rho}}$) and ρ. Then, for $\eta \in \mathbb{C}\Gamma$, set

$$\check{\eta}(\gamma) = \eta(\gamma^{-1})$$

$(\gamma \in \Gamma)$. Recall that Γ acts on the right on $\mathcal{H}$ by

$$\eta \cdot \gamma = \gamma^{-1}\eta$$

while it acts on the right on $\mathbb{C}\Gamma$ by right convolution by Dirac measures. Using the fact that $\overline{\check{p}_{H,\rho}} = p_{H,\rho}^* = p_{H,\rho}$, it is easy to check that the map

$$\Psi : \mathbb{C}\Gamma \bigotimes_{\mathbb{C}H} (\mathbb{C}H \star \overline{p_{H,\rho}}) \to p_{H,\rho} \star \mathbb{C}\Gamma : a \otimes b \mapsto \check{b} \star \check{a}$$

is a map of right $\mathbb{C}\Gamma$-modules, which moreover preserves the $C^*\Gamma$-valued scalar products, i.e.

$$\langle a_1 \otimes b_1 | a_2 \otimes b_2 \rangle_{C^*\Gamma} = (\check{b_1} \star \check{a_1})^* \star (\check{b_2} \star \check{a_2})$$

$(a_1, a_2 \in \mathbb{C}\Gamma,\ b_1, b_2 \in \mathbb{C}H \star p_{H,\rho})$. So Ψ extends to an isometry of Hilbert $C^*\Gamma$-modules, from the completion of $\mathbb{C}\Gamma \bigotimes_{\mathbb{C}H} (\mathbb{C}H \star \overline{p_{H,\rho}})$ to the right ideal $p_{H,\rho}C^*\Gamma$ in $C^*\Gamma$. In more down-to-earth language, $\tilde{\mu}_0^\Gamma(\beta_{H,\rho})$ is just the class $[p_{H,\rho}]$ of the projection $p_{H,\rho}$ in $K_0(C^*\Gamma)$.

In particular for $H = 1$ and $\rho = 1_H$ the trivial one-dimensional representation, we have $\tilde{\mu}_0^\Gamma(\beta_{H,1_H}) = [1]$, the K-theory class of the unit of $C^*\Gamma$. This element has infinite order in $K_0(C^*\Gamma)$, since it maps to 1 under the trivial one-dimensional representation of Γ. Its image $\mu_0^\Gamma(\beta_{H,1_H}) = [1]$ in $K_0(C_r^*\Gamma)$ also has infinite order, since the canonical trace on $C_r^*\Gamma$ maps [1] to 1. This can be rephrased by saying that there is a commutative diagram

$$\begin{array}{ccc} RK_0^\Gamma(\underline{E}\Gamma) & \stackrel{\mu_0^\Gamma}{\to} & K_0(C_r^*\Gamma) \\ \uparrow & & \uparrow \\ H_0(\Gamma, \mathbb{Z}) & = & \mathbb{Z} \end{array}$$

where vertical maps are monomorphisms.

Suppose that Γ is torsion-free; then $RK_0(\underline{E}\Gamma) = RK_0^\Gamma(E\Gamma) = RK_0(B\Gamma)$; also, as mentioned in the Introduction, one has $RK_0(B\Gamma) \bigotimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{n=0}^\infty H_{2n}(\Gamma, \mathbb{C})$. So the preceding diagram shows that the Baum-Connes conjecture holds on the 0-dimensional part of $RK_0^\Gamma(E\Gamma)$.

Suppose at the other extreme that Γ is a finite group. In this case both $K_0^\Gamma(pt)$ and $K_0(C_r^*\Gamma)$ are abstractly isomorphic to the additive group of the representation ring $R(\Gamma)$, i.e. to the free abelian group on the set $\hat{\Gamma}$ of isomorphism classes of irreducible representations of Γ. In the above construction, take $H = \Gamma$ and let ρ run along $\hat{\Gamma}$. Then the $\beta_{\Gamma,\rho}$'s run along a set of generators of $K_0^\Gamma(pt)$ and the $[p_{\Gamma,\rho}]$'s run along a set of generators of $K_0(C_r^*\Gamma)$. On the other hand:

$$RK_1^\Gamma(\underline{E}\Gamma) = 0 = K_1(C_r^*\Gamma)$$

for Γ finite. In other words, we have checked that the Baum-Connes conjecture holds for finite groups.

2.4 Equivalence of two definitions

Here we shall simultaneously prove Proposition 2 and give an alternative construction of the analytical assembly map. First, we take a closer look at the situation described in Lemma 2, i.e. a proper Γ-space X and a Hilbert space $\mathcal{H}$ endowed with a covariant representation π of $C_0(X)$. Next lemma complements Lemma 2.

Lemma 5 *Fix a real-valued $f \in C_c(X)$. For $T \in \mathcal{L}(\mathcal{H})$, set*

$$A_f(T) = \sum_{\gamma\in\Gamma} \gamma\pi(f)T\pi(f)\gamma^{-1}$$

as in Lemma 2. Then

1. *$A_f(T)$ extends continuously to an operator $\mathcal{A}_f(T) \in \mathcal{L}_{C^*\Gamma}(\mathcal{E})$.*

2. *If T is a compact operator on $\mathcal{H}$, then $\mathcal{A}_f(T)$ is a compact operator on $\mathcal{E}$.*

Proof:

1. Lemma 2 shows that $A_f(T)$ is Γ-equivariant and properly supported. So Lemma 4 applies.

2. Consider again the function h of formula (2.1), and the operators S, S^* appearing in the proof of Lemma 3. In the proof of Lemma 4, we saw that $STS^* = \sum_{s\in\Gamma} \pi(h)T\pi(sh)s \otimes \lambda_\Gamma(s)$, provided T is Γ-equivariant, properly supported on $\mathcal{H}$. So replacing T by $A_f(T)$, we get

$$SA_f(T)S^* = \sum_{s\in\Gamma}\sum_{\gamma\in\Gamma} \pi(h.\gamma f)\gamma T\pi(f.\gamma^{-1}sh)\gamma^{-1}s \otimes \lambda_\Gamma(s).$$

This double sum is finite, as a consequence of properness of the Γ-action on X. If T is compact on $\mathcal{H}$, then $SA_f(T)S^*$ extends to an operator on $\mathcal{H} \otimes C^*\Gamma$, which belongs to $\mathcal{K}(\mathcal{H}) \otimes C^*\Gamma = \mathcal{K}(\mathcal{H} \otimes C^*\Gamma)$. So $A_f(T) = S^*(SA_f(T)S^*)S$ extends to a compact operator $\mathcal{A}_f(T)$ on $\mathcal{E}$.

□

We observe now that, if X is a proper, Γ-compact space, then there is a canonical element $[\mathcal{L}_X]$ in the K_0-group of the crossed product $C_0(X) \rtimes \Gamma$. Recall from [Ped79], 7.6.5, that $C_0(X) \rtimes \Gamma$ is the universal C*-completion of $C_c(X \times \Gamma)$, with convolution product given by

$$f_1 \star f_2(x, \sigma) = \sum_{\gamma \in \Gamma} f_1(x, \gamma) f_2(\gamma^{-1}x, \gamma^{-1}\sigma)$$

($f_1,\ f_2 \in C_c(X \times \Gamma)$, $x \in X$, $\sigma \in \Gamma$), and involution given by

$$f^*(x, \sigma) = \overline{f(\sigma^{-1}x, \sigma^{-1})}$$

($f \in C_c(X \times \Gamma)$, $x \in X$, $\sigma \in \Gamma$). Let us give the construction of this element $[\mathcal{L}_X]$ (reminiscent of the Mishchenko line bundle constructed in [Con94]). We first view $C_c(X)$ as a right $C_c(X \times \Gamma)$-module via the formula

$$\eta f(x) = \sum_{\gamma \in \Gamma} \eta(\gamma x) f(\gamma x, \gamma) \tag{2.6}$$

($\eta \in C_c(X)$, $f \in C_c(X \times \Gamma), x \in X$). Moreover the formula

$$\langle \xi | \eta \rangle (x, \gamma) = \overline{\xi(x)} \eta(\gamma^{-1}x) \tag{2.7}$$

($\xi,\ \eta \in C_c(X)$, $x \in X$, $\gamma \in \Gamma$) defines a $C_c(X \times \Gamma)$-valued scalar product on $C_c(X)$, and the following lemma shows that this scalar product is indeed positive-definite.

Lemma 6 *For $\eta \in C_c(X)$, the element $\langle \eta | \eta \rangle$ is positive in the crossed product C*-algebra $C_0(X) \rtimes \Gamma$.*

Proof: Since the Γ-action on X is proper, hence also amenable, by [Ana87] the full crossed product $C_0(X) \rtimes \Gamma$ is isomorphic to the reduced crossed product $C_0(X) \rtimes_r \Gamma$, which we now describe. Consider the trivial field of Hilbert spaces $X \times \ell^2(\Gamma)$ over X: its set of continuous sections is $C_0(X, \ell^2(\Gamma))$, a Hilbert C*-module over $C_0(X)$; we view sections in $C_0(X, \ell^2(\Gamma))$ as functions on $X \times \Gamma$. We define a *-representation of $C_c(X \times \Gamma)$ on $C_0(X, \ell^2(\Gamma))$ by

$$f\xi(x, \gamma) = \sum_{\sigma \in \Gamma} f(\gamma x, \sigma).\xi(x, \sigma^{-1}\gamma)$$

($f \in C_c(X \times \Gamma)$, $\xi \in C_0(X, \ell^2(\Gamma))$, $x \in X$, $\gamma \in \Gamma$). It is a known fact (see e.g. [Kas88], 3.7) that this *-homomorphism extends to a faithful *-representation of $C_0(X) \rtimes_r \Gamma$ on $C_0(X, \ell^2(\Gamma))$. For $\eta \in C_c(X)$, we then have to prove that $\langle \eta | \eta \rangle$ is a positive element in the C*-algebra $\mathcal{L}_{C_0(X)}(C_0(X, \ell^2(\Gamma)))$. By lemma 4.1 in [Lan95], this amounts to proving that, for every $\xi \in C_0(X, \ell^2(\Gamma))$, the scalar product $\langle \langle \eta | \eta \rangle \xi | \xi \rangle$ is a positive element in the C*-algebra $C_0(X)$, i.e. is a non-negative function on X. So, for $x \in X$, we compute:

$$\langle \langle \eta | \eta \rangle \xi | \xi \rangle (x) = \sum_{\gamma \in \Gamma} \overline{(\langle \eta | \eta \rangle \xi)(x, \gamma)} \xi(x, \gamma)$$

$$\begin{aligned}
&= \textstyle\sum_{\gamma \in \Gamma} \sum_{\sigma \in \Gamma} \overline{\langle \eta | \eta \rangle (\gamma x, \sigma) \xi(x, \sigma^{-1}\gamma)} \xi(x, \gamma) \\
&= \textstyle\sum_{\gamma \in \Gamma} \sum_{\sigma \in \Gamma} \eta(\gamma x) \xi(x, \gamma) \overline{\eta(\sigma^{-1}\gamma x) \xi(x, \sigma^{-1}\gamma)} \\
&= \textstyle\sum_{\gamma \in \Gamma} \eta(\gamma x) \xi(x, \gamma) \cdot \sum_{\sigma \in \Gamma} \overline{\eta(\sigma x) \xi(x, \gamma)} \\
&= \left| \textstyle\sum_{\gamma \in \Gamma} \eta(\gamma x) \xi(x, \gamma) \right|^2 \qquad \geq 0.
\end{aligned}$$

This concludes the proof of the lemma. □

Form the completion of $C_c(X)$ with respect to this scalar product, and get a C*-module $\mathcal{L}_X$ over $C_0(X) \rtimes \Gamma$.

Lemma 7 *The identity of $\mathcal{L}_X$ is a rank 1 operator, in the sense of C*-modules.*

Proof: For $\xi, \phi, \psi \in C_c(X)$ and $x \in X$, we get:

$$(\theta_{\phi, \psi}(\xi))(x) = (\phi \langle \psi | \xi \rangle)(x) = (\sum_{\gamma \in \Gamma} \phi(\gamma x) \overline{\psi(\gamma x)}).\xi(x).$$

Let $h \in C_c(X)$ be defined as in equation (2.1); since $\sum_{\gamma\in\Gamma} h^2(\gamma x) = 1$ for every $x \in X$, we see that $\theta_{h,h}$ is the identity on $\mathcal{L}_X$. $\square$

This lemma shows that $\mathcal{L}_X$ defines an element $[\mathcal{L}_X]$ of $K_0(C_0(X) \rtimes \Gamma)$. Notice that since $h\langle h|\xi\rangle = \xi$ for any $\xi \in C_c(X)$, we get $\langle h|h\rangle \star \langle h|\xi\rangle = \langle h|\xi\rangle$ and in particular $\langle h|h\rangle^2 = \langle h|h\rangle$, so that $p = \langle h|h\rangle$ is a projector in $C_c(X\times\Gamma)$. The map $p \star C_c(X \times \Gamma) \to C_c(X) : p \star f \mapsto h \cdot f$ is then well-defined, and identifies the right ideal $p.(C_0(X) \rtimes \Gamma)$ of $C_0(X) \rtimes \Gamma$ with the C*-module $\mathcal{L}_X$.

