Convergence of Baumslag-Solitar groups

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Abstract

We study convergent sequences of Baumslag-Solitar groups in the space of marked groups. We prove that $BS(p,q) \to \mathbb{F}_2$ for $|p|, |q| \to \infty$ and $BS(1,q) \to \mathbb{Z} \wr \mathbb{Z}$ for $|q| \to \infty$. For p fixed, $|p| \geqslant 2$, we show that the sequence $(BS(p,q))_q$ is not convergent and characterize many convergent subsequences. Moreover if X_p is the set of $BS(p,q)$'s for q relatively prime to p and $|q| \geqslant 2$, then the map $BS(p,q) \mapsto q$ extends continuously on $\overline{X_p}$ to a surjection onto invertible p-adic integers.

1 Introduction

Let $\mathcal{G}_2$ be the space of finitely generated marked groups on two generators (see section 2 for definition) and let $\mathbb{F}_2 = \langle a, b \mid \varnothing \rangle$ be the free group on two generators. Baumslag-Solitar groups are defined by the presentations

$$BS(p,q) = \langle a, b \mid ab^p a^{-1} = b^q \rangle$$

for $p, q \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. The purpose of the present paper is to understand how Baumslag-Solitar groups are distributed in $\mathcal{G}_2$. More precisely, we determine convergent sequences and in some cases we are able to give the limit group. In the following results, we mark $\mathbb{F}_2$ and $BS(p,q)$ by $\{a, b\}$.

Theorem 1 *$BS(p,q) \to \mathbb{F}_2$ when $|p|, |q| \to \infty$.*

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In particular, the property of being hopfian is not open in $\mathcal{G}_2$ since $BS(2^k, 3^k)$ is known to be non hopfian for all $k \geqslant 1$ (see [LS77], Chapter IV, Theorem 4.9.) while $\mathbb{F}_2$ is hopfian. Theorem 1 is not so surprising because the length of the relator appearing in the presentation of $BS(p,q)$ tends to ∞ when $|p|, |q| \to \infty$. However, this relator is not the shortest relation in the group for many values of (p,q). To prove Theorem 1, we give a lower bound for the length of shortest relations in the more general setting of HNN-extensions (see section 3).

We now fix the parameter p. In the case $p = \pm 1$, we show that the sequence $(BS(\pm 1, q))_{q\in\mathbb{Z}}$ is convergent and we can identify the limit.

Theorem 2 *Let the wreath product $\mathbb{Z} \wr \mathbb{Z} = \mathbb{Z} \ltimes \oplus_{i\in\mathbb{Z}}\mathbb{Z}$ be marked by the elements $(1,0)$ and $(0,e_0)$ where $e_0 \in \oplus_{i\in\mathbb{Z}}\mathbb{Z}$ is the Dirac mass at 0. Then $BS(\pm 1, q) \to \mathbb{Z} \wr \mathbb{Z}$ when $|q| \to \infty$.*

In the case $|p| \geqslant 2$, we show that the sequence $(BS(p,q))_{q\in\mathbb{Z}}$ is not convergent in $\mathcal{G}_2$. As $\mathcal{G}_2$ is compact, it has convergent subsequences. The last result we state in this introduction, among subsequences, characterizes many convergent ones. However we don't actually know what the limits are. Remark that the result also holds for $p = \pm 1$, even if it is in this case weaker than Theorem 2.

Theorem 3 *Let $p \in \mathbb{Z}^*$ and let $(q_n)_n$ be a sequence of integers relatively prime to p. The sequence $(BS(p,q_n))_n$ is convergent in $\mathcal{G}_2$ if and only if one (and only one) of the following assertions holds:*

(a) $(q_n)_n$ is eventually constant;

(b) $|q_n| \to \infty$ and for all $h \geqslant 1$ the sequence $(q_n)_n$ is eventually constant modulo p^h.

Note that condition (b) precisely means that $|q_n| \to \infty$ and $(q_n)_n$ is convergent in $\mathbb{Z}_p$, the ring of p-adic integers. The link between Baumslag-Solitar groups and p-adic integers can be made more precise. We define X_p to be the set of $BS(p,q)$'s, for q relatively prime to p and $|q| \geqslant 2$ and we denote by $\mathbb{Z}_p^\times$ the set of invertible elements of $\mathbb{Z}_p$.

Theorem 4 *The map $\Psi : X_p \to \mathbb{Z}$; $BS(p,q) \mapsto q$ extends to a uniformly continuous and surjective map $\overline{\Psi} : \overline{X_p} \to \mathbb{Z}_p^\times$. However, $\overline{\Psi}$ is not injective.*

An immediate corollary of Theorem 3 or Theorem 4 is that (for $|p| \geqslant 2$) the sequence $(BS(p,q))_q$ admits uncountably many accumulation points, namely at least one for each invertible p-adic integer.

We end this introduction by a remark on markings of Baumslag-Solitar groups. In this paper we always mark the group $BS(p,q)$ by the generators coming from its canonical presentation given above. Nevertheless, it is also an interesting approach to consider different markings on $BS(p,q)$. For instance, take p and q greater than 2 and relatively prime, so that $\Gamma = BS(p,q)$ is non-hopfian, the epimorphism $\phi : \Gamma \to \Gamma$ given by $a \mapsto a$ and $b \mapsto b^p$ being non-injective (see again [LS77], Chapter IV, Theorem 4.9.). In [ABL$^+$03], the authors consider the sequence of groups $\Gamma_n = \Gamma / \ker(\phi^n)$ (marked by a and b). They show that the sequence $(\Gamma_n)_n$ converges to an amenable group while, being all isomorphic to Γ as groups, the Γ_n's are not amenable. This allow them to prove that Γ is non-amenable, but not uniformly (cf. Proposition 13.3) and also shows that the property of being amenable is not open in $\mathcal{G}_2$.

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2 Preliminaries and examples

We collect in this section some definitions and material which are needed in the rest of the paper. The reader which is familiar with the notions of Presentations, HNN-extensions and topology on the space of marked groups can skip directly to section 3. We nevertheless present at the end of this section symmetric groups as a convergent sequence in $\mathcal{G}_2$. This example does not seem to be published.

Presentations. Let X be a set and $\mathcal{R}$ be a collection of cyclically reduced words in $\mathbb{F}_X$, the free group on X. Recall that the group given by the presentation $\langle X \mid \mathcal{R}\rangle$ is $\Gamma = \mathbb{F}_X/N_{\mathcal{R}}$, where $N_{\mathcal{R}}$ denotes the normal subgroup of $\mathbb{F}_