We shall need the *descent homomorphism*

$$j_\Gamma : KK_i^\Gamma(C_0(X), \mathbb{C}) \to KK_i(C_0(X) \rtimes \Gamma, C^*\Gamma),$$

also known as *induction to the crossed product*, see [Kas95], [Kas88]. For $x = (\mathcal{H}, \pi, F) \in KK_i^\Gamma(C_0(X), \mathbb{C})$, the element $j_\Gamma(x)$ is described as follows. Recall that we constructed the Hilbert $C^*\Gamma$-module $\tilde{\mathcal{H}} = \mathcal{H} \otimes C^*\Gamma$ as a completion of $C_c(\Gamma, \mathcal{H})$ for the $\mathbb{C}\Gamma$-valued scalar product in formula (2.5). There is an isometric left Γ-action on $C_c(\Gamma, \mathcal{H})$ given by

$$(\gamma \cdot \xi)(\sigma) = \gamma(\xi(\gamma^{-1}\sigma))$$

($\xi \in C_c(\Gamma, \mathcal{H})$; $\gamma, \sigma \in \Gamma$); there is also a Γ-covariant left $C_0(X)$-action $\tilde{\pi}$ on $C_c(\Gamma, \mathcal{H})$ given by

$$(\tilde{\pi}(f) \cdot \xi)(\sigma) = \pi(f)(\xi(\sigma))$$

($\xi \in C_c(\Gamma, \mathcal{H})$, $f \in C_c(X)$, $\sigma \in \Gamma$). In the identification of $C_c(\Gamma, \mathcal{H})$ with $\mathcal{H} \otimes \mathbb{C}\Gamma$, this can be re-written $\tilde{\pi}(f) = \pi(f) \otimes 1$, so that it extends to a left action of $C_0(X)$ on $\tilde{\mathcal{H}}$; being Γ-covariant, this action extends to the crossed product $C_0(X) \rtimes \Gamma$; the "integrated" form of that action is:

$$(\tilde{\pi}(\sum_{\gamma\in\Gamma} f_\gamma \cdot \gamma)\xi)(\sigma) = \sum_{\gamma\in\Gamma} \pi(f_\gamma)\gamma(\xi(\gamma^{-1}\sigma))$$

($f_\gamma \in C_0(X)$, $\xi \in C_c(\Gamma, \mathcal{H})$, $\sigma \in \Gamma$). Finally, define the operator $\tilde{F}$ on $C_c(\Gamma, \mathcal{H})$ by

$$(\tilde{F}\xi)(\gamma) = F(\xi(\gamma))$$

($\xi \in C_c(\Gamma, \mathcal{H})$, $\gamma \in \Gamma$). The operator $\tilde{F}$ extends to an operator on $\tilde{\mathcal{H}}$, and the triple $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{F})$ defines the Kasparov element $j_\Gamma(x) \in KK_i(C_0(X) \rtimes \Gamma, C^*\Gamma)$. Next we wish to perform the Kasparov product

$$[\mathcal{L}_X] \otimes_{C_0(X)\rtimes\Gamma} j_\Gamma(x) \in KK_i(\mathbb{C}, C^*\Gamma) = K_i(C^*\Gamma),$$

As above, we take $x = (\mathcal{H}, \pi, F)$ with F properly supported and Γ-equivariant; we denote by $\mathcal{E}$ the C*-module completion of $\pi(C_c(X))\mathcal{H}$, as in Lemma 1. We shall need the peculiar function $h \in C_c(X)$ appearing in equation (2.1).

Lemma 8 $[\mathcal{L}_X] \otimes_{C_0(X)\rtimes\Gamma} j_\Gamma(x)$ *can be represented by the pair* $(\mathcal{E}, \mathcal{A}_h(F))$.

Proof: Since $\mathcal{L}_X$ can be described as the right ideal $p.C_0(X) \rtimes \Gamma$, or alternatively by the *-homomorphism

$$\alpha : \mathbb{C} \to C_0(X) \rtimes \Gamma : 1 \mapsto p,$$

the Kasparov product $[\mathcal{L}_X] \bigotimes_{C_0(X)\rtimes\Gamma} j_\Gamma(x) = \alpha^*(j_\Gamma(x))$ is represented by the triple $(\tilde{\mathcal{H}}, \tilde{\pi}(p), \tilde{F})$, where the action of $\mathbb{C}$ on $\tilde{\mathcal{H}}$ is via the projector $\tilde{\pi}(p)$. Define then a map:

$$\beta : \begin{cases} \tilde{\pi}(p)C_c(\Gamma, \mathcal{H}) & \to & \pi(C_c(X))\mathcal{H} \\ \tilde{\pi}(p)\xi & \mapsto & \sum_{\gamma\in\Gamma} \gamma^{-1}\pi(h)(\xi(\gamma)). \end{cases}$$

For ξ, $\eta \in C_c(\Gamma, \mathcal{H})$, one checks using equations (2.5), (2.4) and (2.2) that the following relation holds in $\mathbb{C}\Gamma$:

$$\langle \tilde{\pi}(p)\xi | \tilde{\pi}(p)\eta \rangle = \langle \sum_{\gamma\in\Gamma} \gamma^{-1}\pi(h)(\xi(\gamma)) | \sum_{\gamma\in\Gamma} \gamma^{-1}\pi(h)(\eta(\gamma)) \rangle.$$

This shows that β is well-defined and extends to an isometric map of $C^*\Gamma$-modules between $\tilde{\pi}(p)\tilde{\mathcal{H}}$ and $\mathcal{E}$. Let us check that β is onto. For $f \in C_c(X)$, $\xi \in C_c(\Gamma, \mathcal{H})$, $\sigma \in \Gamma$, we have:

$$(\tilde{\pi}(\langle h|f \rangle)\xi)(\sigma) = \sum_{\gamma\in\Gamma} \pi(h)\pi(\gamma(f))\gamma(\xi(\gamma^{-1}\sigma)).$$

So that

$$\beta(\tilde{\pi}(p)(\tilde{\pi}(\langle h|f \rangle)\xi)) = \sum_{\gamma,\sigma\in\Gamma} \sigma^{-1}\pi(h^2)\gamma\pi(f)(\xi(\gamma^{-1}\sigma))$$

$$= \sum_{\gamma,\sigma\in\Gamma} \sigma^{-1}\pi(h^2)\sigma\gamma^{-1}\pi(f)(\xi(\gamma)) = \sum_{\gamma\in\Gamma} \gamma^{-1}\pi(f)(\xi(\gamma))$$

where the last equality again follows from (2). For $\eta \in \mathcal{H}$, define then a function $\overline{\eta} \in C_c(\Gamma, \mathcal{H})$ by

$$\overline{\eta} = \begin{cases} \eta & \text{if} \quad \gamma = 1 \\ 0 & \text{if} \quad \gamma \neq 1 \end{cases}$$

Then

$$\beta(\tilde{\pi}(p)(\tilde{\pi}(\langle h|f\rangle)\overline{\eta})) = \pi(f)\eta$$

shows that β is onto. We then transfer $\tilde{F}$ on $\mathcal{E}$ via β, and find by a simple computation

$$\beta\tilde{F}\beta^{-1} = \mathcal{A}_h(F).$$

□

Proof of Proposition 2: We have to show that $\mathcal{F}^2 - 1$ is a compact operator on the C*-module $\mathcal{E}$; on the other hand, we know by lemma 8 that $(\mathcal{A}_h(F))^2 - 1$ is a compact operator on $\mathcal{E}$. So, to prove Proposition 2, it is enough to prove that $\mathcal{F} - \mathcal{A}_h(F)$ is compact on $\mathcal{E}$. We use the fact that F is properly supported; so we find $g \in C_c(X)$ such that $\pi(g)F\pi(h) = F\pi(h)$. Let $f \in C_c(X)$ be real-valued and equal to 1 on $supp(g) \cup supp(h)$, so that $f.g = g$ and $f.h = h$. Then, on $\pi(C_c(X))\mathcal{H}$, one has, using (2.2):

$$\begin{aligned} F - A_h(F) &= \textstyle\sum_{\gamma\in\Gamma}(\gamma^{-1}F\pi(h^2)\gamma - \gamma^{-1}\pi(h)F\pi(h)\gamma) \\ &= \textstyle\sum_{\gamma\in\Gamma}\gamma^{-1}(F\pi(h) - \pi(h)F)\pi(h)\gamma \\ &= \textstyle\sum_{\gamma\in\Gamma}\gamma^{-1}(\pi(f)F\pi(h) - \pi(f)\pi(h)F)\pi(h)\pi(f)\gamma \\ &= \textstyle\sum_{\gamma\in\Gamma}\gamma^{-1}\pi(f)[F,\pi(h)]\pi(h)\pi(f)\gamma = A_f([F,\pi(h)]\pi(h)). \end{aligned}$$

In other words $\mathcal{F} - \mathcal{A}_h(F) = \mathcal{A}_f([F,\pi(h)]\pi(h))$; since $[F,\pi(h)]\pi(h)$ is a compact operator by assumption, lemma 5(2) applies to give the result. □

Notice that the above proof really identifies two Kasparov elements, hence gives an equivalent definition for the analytical assembly map:

Corollary 3 *Let X be a Γ-compact subset of $\underline{E}\Gamma$; for $x \in KK_i^{\Gamma}(C_0(X),\mathbb{C})$, one has:*

$$\tilde{\mu}_i^{\Gamma}(x) = [\mathcal{L}_X] \otimes_{C_0(X)\rtimes\Gamma} j_{\Gamma}(x).$$

Chapter 3

Naturality of the assembly map

3.1 The case of monomorphisms

Let $\alpha : \Gamma_1 \to \Gamma_2$ be a group monomorphism. We first describe how the Baum-Connes geometric group behaves under α_*. Identifying Γ_1 with $\alpha(\Gamma_1)$, we may assume that Γ_1 is a subgroup of Γ_2 and that α denotes inclusion.

Let X be a Γ_1-compact subset of $\underline{E}\Gamma_1$, and let $x = (\mathcal{H}, \pi, F)$ be an element of $KK_i^{\Gamma_1}(C_0(X), \mathbb{C})$, where F is Γ_1-equivariant and properly supported. Our first aim is to describe $\alpha_*(x) \in RK_i^{\Gamma_2}(\underline{E}\Gamma_2)$. Set

$$\tilde{X} = \Gamma_2 \times_{\Gamma_1} X,$$

the quotient of $\Gamma_2 \times X$ by the equivalence relation

$$(\gamma_2\gamma_1, x) \sim (\gamma_2, \gamma_1 x)$$

$(\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2, x \in X)$; the Γ_2-space $\tilde{X}$ is proper and Γ_2-compact. We denote by $[\gamma_2, x]$ the equivalence class of the pair (γ_2, x). Consider now the Hilbert space

$$\tilde{\mathcal{H}} = \{\xi : \Gamma_2 \to \mathcal{H} : \xi(\gamma_2\gamma_1) = \gamma_1^{-1}(\xi(\gamma_2)) \text{ for every } \gamma_1 \in \Gamma_1,\ \gamma_2 \in \Gamma_2,$$

$$\mathit{and} \sum_{\dot{\gamma} \in \Gamma_2/\Gamma_1} \|\xi(\dot{\gamma})\|^2 < \infty\},$$

with Γ_2-action by left translations (this is nothing but the representation induced from Γ_1 to Γ_2).

Notice now that, for $\gamma_2 \in \Gamma_2$, the map

$$\iota_{\gamma_2} : X \to \tilde{X} : x \mapsto [\gamma_2, x]$$

is proper and injective; so for $f \in C_0(\tilde{X})$, the function $f \circ \iota_{\gamma_2}$ belongs to $C_0(X)$. We then define an operator $\tilde{\pi}(f)$ on $\tilde{\mathcal{H}}$ by

$$(\tilde{\pi}(f)\xi)(\gamma_2) = \pi(f \circ \iota_{\gamma_2})(\xi(\gamma_2))$$

($\xi \in \mathcal{H}, \gamma_2 \in \Gamma_2$). It is easy to check that $f \to \tilde{\pi}(f)$ is a Γ_2-covariant representation of $C_0(\tilde{X})$ on $\tilde{\mathcal{H}}$. Define now a Γ_2-equivariant operator $\tilde{F}$ on $\tilde{\mathcal{H}}$ by

$$(\tilde{F}\xi)(\gamma_2) = F(\xi(\gamma_2)).$$

Lemma 9 *The triple $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{F})$ defines an element $\tilde{x} \in KK_i^{\Gamma_2}(C_0(\tilde{X}), \mathbb{C})$, with $\tilde{F}$ properly supported.*

Proof: We may consider $\tilde{\mathcal{H}}$ as the space of ℓ^2-sections of the Hilbert space bundle $\Gamma_2 \times_{\Gamma_1} \mathcal{H}$ over Γ_2/Γ_1; the choice of a transversal for Γ_2/Γ_1 allows us to trivialize this bundle and to identify (in a *non*-Γ_2-equivariant way!) $\tilde{\mathcal{H}}$ with $\ell^2(\Gamma_2/\Gamma_1) \bigotimes \mathcal{H}$; similarly, the same choice of a transversal identifies topologically $\tilde{X}$ with $(\Gamma_2/\Gamma_1) \times X$; under these identifications, $\tilde{F}$ is realized as $1 \bigotimes F$ and, for $f_1 \in C_0(\Gamma_2/\Gamma_1)$, $f_2 \in C_0(X)$, the operator $\tilde{\pi}(f_1 \bigotimes f_2)$ is realized as $M_{f_1} \bigotimes \pi(f_2)$, where M_{f_1} is the operator of multiplication by f_1 on $\ell^2(\Gamma_2/\Gamma_1)$. Since M_{f_1} is a compact operator, it is clear from this realization that $[\tilde{\pi}(f), \tilde{F}]$ and $\tilde{\pi}(f)(\tilde{F}^2 - 1)$ are compact operators for every $f \in C_0(\tilde{Y})$. It is also clear in this realization that $\tilde{F}$ is properly supported. $\square$

Since $\tilde{X}$ is a proper Γ_2-space, by the universal property of $\underline{E}\Gamma_2$, there exists a Γ_2-equivariant continuous map $\phi : \tilde{X} \to \underline{E}\Gamma_2$. The following lemma is perfectly general.

Lemma 10 *Let $\tilde{X}$ be a proper, Γ_2-compact Γ_2-space. Let $\phi : \tilde{X} \to \underline{E}\Gamma_2$ be a continuous, Γ_2-equivariant map. Then ϕ is proper.*

Proof: Let C be a compact subset of $\underline{E}\Gamma_2$: we have to show that $\phi^{-1}(C)$ is compact in $\tilde{X}$. Since $\tilde{X}$ is Γ_2-compact, there exists a compact subset K such that $\tilde{X} = \Gamma_2 \cdot K$. By properness of the Γ_2-action on $\underline{E}\Gamma_2$, the set

$$F = \{\gamma \in \Gamma_2 : C \cap \gamma\phi(K) \neq \emptyset\}$$

is finite. Fix $x \in \phi^{-1}(C)$; let $\gamma \in \Gamma_2$ be such that $\gamma^{-1}x \in K$; then

$$\phi(\gamma^{-1}x) = \gamma^{-1}\phi(x) \in \phi(K) \cap \gamma^{-1}C.$$

This implies $\gamma \in F$, i.e. $x \in F \cdot K$. We have proved $\phi^{-1}(C) \subset F \cdot K$, which shows that $\phi^{-1}(C)$ is compact. $\square$

Set $Y = \phi(\tilde{X})$. Since the map $\phi : \tilde{X} \to Y$ is continuous, proper, Γ_2-equivariant, by functoriality in equivariant K-homology we have $\phi_*(\tilde{x}) = (\tilde{\mathcal{H}}, \tilde{\pi} \circ \phi^*, \tilde{F}) \in KK_i(C_0(Y), \mathbb{C})$, and we set

$$\phi_*(\tilde{x}) = \alpha_*(x).$$

Remark By restricting the Γ_2-action on $\underline{E}\Gamma_2$ to Γ_1, we get a universal proper Γ_1-space (see [BCH94], (1.9)). So we could take $\underline{E}\Gamma_1 = \underline{E}\Gamma_2$; in this case, we could take for Y the Γ_2-saturation of X in $\underline{E}\Gamma_2$, and for ϕ the map:

$$\phi : \tilde{X} \to Y : [\gamma_2, x] \mapsto \gamma_2 x.$$

However, we will not assume $\underline{E}\Gamma_1 = \underline{E}\Gamma_2$ in order to get more flexibility (e.g. in Chapter 4, we deal with the case $\Gamma_1 = \mathbb{Z}$, and we prefer to take $\underline{E}\Gamma_1 = \mathbb{R}$ rather than $\underline{E}\Gamma_1 = \underline{E}\Gamma_2$).

To describe $\tilde{\mu}_i^{\Gamma_2}(\alpha_*(x))$, we have to take the completion $\tilde{\mathcal{E}}$ of $\tilde{\pi}(\phi^*(C_c(Y)))\tilde{\mathcal{H}}$ with respect to the $\mathbb{C}\Gamma_2$-valued scalar product

$$\langle \xi_1 | \xi_2 \rangle (\gamma_2) = \langle \xi_1 | \gamma_2 \xi_2 \rangle$$

($\xi_1, \xi_2 \in \tilde{\pi}(\phi^*(C_c(Y)))\mathcal{H}$, $\gamma_2 \in \Gamma_2$), and with the operator $\tilde{\mathcal{F}}$ that extends $\tilde{F}$ continuously (see Lemma 4).

On the other hand, to describe $\alpha_*(\tilde{\mu}_i^{\Gamma_1}(x))$, we first consider (as in Lemma 3) the completion $\mathcal{E}$ of $\pi(C_c(X))\mathcal{H}$ with respect to the $\mathbb{C}\Gamma_1$-valued scalar product

$$\langle \eta_1 | \eta_2 \rangle (\gamma_1) = \langle \eta_1 | \gamma_1 \eta_2 \rangle$$

($\eta_1, \eta_2 \in \mathcal{H}$, $\gamma_1 \in \Gamma_1$). The element $\alpha_*(\tilde{\mu}_i^{\Gamma_1}(x))$ is then described by the pair $(\mathcal{E} \bigotimes_{C^*\Gamma_1} C^*\Gamma_2, \mathcal{F} \bigotimes 1)$, where $C^*\Gamma_2$ is viewed as a left module over $C^*\Gamma_1$, and as a right module over itself. The $C^*\Gamma_2$-valued scalar product on $\mathcal{E} \bigotimes_{C^*\Gamma_1} C^*\Gamma_2$ is given by

$$\langle \eta_1 \otimes b_1 | \eta_2 \otimes b_2 \rangle_{C^*\Gamma_2} = b_1^* \langle \eta_1 | \eta_2 \rangle_{C^*\Gamma_1} b_2$$

($\eta_1, \eta_2 \in \mathcal{E}$, $b_1, b_2 \in C^*\Gamma_2$; see [Lan95], 4.5).

Proof of Theorem 1, case of a monomorphism

We have to show $\alpha_*(\tilde{\mu}_i^{\Gamma_1}(x)) = \tilde{\mu}_i^{\Gamma_2}(\alpha_*(x))$ in $KK_i(\mathbb{C}, C^*\Gamma_2) = K_i(C^*\Gamma_2)$. Following a similar construction due to Rieffel ([Rie74], p. 228), we define a map

$$\Psi : \begin{cases} \pi(C_c(X))\mathcal{H} \bigotimes_{\mathbb{C}\Gamma_1} \mathbb{C}\Gamma_2 & \to & \tilde{\mathcal{H}} \\ \eta \otimes b & \mapsto & (\gamma_2 \mapsto \sum_{\gamma_1 \in \Gamma_1} b(\gamma_1^{-1}\gamma_2^{-1})\gamma_1 \eta \end{cases}$$

($\eta \in \pi(C_c(X))\mathcal{H}$, $b \in \mathbb{C}\Gamma_2$, $\gamma_2 \in \Gamma_2$). It follows from Theorem 5.12 in [Rie74] that Ψ is well-defined, i.e., for $\gamma_1 \in \Gamma_1$:

$$\Psi(\eta\gamma_1 \otimes b) = \Psi(\eta \otimes \gamma_1 b).$$

It is readily checked that Ψ is a $\mathbb{C}\Gamma_2$-module map. Now, for $\gamma, \gamma_2 \in \Gamma_2$, one has:

$$\Psi(\eta \otimes \gamma)(\gamma_2) = \begin{cases} \gamma_2^{-1}\gamma^{-1}\eta & if \;\; \gamma.\gamma_2 \in \Gamma_1 \\ 0 & otherwise \end{cases}$$

It also follows from this that the range of Ψ is contained in $\tilde{\pi}(\phi^*(C_c(Y)))\tilde{\mathcal{H}}$; to see it, suppose that $\eta = \pi(f)\xi$ (with $f \in C_c(X)$, $\xi \in \mathcal{H}$), and choose $g \in C_c(Y)$ such that $g = 1$ on $\gamma^{-1}(\phi(\iota_1(supp\, f)))$. Then

$$\Psi(\eta \otimes \gamma) = \tilde{\pi}(\phi^* g)\Psi(\eta \otimes \gamma) \in \tilde{\pi}(\phi^*(C_c(Y)))\tilde{\mathcal{H}}$$

Let $\{s_i\}_{i \in I}$ be a transversal for Γ_2/Γ_1. The inverse map

$$\Psi^{-1} : \tilde{\pi}(\phi^*(C_c(Y)))\tilde{\mathcal{H}} \to \pi(C_c(X))\mathcal{H} \otimes_{\mathbb{C}\Gamma_1} \mathbb{C}\Gamma_2$$

is given by:

$$\Psi^{-1}(\tilde{\xi}) = \sum_{i \in I} \tilde{\xi}(s_i) \otimes s_i^{-1} \tag{3.1}$$

($\tilde{\xi} \in \tilde{\pi}(\phi^*(C_c(Y)))\tilde{\mathcal{H}}$); using the fact that there are finitely many s_i's such that $\iota_{s_i}(X)$ meets a given compact subset of $\tilde{X}$, one sees that the sum in (8) is actually a finite sum. Next, for $\eta, \eta\prime \in \pi(C_c(X))\mathcal{H}$ and $\gamma, \gamma\prime, \gamma_2 \in \Gamma_2$, we compute:

$$\begin{aligned} & \langle \Psi(\eta \otimes \gamma) | \Psi(\eta\prime \otimes \gamma\prime) \rangle (\gamma_2) = \langle \Psi(\eta \otimes \gamma) | \gamma_2 \Psi(\eta\prime \otimes \gamma\prime) \rangle \\ = & \sum_{\dot{\sigma} \in \Gamma_2/\Gamma_1} \langle \Psi(\eta \otimes \gamma) | \Psi(\eta\prime \otimes \gamma\prime)(\gamma_2^{-1}\sigma) \rangle \\ = & \sum_{\substack{\dot{\sigma} \in \Gamma_2/\Gamma_1 \\ \gamma\sigma \in \Gamma_1 \\ \gamma\prime\gamma_2^{-1}\sigma \in \Gamma_1}} \langle \sigma^{-1}\gamma\eta | \sigma^{-1}\gamma_2\gamma\prime\eta\prime \rangle \end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} \langle \eta | \gamma\gamma_2\gamma'^{-1}\eta' \rangle & if \;\; \gamma\gamma_2\gamma'^{-1} \in \Gamma_1 \\ 0 & otherwise \end{cases} \\
&= \langle \eta | \eta' \rangle_{C^*\Gamma_1}(\gamma\gamma_2\gamma'^{-1}) \\
&= (\gamma^{-1}\langle \eta | \eta' \rangle_{C^*\Gamma_1}\gamma')(\gamma_2) \\
&= \langle \eta \otimes \gamma | \eta' \otimes \gamma' \rangle_{C^*\Gamma_2}(\gamma_2).
\end{aligned}
$$

This means that Ψ extends to a unitary isomorphism of $C^*\Gamma_2$-modules between $\mathcal{E} \otimes_{C^*\Gamma_1} C^*\Gamma_2$ and $\tilde{\mathcal{E}}$. Moreover, an easy computation gives

$$\Psi(\mathcal{F} \bigotimes 1)\Psi^{-1} = \tilde{\mathcal{F}},$$

which concludes the proof. □

3.2 The case of epimorphisms

Let $\alpha : \Gamma_1 \to \Gamma_2$ be a group epimorphism; set $N = \ker \alpha$. By identifying Γ_1/N with Γ_2, we may assume that α is the quotient-map. We also denote by α the induced algebra homomorphisms $\mathbb{C}\Gamma_1 \twoheadrightarrow \mathbb{C}\Gamma_2$ and $C^*\Gamma_1 \twoheadrightarrow C^*\Gamma_2$.

We first describe how the left-hand side of the assembly map behaves under α_*; we were inspired by Kasparov's descent homomorphism in Theorem 3.4 of [Kas88]. Let then X be a Γ_1-compact subset of $\underline{E}\Gamma_1$, and $x = (\mathcal{H}, \pi, F)$ be an element of $KK_i^{\Gamma_1}(C_0(X), \mathbb{C})$, with F properly supported and Γ_1-equivariant. Set $\tilde{X} = N\backslash X$: this is a proper, Γ_2-compact Γ_2-space.

Consider on $\pi(C_c(X))\mathcal{H}$ the scalar product

$$\ll \xi | \eta \gg = \sum_{n \in N} \langle \xi | n\eta \rangle$$

($\xi, \eta \in \pi(C_c(X))\mathcal{H}$); it is non-negative, since by Lemma 3 the function $n \mapsto \langle \xi | n\xi \rangle$ defines a positive element in C^*N; so by applying the trivial representation we get $\ll \xi | \xi \gg \geq 0$. Let $\tilde{\mathcal{H}}$ be the separation-completion of $\pi(C_c(X))\mathcal{H}$ for this scalar product. The natural action of Γ_1 on $\pi(C_c(X))\mathcal{H}$ is isometric (because N is normal in Γ_1); hence it extends to a unitary representation of Γ_1 on $\tilde{\mathcal{H}}$ which is trivial on N, hence factors through a unitary representation of Γ_2. Let T be a bounded operator on $\mathcal{H}$, preserving $\pi(C_c(X))\mathcal{H}$ and N-equivariant; it follows from Lemma 4 that, for some $K > 0$ and every $\xi \in \pi(C_c(X))\mathcal{H}$, the element

$$K\langle \xi | \xi \rangle(\cdot) - \langle T\xi | T\xi \rangle(\cdot)$$

is a positive element in C^*N. Summing over N, this implies

$$\ll T\xi|T\xi \gg \leq K \ll \xi|\xi \gg,$$

so that T extends to a bounded operator $\tilde{T}$ on $\tilde{\mathcal{H}}$. In particular, F provides a Γ_2-equivariant operator $\tilde{F}$ on $\tilde{\mathcal{H}}$.

Any function on $\tilde{X}$ can be lifted to an N-invariant function on X. Viewing in this way $C_0(\tilde{X})$ as an algebra of multipliers of $C_0(X)$, and extending the representation π to multipliers, as in [Ped79] 3.12.10, we get an algebra of operators on $\mathcal{H}$ that preserve $\pi(C_c(X))\mathcal{H}$ and commute with N, so that the preceding observation provides a Γ_2-covariant representation $\tilde{\pi}$ of $C_0(\tilde{X})$ on $\tilde{\mathcal{H}}$.

Lemma 11 *The triple $(\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{F})$ defines an element $\tilde{x} \in KK_i^{\Gamma_2}(C_0(\tilde{X}), \mathbb{C})$, with $\tilde{F}$ properly supported.*

Proof: We consider the C^*N-module $\mathcal{E}$ obtained by completing $\pi(C_c(X))\mathcal{H}$ with respect to the scalar product given by equation (2.4) (with Γ replaced by N); it is clear from the definitions that

$$\tilde{\mathcal{H}} = \mathcal{E} \otimes_{C^*N} \mathbb{C},$$

where $\mathbb{C}$ is a left C^*N-module via the augmentation map (i.e. the character coming from the trivial representation of N). For $T \in \mathcal{L}(\mathcal{H})$ preserving $\pi(C_c(X))\mathcal{H}$ and N-equivariant, one has, using the notation from Lemma 4:

$$\tilde{T} = \mathcal{T} \otimes 1.$$

We have to prove that, for every $\tilde{h} \in C_0(\tilde{X})$, the operators $[\tilde{\pi}(\tilde{h}), \tilde{F}]$ and $\tilde{\pi}(\tilde{h})(\tilde{F}^2 - 1)$ are compact. By linearity and density, we may assume that $\tilde{h}$ belongs to $C_c(\tilde{X})$ and that $\tilde{h}$ is real-valued. Find $h \in C_c(X)$, real-valued, such that

$$\tilde{h} = \sum_{n \in N} n(h). \tag{3.2}$$

As in the proof of Proposition 2, let $f \in C_c(X)$ be a plateau function such that $f.h = h$ and $\pi(f)F\pi(h) = F\pi(h)$. Then

$$\begin{aligned}
[\pi(\tilde{h}), F] &= \sum_{n \in N} n[\pi(h), F]n^{-1} \\
&= \sum_{n \in N} n\pi(f)[\pi(h), F]\pi(f)n^{-1} \\
&= A_f([\pi(h), F])
\end{aligned}$$

(notation as in Lemma 5). It is a consequence of Lemma 5(2) that the extension $\mathcal{A}_f([\pi(h), F])$ of $A_f([\pi(h), F])$ to $\mathcal{E}$, is a compact operator on $\mathcal{E}$. Since

$$[\tilde{\pi}(\tilde{h}), \tilde{F}] = \mathcal{A}_f([\pi(h), F]) \otimes 1,$$

it follows from 4.7 in [Lan95], that $[\tilde{\pi}(\tilde{h}), \tilde{F}]$ is compact on $\mathcal{H}$. A similar argument works for compactness of $\tilde{\pi}(\tilde{h})(\tilde{F}^2 - 1)$.
It remains to prove that $\tilde{F}$ is properly supported. Thus, fix $\tilde{h} \in C_c(\tilde{X})$, and choose $h \in C_c(X)$ as in (3.2). Since F is properly supported, one finds $g \in C_c(X)$ such that $\pi(g)F\pi(h) = F\pi(h)$ and $g.h = h$. As above, one computes:

$$F\pi(\tilde{h}) = A_g(F\pi(h)).$$

Let $\tilde{K}$ be the image of *supp* g in $\tilde{X}$; let $\tilde{f} \in C_c(\tilde{X})$ be equal to 1 on $\tilde{K}$, so that $\tilde{f}.g = g$ (where $\tilde{f}$ is now viewed as a multiplier of $C_0(X)$). Since $\pi(\tilde{f})$ is N-equivariant, one has

$$\pi(\tilde{f})A_g(F\pi(h)) = A_g(F\pi(h)),$$

i.e.

$$\tilde{\pi}(\tilde{f})\tilde{F}\tilde{\pi}(\tilde{h}) = \tilde{F}\tilde{\pi}(\tilde{h}).$$

This concludes the proof of the lemma. □

Since $\tilde{X}$ is a proper, Γ_2-compact Γ_2-space, there exists a Γ_2-equivariant continuous map $\phi : \tilde{X} \to \underline{E}\Gamma_2$, which is unique up to Γ_2-equivariant homotopy. By Lemma 10, this map ϕ is proper. Set $Y = \phi(\tilde{X})$; applying functoriality in equivariant K-homology we get $\phi_*(\tilde{x}) = (\tilde{\mathcal{H}}, \tilde{\pi} \circ \phi^*, \tilde{F}) \in KK_i^{\Gamma_2}(C_0(Y), \mathbb{C})$, and we set

$$\phi_*(\tilde{x}) = \alpha_*(x).$$

Remark: The assumption of Γ_1-compactness was used in the proof of Lemma 11 only to ensure that the space X is locally compact. So, if X is a locally compact, proper Γ_1-space, what lemma 11 actually does is constructing a homomorphism

$$\alpha_* : KK_i^{\Gamma_1}(C_0(X), \mathbb{C}) \to KK_i^{\Gamma_2}(C_0(N \backslash X), \mathbb{C}).$$

We record this for future reference.

Example 2

Let $\alpha : \Gamma \to \{1\}$ be the only homomorphism to the trivial group; then $\alpha : C^*\Gamma \to \mathbb{C}$ is the augmentation map, i.e. the character of the trivial representation. It follows from the proof of lemma 11 that the "geometric" map $\alpha_{*,g} : RK_0^\Gamma(\underline{E}\Gamma) \to K_0(pt) = \mathbb{Z}$ is actually given by

$$\alpha_{*,g} = \alpha_{*,a} \circ \tilde{\mu}_0^\Gamma,$$

where $\alpha_{*,a} : K_0(C^*\Gamma) \to K_0(\mathbb{C}) = \mathbb{Z}$ is the "analytical" map. This means that, in this case, Theorem 1 is essentially built in the definition of $\alpha_{*,g}$. *Assume that the Baum-Connes conjecture is true for the group* Γ. Then $\lambda_\Gamma : K_0(C^*\Gamma) \to K_0(C_r^*\Gamma)$ is onto, with a canonical splitting given by $\tilde{\mu}_0^\Gamma \circ (\mu_0^\Gamma)^{-1}$; loosely speaking, there is a canonical copy of $K_0(C_r^*\Gamma)$ inside $K_0(C^*\Gamma)$; this means that *any* representation of Γ, and in particular the trivial representation α, is defined on $K_0(C_r^*\Gamma)$. It seems to be an interesting *question* to define, without appealing to the Baum-Connes conjecture, a map

$$\alpha_? : K_0(C_r^*\Gamma) \to \mathbb{C}$$

such that, on the image of $\tilde{\mu}_0^\Gamma$, one has:

$$\alpha_{*,a} = \alpha_? \circ (\lambda_\Gamma)_*.$$

If Γ is torsion-free, then the answer is given by $\alpha_? = (\tau_\Gamma)_*$, the homomorphism $K_0(C_r^*\Gamma) \to \mathbb{C}$ associated with the canonical trace τ_Γ on $C_r^*\Gamma$. Indeed the equality of the two maps $\alpha_{*,a} \circ \tilde{\mu}_0^\Gamma$ and $(\tau_\Gamma)_* \circ \mu_0^\Gamma$, from $RK_0^\Gamma(E\Gamma)$ to $\mathbb{Z}$, is proved in Theorem 3.3.1 in [Pie00] (see also Theorem 6.14 of [Mis]), and can be seen as a version of Atiyah's L^2-index theorem.

Proof of Theorem 1, case of an epimorphism

We have to show $\alpha_*(\tilde{\mu}_i^{\Gamma_1}(x)) = \tilde{\mu}_i^{\Gamma_2}(\alpha_*(x))$ in $KK_i(\mathbb{C}, C^*\Gamma_2) = K_i(C^*\Gamma_2)$.

The Hilbert $C^*\Gamma_2$ module underlying $\alpha_*(\tilde{\mu}_i^{\Gamma_1}(x))$ is (as in section 3.1) the tensor product $\mathcal{E} \otimes_{C^*\Gamma_1} C^*\Gamma_2$, where $C^*\Gamma_2$ is viewed as a left module over $C^*\Gamma_1$ via α, and as a right module over itself. The $C^*\Gamma_2$-valued scalar product on $\mathcal{E} \otimes_{C^*\Gamma_1} C^*\Gamma_2$ is given by

$$\langle \eta_1 \otimes b_1 | \eta_2 \otimes b_2 \rangle_{C^*\Gamma_2} = b_1^* \alpha(\langle \eta_1 | \eta_2 \rangle_{C^*\Gamma_1}) b_2$$

($\eta_1, \eta_2 \in \mathcal{E}$, $b_1, b_2 \in C^*\Gamma_2$; see 4.5 in [Lan95]). The operator giving the K-theory element is $\mathcal{F} \otimes 1$.

On the other hand the $C^*\Gamma_2$-module underlying $\tilde{\mu}_i^{\Gamma_2}(\alpha_*(x))$ is defined as the completion $\tilde{\mathcal{E}}$ of $\tilde{\pi}(\phi^*(C_c(Y)))\tilde{\mathcal{H}}$ with respect to the $\mathbb{C}\Gamma_2$-scalar product

$$\ll \xi_1|\xi_2 \gg (\gamma_2) = \ll \xi_1|\gamma_2\xi_2 \gg$$

$(\xi_1, \xi_2 \in \tilde{\pi}(\phi^*(C_c(Y)))\tilde{\mathcal{H}}$, $\gamma_2 \in \Gamma_2)$. The operator $\tilde{\mathcal{F}}$ giving the K-theory element is the continuous extension of $\tilde{F}$ given by Lemma 4.

For $\eta \in \pi(C_c(X))\mathcal{H}$, we denote by $\tilde{\eta}$ the image of η in $\tilde{\mathcal{H}}$. We notice that $\tilde{\eta}$ really belongs to $\tilde{\pi}(\phi^*(C_c(Y)))\tilde{\mathcal{H}}$; indeed, for $\eta = \pi(f)\xi$, let K be the image of *supp* f in $\tilde{X}$, and let $g \in C_c(Y)$ be equal to 1 on $\phi(K)$. Then $\phi^*(g).f = f$ and

$$\tilde{\pi}(\phi^* g)\widetilde{\pi(f)\xi} = \widetilde{\pi(f)\xi} = \tilde{\eta}.$$

We define a map

$$\Psi : \begin{cases} \pi(C_c(X)\mathcal{H} \otimes_{\mathbb{C}\Gamma_1} \mathbb{C}\Gamma_2 & \to \quad \tilde{\pi}(\phi^*(C_c(Y)))\tilde{\mathcal{H}} \\ \eta \otimes \gamma_2 & \mapsto \quad \gamma_2^{-1}\tilde{\eta} \end{cases}$$

$(\eta \in \pi(C_c(X)\mathcal{H}$, $\gamma_2 \in \Gamma_2)$. Next, for $\gamma, \gamma', \gamma_2 \in \Gamma_2$, let $\gamma_1 \in \Gamma_1$ be any group element such that $\alpha(\gamma_1) = \gamma\gamma_2\gamma'^{-1}$. Then, for $\eta, \eta' \in \pi(C_c(X))\mathcal{H}$ we compute:

$$\begin{array}{rc} & \langle \eta \otimes \gamma|\eta' \otimes \gamma'\rangle(\gamma_2) = (\gamma^{-1}\alpha(\langle \eta|\eta'\rangle_{C^*\Gamma_1})\gamma')(\gamma_2) \\ = & \alpha(\langle \eta|\eta'\rangle_{C^*\Gamma_1})(\gamma\gamma_2\gamma'^{-1}) \\ = & \sum_{n\in N}\langle \eta|\eta'\rangle_{C^*\Gamma_1}(n\gamma_1) \\ = & \sum_{n\in N}\langle \eta|n\gamma_1\eta'\rangle \\ = & \ll \tilde{\eta}|\gamma\gamma_2\gamma'^{-1}\widetilde{\eta'} \gg \\ = & \ll \gamma^{-1}\tilde{\eta}|\gamma'^{-1}\widetilde{\eta'} \gg (\gamma_2) \\ = & \ll \Psi(\eta \otimes \gamma)|\Psi(\eta' \otimes \gamma') \gg (\gamma_2) \end{array}$$

This means that Ψ extends to a unitary isomorphism of $C^*\Gamma_2$-modules between $\mathcal{E} \otimes_{C^*\Gamma_1} C^*\Gamma_2$ and $\tilde{\mathcal{E}}$. Notice that, for $f \in C_c(X)$, $g \in C_c(Y)$, $\xi \in \mathcal{H}$, one has:

$$\tilde{\pi}(\phi^* g)\widetilde{\pi(f)\xi} = \pi(\widetilde{\phi^* g.f})\xi = \Psi(\pi(\phi^* g.f)\xi \otimes 1).$$

This shows that Ψ has dense image, so that Ψ is onto. Finally, an easy computation gives

$$\Psi(\mathcal{F} \otimes 1)\Psi^{-1} = \tilde{\mathcal{F}},$$

which concludes the proof of Theorem 1. □

3.3 An application to free actions

Let X be a locally compact, proper Γ-space; let α be the homomorphism from Γ to the trivial group. By the remark following Lemma 11, there is a homomorphism

$$\alpha_* : KK_i^\Gamma(C_0(X), \mathbb{C}) \to KK_i(C_0(\Gamma\backslash X), \mathbb{C}).$$

Corollary 4 *If Γ acts properly and freely on X, then α_* is an isomorphism.*

It is of course well-known that, in the case of a free action:

$$KK_i^\Gamma(C_0(X), \mathbb{C}) \simeq KK_i(C_0(\Gamma\backslash X), \mathbb{C})$$

(see [Rie82]). This is usually proved by identifying $KK_i^\Gamma(C_0(X), \mathbb{C})$ with $KK_i(C_0(X) \rtimes \Gamma, \mathbb{C})$, and then appealing to freeness of the action to conclude that $C_0(X) \rtimes \Gamma$ is Morita equivalent to $C_0(\Gamma\backslash X)$. We think that the interest of Corollary 4 is to provide an explicit and easily described isomorphism. Thanks are due to S. Echterhoff for his help in the following proof.

Proof of Corollary 4: We are going to show that α_* coincides with the isomorphism obtained via Morita equivalence, as indicated above. The imprimitivity bimodule realizing the Morita equivalence between $C_0(\Gamma\backslash X)$ and $C_0(X) \rtimes \Gamma$ is a suitable completion $\overline{C_c(X)}$ of $C_c(X)$, with the obvious left action of $C_c(\Gamma\backslash X)$, the right $C_c(X \times \Gamma)$-module structure given by formula (5), and the $C_c(X \times \Gamma)$-valued scalar product given by formula (6). The bimodule $\overline{C_c(X)}$ defines an element $[\overline{C_c(X)}]$ in the Kasparov group $KK_0(C_0(\Gamma\backslash X), C_0(X) \rtimes \Gamma)$.

Consider the Kasparov elements

$$x = (\mathcal{H}, \pi, F) \in KK_i^\Gamma(C_0(X), \mathbb{C})$$

and

$$\alpha_*(x) = (\tilde{\mathcal{H}}, \tilde{\pi}, \tilde{F}) \in KK_i(C_0(\Gamma\backslash X), \mathbb{C}),$$

as in Lemma 11. View x as an element of $KK_i(C_0(X) \rtimes \Gamma, \mathbb{C})$. We want to show that

$$[\overline{C_c(X)}] \otimes_{C_0(X)\rtimes\Gamma} x = \alpha_*(x). \tag{3.3}$$

For $f \in C_c(X)$ and $\xi \in \mathcal{H}$, denote by $[\pi(f)\xi]$ the image of $\pi(f)\xi$ in $\tilde{\mathcal{H}}$. An easy check shows that the map

$$C_c(X) \otimes \mathcal{H} \to \tilde{\mathcal{H}} : f \otimes \xi \mapsto [\pi(f)\xi]$$

extends to a unitary isomorphism between the Hilbert spaces $\overline{C_c(X)}\otimes_{C_0(X)\rtimes\Gamma}$ $\mathcal{H}$ and $\tilde{\mathcal{H}}$, which moreover intertwines the representations of $C_0(\Gamma\backslash X)$.

To show that the operator $\tilde{F}$ realizes the Kasparov product, we use the connection formalism of Connes and Skandalis [CS84]. For $f \in C_c(X)$, consider the map

$$\theta_f : \mathcal{H} \to \tilde{\mathcal{H}} : \xi \mapsto [\pi(f)\xi].$$

We have to show that $\tilde{F}$ is an F-connection, i.e. that the operator $\theta_f F - \tilde{F}\theta_f$ is a compact operator from $\mathcal{H}$ to $\tilde{\mathcal{H}}$. Set $T = [\pi(f), F]$, a compact operator on $\mathcal{H}$. Notice that, for $\xi \in \pi(C_c(X))\mathcal{H}$, one has

$$(\theta_f F - \tilde{F}\theta_f)\xi \;=\; [T\xi]. \tag{3.4}$$

Since F is properly supported, there exists $g \in C_c(X)$ such that $\pi(g)T\pi(g) = T$.

We claim that the map

$$\overline{\pi(g)\mathcal{H}} \to \tilde{\mathcal{H}} : \pi(g)\eta \mapsto [\pi(g)\eta]$$

is bounded. Indeed

$$\ll \pi(g)\eta|\pi(g)\eta \gg = \sum_{\gamma\in\Gamma} \langle \gamma\pi(g)\eta|\pi(g)\eta\rangle.$$

But the summation in the right hand side is over the finite set

$$F = \{\gamma \in \Gamma : \gamma(\mathit{supp}\, g) \,\cap\, \mathit{supp}\, g \neq \emptyset\},$$

so that, by the Cauchy-Schwarz inequality:

$$\ll \pi(g)\eta|\pi(g)\eta \gg \;\leq\; \mathit{card}\, F \cdot \|\pi(g)\eta\|^2,$$

which proves the claim.

Consider now the product of the following three operators:

$$\begin{array}{ll} \mathcal{H} \to \overline{\pi(g)\mathcal{H}} : & \xi \mapsto \pi(g)\xi; \\ \overline{\pi(g)\mathcal{H}} \to \overline{\pi(g)\mathcal{H}} & \zeta \mapsto T\zeta; \\ \overline{\pi(g)\mathcal{H}} \to \tilde{\mathcal{H}} & \pi(g)\eta \mapsto [\pi(g)\eta]. \end{array}$$

It follows from equation (3.4) that this product is exactly $\theta_f F - \tilde{F}\theta_f$. Since T is compact and the two other operators are bounded, it follows that $\theta_f F - \tilde{F}\theta_f$ is compact: this proves formula (3.3), and hence concludes the proof. $\square$

Chapter 4

Interlude: the group of integers

For the group $\mathbb{Z}$ of integers, we take $\underline{E}\mathbb{Z} = E\mathbb{Z} = \mathbb{R}$ (the real line) and $B\mathbb{Z} = S^1$ (the circle). By Fourier series (see [Ped79], 7.1.6), we identify $C^*\mathbb{Z}$ with $C(S^1)$.

Proposition 3 *For $i = 0, 1$, the map*

$$\mu_i^{\mathbb{Z}} : K_i^{\mathbb{Z}}(\mathbb{R}) = K_i(S^1) \to K_i(C^*\mathbb{Z})$$

is an isomorphism.

4.1 Proof, case $i = 0$

This is the easy case. We have $K_0(S^1) = \mathbb{Z}$, generated by the character of $C(S^1)$ given by evaluation at a given base-point (or, dually, generated by the inclusion of the base-point). When lifted to $\mathbb{R}$, this gives exactly the element $\beta_{\{0\}} \in K_0^{\mathbb{Z}}(\mathbb{R})$ described in Example 1. On the other hand, $K_0(C(S^1)) = \mathbb{Z}$, generated by the class of the constant function 1 (or, dually, generated by the trivial one-dimensional vector bundle over S^1). The result then follows from Example 1 above. $\square$

4.2 Proof, case $i = 1$

Here again, both groups $K_1(S^1)$ and $K_1(C(S^1))$ are isomorphic to $\mathbb{Z}$. To describe the generator of $K_1(S^1)$, we identify S^1 with $\mathbb{R}/\mathbb{Z}$ and consider the Hilbert space $L^2(S^1)$ with the trigonometric basis $(\exp(2\pi i n\theta))_{n\in\mathbb{Z}}$. Consider the operator F, diagonal in that basis, given by

$$F = diag(sign(n))_{n\in\mathbb{Z}}.$$

Let M be the representation of $C(S^1)$ by pointwise multiplication on $L^2(S^1)$. Then the triple $(L^2(S^1), M, F)$ defines the "Toeplitz" generator of $K_1(S^1) = KK_1(C(S^1), \mathbb{C})$.

To proceed, it will be convenient to work in the context of unbounded Kasparov elements (in the sense of Baaj-Julg [BJ83]). The unbounded picture of $(L^2(S^1), M, F)$ is $(L^2(S^1), M, D)$, where

$$D = \frac{1}{i} \cdot \frac{d}{d\theta};$$

indeed the phase of the operator D is just F. To say that $(L^2(S^1), M, D)$ is an unbounded Kasparov module means that:

1. D is a densely defined, self-adjoint operator;

2. $M(f)(1 + D^2)^{-1}$ is a compact operator for every $f \in C(S^1)$;

3. $[M(f), D]$ is a bounded operator for every f in a dense subalgebra of $C(S^1)$ (here $C^\infty(S^1)$).

Working with D has the advantage of being independant of the choice of any particular basis.

Via the isomorphism $K_1(S^1) \simeq K_1^{\mathbb{Z}}(\mathbb{R})$, the triple $(L^2(S^1), M, D)$ goes to the triple $(L^2(\mathbb{R}), \tilde{M}, \tilde{D})$, where $\tilde{M}$ is the $\mathbb{Z}$-covariant representation of $C_0(\mathbb{R})$ by pointwise multiplication on $L^2(\mathbb{R})$, and

$$\tilde{D} = \frac{1}{i} \cdot \frac{d}{dt}.$$

As a domain for $\tilde{D}$, we take the Schwartz space $\mathcal{S}(\mathbb{R})$.

We pause at this point to notice that, since the Fourier transform of the operator $\tilde{D}$ is the operator $\tilde{E}$ of multiplication by the dual variable λ (up to a factor 2π), the most natural bounded version of $\tilde{D}$ would be H, the convolution operator by the function whose Fourier transform is the sign function, which is nothing but the Hilbert transform on $L^2(\mathbb{R})$. One difficulty here is that H is *not* properly supported! (This can be seen as a good reason to consider the unbounded picture for Kasparov elements...)

To proceed, we shall appeal in a systematic way to the Fourier transform $\xi \mapsto \hat{\xi}$, where for $\xi \in \mathcal{S}(\mathbb{R})$ and $\lambda \in \mathbb{R}$:

$$\hat{\xi}(\lambda) = \int_{\mathbb{R}} \xi(t) \exp(-2\pi i \lambda t)\, dt.$$

Fourier transforming the Kasparov triple $(L^2(\mathbb{R}), \tilde{M}, \tilde{D})$ yields the Kasparov triple $(L^2(\mathbb{R}), \Lambda, \tilde{E})$ where Λ is the representation of $C_0(\mathbb{R})$ by convolution by Fourier transforms, and $n \in \mathbb{Z}$ now acts (on the left) by pointwise multiplication by the function $\lambda \mapsto \exp(-2\pi i n \lambda)$. The domain of $\tilde{E}$ is $\mathcal{S}(\mathbb{R})$. The $C^*\mathbb{Z}$-valued scalar product on $\mathcal{S}(\mathbb{R})$ is given by

$$\langle \xi_1 | \xi_2 \rangle (n) \;=\; \langle \xi_1 | n(\xi_2) \rangle \;=\; \int_{\mathbb{R}} \overline{\xi_1(\lambda)} \exp(-2\pi i n \lambda) \xi_2(\lambda)\, d\lambda$$

($\xi_1, \xi_2 \in \mathcal{S}(\mathbb{R})$, $n \in \mathbb{Z}$). Under the identification $C^*\mathbb{Z} \to C(S^1)$ given by Fourier series, i.e.

$$a \mapsto \;(\theta \mapsto \sum_{n\in\mathbb{Z}} a(n) \exp(2\pi i n \theta))$$

($a \in \mathbb{C}\mathbb{Z}$, $\theta \in S^1$), this becomes a $C(S^1)$-valued scalar product:

$$\langle \xi_1 | \xi_2 \rangle (\theta) = \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \overline{\xi_1(\lambda)} \exp(2\pi i n(\theta - \lambda)) \xi_2(\lambda)\, d\lambda.$$

Let $\mathcal{E}$ be the completion of $\mathcal{S}(\mathbb{R})$ for this $C(S^1)$-valued scalar product. Then $\mu_1^{\mathbb{Z}}(L^2(\mathbb{R}), \Lambda, \tilde{E})$ is described by $(\mathcal{E}, \tilde{E}) \in KK_1(\mathbb{C}, C(S^1))$. We now want to describe the Hilbert $C(S^1)$-module $\mathcal{E}$ as a continuous field of Hilbert spaces over S^1.

For that purpose, consider the Hilbert bundle over S^1 induced by the left regular representation of $\mathbb{Z}$; in other words, the total space of this bundle is $\mathbb{R} \times_{\mathbb{Z}} \ell^2(\mathbb{Z})$ and the space of continuous sections is

$$\mathcal{E}' = \{\eta : \mathbb{R} \to \ell^2(\mathbb{Z}),\ \text{continuous}\,,\ \eta(\lambda{+}1)_n = \eta(\lambda)_{n+1} \text{ for every } n \in \mathbb{Z},\, \lambda \in \mathbb{R}\};$$

a section is determined by its values for $-\frac{1}{2} \le \lambda < \frac{1}{2}$. Pointwise scalar product turns $\mathcal{E}'$ into a $C(S^1)$-Hilbert module:

$$\langle \eta_1 | \eta_2 \rangle (\theta) = \sum_{n\in\mathbb{Z}} \overline{\eta_1(\tilde{\theta})_n} \eta_2(\tilde{\theta})_n$$

($\eta_1, \eta_2 \in \mathcal{E}'$); here $\tilde{\theta}$ is any real number which lifts $\theta \in S^1$.

Consider the map $\Psi : \mathcal{S}(\mathbb{R}) \to \mathcal{E}'$ defined by

$$((\Psi(\xi))(\lambda))_n = \xi(\lambda + n)$$

($\xi \in \mathcal{S}(\mathbb{R})$, $\lambda \in \mathbb{R}$, $n \in \mathbb{Z}$). Let $n \in \mathbb{Z}$ act on $\mathcal{E}'$ (on the right) by pointwise multiplication by $\theta \mapsto \exp(2\pi i n\theta)$. It is clear that Ψ is a $\mathbb{C}\mathbb{Z}$-module map. Moreover Ψ is isometric with respect to $C(S^1)$-valued scalar products; indeed:

$$\begin{array}{rc} & \langle \Psi(\xi_1)|\Psi(\xi_2)\rangle(\theta) = \sum_{n\in\mathbb{Z}} \overline{\xi_1(\tilde{\theta}+n)}\xi_2(\tilde{\theta}+n) \\ = & \sum_{n\in\mathbb{Z}} (\overline{\xi_1}\xi_2)(\tilde{\theta}+n) \\ = & \sum_{n\in\mathbb{Z}} \widehat{\overline{\xi_1}\xi_2}(n) \exp(2\pi i n\theta) \\ = & \sum_{n\in\mathbb{Z}} \exp(2\pi i n\theta) \int_{\mathbb{R}} (\overline{\xi_1}\xi_2)(\lambda) \exp(-2\pi i n\lambda)\, d\lambda \\ = & \langle \xi_1|\xi_2\rangle(\theta), \end{array}$$

where the third equality follows from the Poisson summation formula. It is clear that Ψ has dense image since, on the suitable subspace of smooth sections of $\mathbb{R} \times_{\mathbb{Z}} \ell^2(\mathbb{Z})$ with rapid decay in the fibers, Ψ can be inverted by setting

$$(\Psi^{-1}(\eta))(\lambda) = \eta(\lambda)_0$$

($\eta \in \mathcal{E}'$, $\lambda \in \mathbb{R}$). So Ψ extends to an isometric isomorphism of Hilbert $C(S^1)$-modules. Set $\mathcal{F} = \Psi\tilde{E}\Psi^{-1}$; then

$$(\mathcal{F}\eta)(\lambda)_n = 2\pi(\lambda + n) \cdot \eta(\lambda)_n.$$

We still have to identify the unbounded Kasparov element $(\mathcal{E}', \mathcal{F}) \in KK_1(\mathbb{C}, C(S^1))$ with the generator of $K_1(C(S^1))$. For B a unital C*-algebra, Kucerovsky ([Kuc94], Chapter 6) has shown that, when $KK_1(\mathbb{C}, B)$ is described by means of unbounded elements, one may realize explicitly the isomorphism

$$KK_1(\mathbb{C}, B) \simeq K_1(B)$$

by means of the Cayley transform. If $(\mathcal{E}, \mathcal{D})$ is an unbounded element of $KK_1(\mathbb{C}, B)$, meaning that $\mathcal{E}$ is a Hilbert B-module and that $\mathcal{D}$ is an unbounded self-adjoint operator on $\mathcal{E}$ such that $(\mathcal{D}^2 + 1)^{-1}$ is compact in the sense of C*-modules, i.e. it belongs to the ideal $\mathcal{K}(\mathcal{E})$ of compact operators; then $U = \frac{i\mathcal{D}+1}{i\mathcal{D}-1}$ is a unitary operator equal to 1 modulo $\mathcal{K}(\mathcal{E})$. Hence U defines an element in $K_1(\mathcal{K}(\mathcal{E})) = K_1(B)$.

We now come back to the element $(\mathcal{E}', \mathcal{F}) \in KK_1(\mathbb{C}, C(S^1))$. The Cayley transform of $\mathcal{F}$ is the unitary operator U_1 on $\mathcal{E}'$ given by

$$\begin{aligned}(U_1\eta)(\lambda)_n &= \tfrac{2\pi i(\lambda+n)+1}{2\pi i(\lambda+n)-1} \cdot \eta(\lambda)_n \\ &= -(\exp(2i \arctan 2\pi(\lambda+n))) \cdot \eta(\lambda)_n\end{aligned}$$

$(\eta \in \mathcal{E}')$. Consider now the function

$$f_0(\lambda) = \begin{cases} -1 & if \quad \lambda \leq -\frac{1}{2} \\ 2\lambda & if \quad -\frac{1}{2} \leq \lambda \leq \frac{1}{2} \\ 1 & if \quad \frac{1}{2} \leq \lambda \end{cases}$$

and the family of unitary operators U_s $(s \in [0,1])$ on $\mathcal{E}'$ given by

$$(U_s\eta)(\lambda)_n = -(\exp(2is \arctan 2\pi(\lambda+n) + \pi i(1-s)f_0(\lambda+n))) \cdot \eta(\lambda)_n.$$

For every $s \in [0,1]$, the operator $U_s - 1$ belongs to $\mathcal{K}(\mathcal{E}')$, so this family defines a unique element in $K_1(C(S^1))$. Let us look at U_0; for $-\frac{1}{2} \leq \lambda < \frac{1}{2}$, we get

$$(U_0\eta)(\lambda)_n = \begin{cases} \eta(\lambda)_n & if\; n \neq 0 \\ -exp\, 2\pi i\lambda \cdot \eta(\lambda)_0 & if\; n = 0. \end{cases}$$

This makes it clear that the class of U_0 in $K_1(C(S^1))$ is nothing but the canonical generator (indeed, parametrizing S^1 with $[0,1[$ instead of $[-\frac{1}{2}, \frac{1}{2}[$, with $\mu = \lambda + \frac{1}{2}$ we get $-exp\, 2\pi i\lambda = exp\, 2\pi i\mu$). $\square$

Chapter 5

Lowest dimensional part of μ_1^Γ

Recall that Γ^{ab} denotes the abelianized group of Γ, and that $\kappa_\Gamma : \Gamma^{ab} \to K_1(C_r^*\Gamma)$ denotes the canonical homomorphism, induced by the map $\tilde{\kappa}_\Gamma : \Gamma \to K_1(C_r^*\Gamma)$ coming from the inclusion of Γ in the unitary group of $C_r^*\Gamma$.

In this chapter, we will construct a homomorphism

$$\beta_t : \Gamma^{ab} \to RK_1^\Gamma(\underline{E}\Gamma)$$

such that $\mu_1^\Gamma \circ \beta_t = \kappa_\Gamma$. This was previously done by Natsume [Nat88], under the assumption that Γ is torsion-free; this assumption was needed in order to be able to replace $RK_1^\Gamma(\underline{E}\Gamma)$ by $RK_1(B\Gamma)$.

5.1 Definition of β_t

Our definition of β_t will be in two steps: first, we define a (set-theoretic!) map $\tilde{\beta}_t : \Gamma \to RK_1^\Gamma(\underline{E}\Gamma)$; next, we prove that $\tilde{\beta}_t$ is a group homomorphism. Since the target group $RK_1^\Gamma(\underline{E}\Gamma)$ is abelian, $\tilde{\beta}_t$ factors through the desired homomorphism β_t.

To define $\tilde{\beta}_t$, we notice that every element $\gamma \in \Gamma$ defines a unique group homomorphism $\alpha_\gamma : \mathbb{Z} \to \Gamma$ by the requirement

$$\alpha_\gamma(1) = \gamma.$$

As in the proof of Proposition 3, let $x = (L^2(S^1), M, D)$ be the (unbounded picture of the) generator of $RK_1^{\mathbb{Z}}(\underline{E}\mathbb{Z}) = K_1(S^1) \simeq \mathbb{Z}$. By functoriality (see Theorem 1), we get a homomorphism $(\alpha_\gamma)_* : RK_1^{\mathbb{Z}}(\underline{E}\mathbb{Z}) \to RK_1^\Gamma(\underline{E}\Gamma)$ and we set:

$$\tilde{\beta}_t(\gamma) = (\alpha_\gamma)_*(x).$$

We begin by giving another description of this map $\tilde{\beta}_t$. Recall that $E\Gamma$ denotes the universal covering space of $B\Gamma$. Since the Γ-action on $E\Gamma$ is proper, there is a Γ-equivariant continuous map $\phi : E\Gamma \to \underline{E}\Gamma$, unique up to homotopy. Lemma 10 shows that, restricted to any Γ-compact subset $\tilde{X}$ of $E\Gamma$, the map ϕ is proper. By functoriality we have a map

$$\phi_* : KK_1^\Gamma(C_0(\tilde{X}), \mathbb{C}) \to RK_1^\Gamma(\underline{E}\Gamma).$$

Let X be the image of $\tilde{X}$ under the covering map $E\Gamma \to B\Gamma$. Since Γ acts freely on $\tilde{X}$, we have an identification:

$$KK_1^\Gamma(C_0(\tilde{X}), \mathbb{C}) \simeq KK_1(C_0(X), \mathbb{C}).$$

Fix $\gamma \in \Gamma$, and view γ as a loop in $B\Gamma$, i.e. as a continuous map $\gamma : S^1 \to B\Gamma$; set $X = \gamma(S^1)$. Then $\gamma_*(x) \in KK_1(C_0(X), \mathbb{C})$; applying ϕ_* we get an element in $RK_1^\Gamma(\underline{E}\Gamma)$. Thus we define:

$$\tilde{B}_t : \Gamma \to RK_1^\Gamma(\underline{E}\Gamma) : \gamma \mapsto \phi_*(\gamma_*(x)).$$

It is clear from the definition that $\tilde{B}_t$ factors through the canonical map $\phi_* : RK_1(B\Gamma) \to RK_1^\Gamma(\underline{E}\Gamma)$.

Lemma 12 *The maps $\tilde{\beta}_t$ and $\tilde{B}_t$ coincide. Moreover they vanish on torsion elements of Γ.*

Proof: Fix $\gamma \in \Gamma$; we distinguish 2 cases.

1) γ has infinite order in Γ. Then $\tilde{\beta}_t(\gamma) = (\alpha_\gamma)_*(x)$ is described by first considering $\tilde{X} = \Gamma \times_{\mathbb{Z}} \mathbb{R}$ with the Kasparov triple $\tilde{x}$ induced from $(L^2(\mathbb{R}), \tilde{M}, \tilde{D})$. Let $\psi : \tilde{X} \to \underline{E}\Gamma$ be a Γ-equivariant continuous map; from lemma 9, it follows that

$$(\alpha_\gamma)_*(x) = \psi_*(\tilde{x}).$$

On the other hand, since Γ acts freely on $\tilde{X}$, the map ψ factors through $E\Gamma$, i.e. there exists a Γ-equivariant continuous map $\tilde{\psi} : \tilde{X} \to E\Gamma$ such that $\psi = \phi \circ \tilde{\psi}$. Then $\gamma_*(x)$ is described by $\tilde{\psi}_*(\tilde{x})$, so that by functoriality:

$$\tilde{B}_t(\gamma) = \phi_*(\gamma_*(x)) = \phi_*(\tilde{\psi}_*(\tilde{x})) = \psi_*(\tilde{x}) = (\alpha_\gamma)_*(x) = \tilde{\beta}_t(\gamma).$$

2) γ has finite order $n \geq 1$. Since α_γ factors through $\mathbb{Z}/n\mathbb{Z}$, it follows by functoriality that $(\alpha_\gamma)_* : RK_1^{\mathbb{Z}}(\underline{E}\mathbb{Z}) \to RK_1^\Gamma(\underline{E}\Gamma)$ factors through

$$RK_1^{\mathbb{Z}/n\mathbb{Z}}(\underline{E}\mathbb{Z}/n\mathbb{Z}) = K_1^{\mathbb{Z}/n\mathbb{Z}}(pt) = 0.$$

Hence $\tilde{\beta}_t(\gamma) = 0$.

Consider now the map $\gamma_* : RK_1^{\mathbb{Z}}(E\mathbb{Z}) \to RK_1^{\Gamma}(E\Gamma)$; it factors through a map $\tilde{\gamma}_* : RK_1^{\mathbb{Z}/n\mathbb{Z}}(E\mathbb{Z}/n\mathbb{Z}) \to RK_1^{\Gamma}(E\Gamma)$. Denote by ϕ_n the unique map from $E\mathbb{Z}/n\mathbb{Z}$ to $\underline{E}\mathbb{Z}/n\mathbb{Z} = pt$. Because of naturality, there is a commutative diagram

$$\begin{array}{ccc} RK_1^{\mathbb{Z}/n\mathbb{Z}}(E\mathbb{Z}/n\mathbb{Z}) & \stackrel{\tilde{\gamma}_*}{\to} & RK_1^{\Gamma}(E\Gamma) \\ (\phi_n)_* \downarrow & & \downarrow \phi_* \\ K_1^{\mathbb{Z}/n\mathbb{Z}}(pt) & \to & RK_1^{\Gamma}(\underline{E}\Gamma) \end{array}$$

But again $K_1^{\mathbb{Z}/n\mathbb{Z}}(pt) = 0$, so that $\tilde{B}_t(\gamma) = \phi_*(\gamma_*(x)) = 0$. □

Proposition 4 *The map $\tilde{\beta}_t : \Gamma \to RK_1^{\Gamma}(\underline{E}\Gamma)$ is a group homomorphism.*

Proof: In view of the preceding lemma, we have to show that $\tilde{B}_t : \Gamma \to RK_1^{\Gamma}(\underline{E}\Gamma)$ is a group homomorphism. Fix $\gamma_1, \gamma_2 \in \Gamma$ and view γ_1, γ_2 as continuous maps $S^1 \to B\Gamma$. Set $X = \gamma_1(S^1) \cup \gamma_2(S^1)$. For $i = 1, 2$, the element $(\gamma_i)_*(x) \in K_1(X)$ is described by the triple $(L^2(S^1), \pi_i, D)$ where, for $f \in C(X)$, the operator $\pi_i(f)$ is pointwise multiplication by $f \circ \gamma_i$ on $L^2(S^1)$, and $D = \frac{1}{i} \cdot \frac{d}{d\theta}$. Similarly, $(\gamma_1\gamma_2)_*(x)$ is described by $(L^2(S^1), \pi, D)$, where $\pi(f)$ is pointwise multiplication by $f \circ \gamma_1\gamma_2$; here $\gamma_1\gamma_2 : S^1 \to X$ denotes the product loop of γ_1 and γ_2. It will be enough to show that

$$(\gamma_1\gamma_2)_*(x) = (\gamma_1)_*(x) + (\gamma_2)_*(x)$$

in $K_1(X) = KK_1(C_0(X), \mathbb{C})$. For this, consider the doubling unitary:

$$U : \left\{ \begin{array}{ccc} L^2(S^1) \oplus L^2(S^1) & \to & L^2(S^1) \\ (\xi_1, \xi_2) & \mapsto & \left(\theta \mapsto \left\{ \begin{array}{ll} \sqrt{2}\xi_1(2\theta) & if\, 0 \leq \theta \leq \frac{1}{2}; \\ \sqrt{2}\xi_2(2\theta - 1) & if\, \frac{1}{2} \leq \theta \leq 1. \end{array} \right. \right) \end{array} \right.$$

The inverse of U is given on $\xi \in L^2(S^1)$ by the formulae:

$$\left\{ \begin{array}{lcl} (U^*\xi)_1(\theta) & = & \frac{1}{\sqrt{2}}\xi(\frac{\theta}{2}) \\ (U^*\xi)_2(\theta) & = & \frac{1}{\sqrt{2}}\xi(\frac{\theta+1}{2}), \end{array} \right.$$

for $\theta \in S^1$. Since the product loop $\gamma_1\gamma_2$ is given by

$$(\gamma_1\gamma_2)(\theta) = \left\{ \begin{array}{ll} \gamma_1(2\theta) & if\, 0 \leq \theta \leq \frac{1}{2} \\ \gamma_2(2\theta - 1) & if\, \frac{1}{2} \leq \theta \leq 1, \end{array} \right.$$

one sees that $U(\pi_1 \oplus \pi_2)U^* = \pi$, and that $U(D \oplus D)U^* = \frac{D}{2}$. This shows that $(\gamma_1)_*(x) + (\gamma_2)_*(x)$ is unitarily equivalent to the triple $(L^2(S^1), \pi, \frac{D}{2})$, which in turn is trivially homotopic to $(\gamma_1\gamma_2)_*(x)$. The result follows. $\square$

As we already mentioned, since $RK_1^\Gamma(\underline{E}\Gamma)$ is an abelian group, the homomorphism $\tilde{\beta}_t : \Gamma \to RK_1^\Gamma(\underline{E}\Gamma)$ factors through a homomorphism

$$\beta_t : \Gamma^{ab} \to RK_1^\Gamma(\underline{E}\Gamma).$$

5.2 Proof of Theorem 2

Since the homomorphism β_t is already constructed, it remains to prove $\kappa_\Gamma = \mu_1^\Gamma \circ \beta_t$. Clearly, it suffices to prove that, as group homomorphisms $\Gamma \to K_1(C_r^*\Gamma)$, one has

$$\tilde{\kappa}_\Gamma = \mu_1^\Gamma \circ \tilde{\beta}_t.$$

Fix $\gamma \in \Gamma$, and denote again by $\alpha_\gamma : \mathbb{Z} \to \Gamma$ the unique homomorphism such that $\alpha_\gamma(1) = \gamma$. Consider the diagram:

$$\begin{tikzcd}
\mathbb{Z} \arrow[rrr, "\alpha_\gamma"] \arrow[dr, "\kappa_{\mathbb{Z}}"] \arrow[dd, "\beta_t"'] & & & \Gamma \arrow[dl, "\tilde{\kappa}_\Gamma"'] \arrow[dd, "\tilde{\beta}_t"] \\
& K_1(C^*\mathbb{Z}) \arrow[r, "(\alpha_\gamma)_*"] & K_1(C_r^*\Gamma) & \\
RK_1^{\mathbb{Z}}(\underline{E}\mathbb{Z}) \arrow[ur, "\mu_1^{\mathbb{Z}}"'] \arrow[rrr, "(\alpha_\gamma)_*"] & & & RK_1^\Gamma(\underline{E}\Gamma) \arrow[ul, "\mu_1^\Gamma"]
\end{tikzcd}$$

We have $(\alpha_\gamma)_* \circ \kappa_{\mathbb{Z}} = \tilde{\kappa}_\Gamma \circ \alpha_\gamma$ trivially, $\tilde{\beta}_t \circ \alpha_\gamma = (\alpha_\gamma)_* \circ \beta_t$ by the very definition of $\tilde{\beta}_t$, and $(\alpha_\gamma)_* \circ \mu_1^{\mathbb{Z}} = \mu_1^\Gamma \circ (\alpha_\gamma)_*$ by Theorem 1. By diagram chasing, one sees that $\tilde{\kappa}_\Gamma = \mu_1^\Gamma \circ \tilde{\beta}_t$ follows from the analogous result for $\mathbb{Z}$, i.e. $\kappa_{\mathbb{Z}} = \mu_1^{\mathbb{Z}} \circ \beta_t$. This in turn follows from Proposition 3 and its proof. $\square$

Remarks:

1) If follows from Theorem 2 that, if γ is a torsion element in Γ, then $\tilde{\kappa}_\Gamma(\gamma) = 0$. This fact can easily be proved directly, see Proposition 2 in [BV96].

2) The *strong Novikov conjecture* is the statement that

$$\mu_i^\Gamma \circ \phi_* : RK_i(B\Gamma) \simeq RK_i^\Gamma(E\Gamma) \to K_i(C_r^*\Gamma)$$

is rationally injective for $i = 0, 1$ (see [BCH94], p. 276). Since β_t factors through ϕ_*, and since κ_Γ is rationally injective (see [EN87], [BV96]), it follows from Theorem 2 that $\mu_1^\Gamma \circ \phi_*$ is always rationally injective on the image of $\Gamma^{ab} \simeq H_1(\Gamma, \mathbb{Z})$ in $RK_1(B\Gamma)$, i.e. on the lowest dimensional part of $\mu_1^\Gamma \circ \phi_*$.

Chapter 6

Appendix by Dan KUCEROVSKY

6.1 The assembly map in the unbounded picture

The starting point for Kasparov KK-theory is an abstraction and axiomatization of the main properties of zeroth order elliptic operators, whereas in unbounded Kasparov theory (denoted Ψ), one uses first order operators instead.

The Kasparov product is a generalization of the "sharp product" introduced by Atiyah and Singer (1961) in their proof of an index theorem, and this sharp product is easier to define for first order operators than for zeroth order operators. To be precise, it has been proven by Baaj and Julg (1983) that the Kasparov product in fact reduces to a sharp product (a graded tensor product) when written in terms of unbounded operators: their result is that

$$[D_1] \otimes [D_2] = [D_1 \otimes 1 + 1 \otimes D_2]$$

in certain special cases.

Kasparov's original approach (1980) to the product was to show the existence of operators M and N such that

$$[F_1] \otimes [F_2] = [M^{1/2}(F_1 \otimes 1) + N^{1/2}(1 \otimes F_2)]$$

The operators N and M have very special properties and are not easy to construct explicitly, furthermore, there are some technical complications coming from the fact that the tensor product $1 \otimes F_2$ is not well-defined in certain cases of interest, one therefore has to stabilize the Hilbert modules involved and this makes it even more difficult to explicitly determine the product cycle.

It seems plausible that M and N would be easier to construct in the unbounded picture. However, here the matter rested until the discovery of the connection approach to the Kasparov product (Connes and Skandalis (1984)). Connes and Skandalis found the following criterion for $(E_1 \otimes E_2, \phi_1 \otimes 1, F)$ to be the Kasparov product of (E_1, ϕ_1, F_1) and (E_2, ϕ_2, F_2) :

1. $FT_x - (-1)^{\partial x} T_x F_2$ is compact (where T_x is the tensoring operator $T_x : y \mapsto x \otimes y$).

2. $\phi_1(a)[F, F_1 \otimes 1]\phi_1(a)^*$ is positive modulo compact operators for all $a \in A$.

Given this result, it is possible to see what the counterpart in terms of unbounded cycles should be. Roughly speaking, the result of using unbounded cycles instead of bounded ones is that, quite generally, bounded operators are replaced by unbounded ones, and compactness conditions are replaced by boundedness conditions. This has the advantage that not only is boundedness often easier to prove than compactness, but the Kasparov product can be simpler to compute.

We briefly summarize both the bounded and unbounded pictures of Kasparov theory in the following table. In the table, a generic cycle in bounded KK-theory is denoted (E, ϕ, F), a Kasparov product in the bounded picture is denoted $(E_1, \phi_1, F_1) \otimes (E_2, \phi_2, F_2) = (E_1 \otimes E_2, \phi_1 \otimes 1, F)$, and the corresponding objects in the unbounded picture are denoted similarly but with D, D_1, and D_2 instead of F, F_1 and F_2.

Bounded picture: $KK(A,B)$	**Unbounded picture: $\Psi(A,B)$**
Fredholm condition:	
$\phi(a)(1 - F^2)$ is compact	$\phi(a)(\lambda - D)^{-1}$ is compact for some λ.

Commutator condition:

$[F, \phi(a)]$ is compact	$[D, \phi(a)]$ is bounded

Self-adjointness condition:

$\phi(a)(F^* - F)$ is compact	$D^* - D$ is bounded.

Connection condition (where T_x is the tensoring operator $T_x : y \mapsto x \otimes y$.):

$FT_x - (-1)^{\partial x} T_x F_2$ is compact.	$DT_x - (-1)^{\partial x} T_x D_2$ is bounded. It is enough to have boundedness for all x in some dense subset of $\phi_1(A)E_1$.

Positivity condition:

$\phi(a)[F, F_1 \otimes 1]\phi(a)^*$ is positive modulo compact operators	$[D, D_1 \otimes 1]$ is bounded below. It is enough to have semiboundedness in the sense of quadratic forms on the domain of definition.

Degenerate cycles

The representation ϕ commutes with F and $F^2 = 1$.	The representation ϕ commutes with D and D has a gap in its spectrum.

"Compact perturbation" of Kasparov cycles:

Cycles (E, ϕ, F) and (E, ϕ, F') are equivalent if $\phi(a)(F - F')$ is compact.	The corresponding condition in the unbounded picture is that $D - D'$ is bounded on the common domain of D and D'.

Cycle homotopy:

A cycle in $KK(A, B[0,1])$ defines a homotopy of the cycles given by evaluation at the endpoints of [0,1].	In the unbounded picture, the definition is the same, except that the cycle defining the homotopy is unbounded.

Operator homotopy:

The special case of cycle homotopy in which the representation ϕ and Hilbert module E remain constant through the homotopy; and the operators F_0,F_1 are linked by a norm-continuous path in $\mathcal{L}(E)$.	In the unbounded picture, operator homotopies are also a special case of cycle homotopy, with the extra condition that the Cayley transform of the operator implementing the homotopy must correspond to a norm-continuous family of operators.

Equivariant cycles.

Let α_g be the action of a locally compact second countable topological group, covariant with respect to the representation ϕ.

In the bounded picture, the requirement is that the function $g \mapsto (\alpha_g(F) - F)\phi(a)$ is norm-continuous and compact at every point.	In the unbounded picture, the function $g \mapsto (\alpha_g(D) - D)$ is bounded at each point and pointwise continuous in the sense that $g \mapsto (\alpha_g(D) - D)e$ is continuous for each e in the domain of D.

The equivalence relation for Kasparov cycles can be given in any of several standard forms; namely:

1. Cycle homotopy; or

2. Operator homotopy plus addition of degenerate cycles; or

3. Compact perturbation plus addition of degenerate cycles and unitary equivalence.

The equivalence of these three forms is a nonobvious but very useful result. This ends our brief outline of bounded and unbounded KK-theory. For a more precise description, we refer the reader to [Bla86], [Kas81], [Kas95] for the bounded theory, and [BJ83], [Kuc94], [Kuc97, Kuc] for the unbounded theory.

We now discuss Chapter 2 of Valette's Notes from the point of view of unbounded KK-theory. The main point of that chapter is to show that a certain map from a direct limit of equivariant K-homology groups to equivariant K-theory is well-defined. It is sufficient to define a map

$$\tilde{\mu}^\Gamma : K_i^\Gamma(X) \longrightarrow K_i(C^*\Gamma)$$

for Γ-compact subsets X of $\underline{E}\Gamma$, provided that this map commutes with the inclusion maps used to define the direct limit $RK_i^\Gamma(\underline{E}\Gamma)$. In terms of unbounded Kasparov theory, the explicit definition of the map $\tilde{\mu}^\Gamma$ is:

$$\begin{array}{rcl} \tilde{\mu}^\Gamma : \Psi_\Gamma(C_0(X), \mathbb{C}) & \longrightarrow & \Psi_\Gamma(\mathbb{C}, C^*\Gamma) \\ (\mathcal{H}, \pi, D) & \mapsto & (\mathcal{E}, D) \end{array}$$

where $\mathcal{E}$ is the $\mathbb{Z}_2$-graded Hilbert $C^*\Gamma$ module obtained by completing $\pi(C_c(X))\mathcal{H}$ with respect to the $C^*\Gamma$-valued inner product $\langle \xi_1, \xi_2 \rangle (\gamma) := \langle \xi_1, \gamma\xi_2 \rangle$ as in section 2.3. It is not obvious that this map $\tilde{\mu}^\Gamma$ is well-defined, and the easiest way to see that it does map a cycle to a cycle is by an indirect approach.

We define a large semigroup $U_\Gamma(A, B)$ that contains $\Psi_\Gamma(A, B)$:

Definition 5 *The semigroup $U_\Gamma(A, B)$ is given by triples $(E \oplus E, \phi, \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix})$ with the direct sum operation, where E is any Hilbert B-module with an action of Γ, ϕ is an (equivariant) representation of A on E, and D is a properly supported equivariant regular operator[1] on E with $[D, \phi(a)]$ bounded for all a in some dense subset of A.*

We don't need the equivariance condition, assuming the boundedness of $[D, \gamma]$ would be enough, but this additional condition, which is also assumed in the main part of this paper, makes the calculations more clear by suppressing some terms involving $[D, \gamma]$ that would otherwise arise. The reason for introducing this semigroup is only that it is convenient to have a large semigroup for which some form of the Kasparov equivalence relation makes sense. In fact, we could prove our main result without any reference to this semigroup, which is only used as a convenient setting for proving compactness of the resolvent of a certain operator.

Definition 6 *The equivalence relation ubd on $U(A, B)$ is generated by unitary equivalence, perturbation by bounded operators, and addition of degenerate Kasparov cycles.*

[1] The definition of regularity is that an operator is regular if and only if its graph is orthogonally complemented. It follows that a bounded operator is regular if and only if it is adjointable. The reason for preferring regularity to adjointability is that regularity is defined in terms of the graph of the operator, and hence the definition can be immediately generalized to the case of unbounded operators. All Hilbert space operators are regular, but there are usually many non-regular operators on a Hilbert module.

Then we see that the Kasparov group is stable under equivalence within $U_\Gamma(A, B)$:

Proposition 5 *$U_\Gamma(A,B)/ubd$ contains $\Psi_\Gamma(A,B)$.*

This proposition is an equivariant version of a lemma in [Kuc].

Proof: In terms of cycles, this proposition says that a triple $(E \oplus E, \phi, \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix})$ is an unbounded cycle if and only if it is equivalent to an unbounded cycle. This is obvious except for the compactness of the resolvent, which follows from the facts that unitary equivalence preserves the compact operators on a Hilbert module, and that $\phi(a)(\lambda - T)^{-1}$ is compact for some λ if and only if this is true for $\phi(a)(\lambda' - T + B)^{-1}$, where B is some bounded operator. $\square$

The map $\tilde{\mu}^\Gamma$ is certainly at least a map into $U_\Gamma(\mathbb{C}, C^*\Gamma)$, and we will use a unitary equivalence and bounded perturbation to show that the image is in $\Psi^\Gamma(\mathbb{C}, C^*\Gamma)$ after equivalence.

We remind the reader that $\tilde{\mathcal{H}}$ is the $C^*\Gamma$-module completion of $C_c(\Gamma, \mathcal{H})$ with respect to the $\mathbb{C}\Gamma$-valued inner product $\langle \xi, \mu \rangle (\sigma) := \sum_{\gamma \in \Gamma} \langle \xi(\gamma), \mu(\gamma\sigma) \rangle$, and that in lemmas 6 and 7 we defined another inner product $C_c(X) \times C_c(X) \longrightarrow C_c(X \times \Gamma)$ and then constructed a projection $p \in C_c(X \times \Gamma)$ by defining $p := \langle h, h \rangle$ where $h \in C_c(X)$ is a function with $\sum_{\gamma \in \Gamma} h(\gamma x)^2 = 1$ for all x. We now check that the function h can be chosen to have bounded commutator with a given unbounded operator that comes from a triple in $U_\Gamma(\mathbb{C}, C^*\Gamma)$.

Recalling that the function h^2 can be constructed by choosing any positive f which is nonzero on a compact global slice [Pal61] (which exists since X is a Γ-compact Γ-space with proper action of Γ) and then performing an averaging operation that will be described; we can also assume that given an unbounded selfadjoint operator D that comes from a triple, the commutator $[f, D]$ is bounded on the domain of D. The function h^2 is defined by

$$h^2(x) := \frac{f(x)^2}{\sum_{\gamma \in \Gamma} f(\gamma x)^2}.$$

Since D is properly supported, there is a g such that $g[f^2, D] = [f^2, D]$, and taking adjoints shows that $g[f^2, D]g = [f^2, D]$. In particular, the function

$f_\Gamma^2 := \sum_{\gamma\in\Gamma} \gamma f^2 \gamma^{-1}$ has bounded commutator with D, since $A_g([f^2, D]) = [f_\Gamma^2, D]$ is bounded. Therefore

$$\pm i[1/f_\Gamma, D] \leq \|[f_\Gamma^2, D]\| f_\Gamma^{-3}$$

is bounded, by a form of Powers' identity[Pow75, Kuc], and hence $[h, D] = [f/f_\Gamma, D]$ is bounded.

Lemma 13 *There is a unitary* $\beta : \tilde{\pi}(p)\tilde{\mathcal{H}} \longrightarrow \mathcal{E}$ *which almost commutes with operators coming from triples* $(\mathcal{E}, D)$ *in* $U_\Gamma(\mathbb{C}, C^*\Gamma)$, *in the sense that* $\beta\tilde{\pi}(p)D - D\beta : \tilde{\pi}(p)\tilde{\mathcal{H}} \longrightarrow \mathcal{E}$ *is bounded.*

Proof: The map β is defined in the proof of lemma 8, where it is also shown that $\langle \beta x, \beta x\rangle_{\mathcal{E}} = \langle x, x\rangle_{\tilde{\mathcal{H}}}$, which implies by the polarization identity that β is an unitary.

For the second part of the lemma, we recall that $[\pi(h), D]$ can be assumed bounded on the domain of D, and since β is unitary it is enough to show that $\beta\tilde{\pi}(p)D\beta^* - D : \mathcal{E} \longrightarrow \mathcal{E}$ is bounded. Suppressing the representation $\tilde{\pi}$, we write

$$\begin{array}{rcl} \beta p D \beta^* & = & \sum_{\gamma\in\Gamma} \gamma h D h \gamma^{-1} \\ & = & A_g([h, D]h) + \sum_{\gamma\in\Gamma} \gamma D h^2 \gamma^{-1} \;\;, \\ & = & A_g([h, D]h) + D \end{array}$$

where we have first used the proper support of D to find a $g \in C_0(X)$ such that $hg = h$ and $gDh = Dh$, and then we have used the equivariance of D and the averaging property of h to show that $\sum_{\gamma\in\Gamma} \gamma D h^2 \gamma^{-1} = D$. Since $A_g([h, D]h)$ is bounded, we are done. □

Remark: Since $A_h(T) = \beta T \beta^*$ for $T : \mathcal{H} \longrightarrow \mathcal{H}$, we see that the "averaging operator" A_f is actually a unitary transformation in the special case $f = h$, and therefore can be applied without losing information. Furthermore, A_h commutes with the functional calculus.

We can now give a quick proof that an element $(\mathcal{E}, D)$ of the image of $\tilde{\mu}^\Gamma$ is a cycle. First of all, if we consider a map from $\Psi_\Gamma(C_0(X), \mathbb{C})$ to $\Psi_\Gamma(\mathbb{C}, C^*\Gamma)$ given by $(\mathcal{H}, \pi, D) \mapsto (\tilde{\mathcal{H}}, \tilde{\pi}(p) : \mathbb{C} \longrightarrow \mathcal{L}, D)$, then it is quite clear that an element of the image is in fact a cycle. The main question is whether or not the resolvent $R := \tilde{\pi}(p)(i + D)^{-1}$ is compact on $\tilde{\mathcal{H}}$, but regarding $\tilde{\mathcal{H}}$ as a function space over Γ, we see that R is compact at every point, and is compactly supported because

$$\tilde{\pi}(p)\xi := \pi(x \mapsto \overline{h(x)}h(\gamma^{-1}x))\xi(\gamma), \quad \xi \in C_c(\Gamma, \mathcal{H})$$

where h is compactly supported and Γ acts properly.

Now we use the preceeding lemma to give the equivalences

$$\begin{aligned}(\tilde{\mathcal{H}}, \tilde{\pi}(p) : \mathbb{C} \longrightarrow \mathcal{L}, D) &\sim_d (\tilde{\mathcal{H}}, \tilde{\pi}(p)D) \\ &\sim_u (\mathcal{E}, \beta\tilde{\pi}(p)D\beta^*) \\ &\sim_b (\mathcal{E}, D)\end{aligned}$$

proving that the map from $\Psi_\Gamma(C_0(X), \mathbb{C})$ to $\Psi_\Gamma(\mathbb{C}, C^*\Gamma)$ that we just gave is in fact the Baum-Connes map $\tilde{\mu}^\Gamma$.

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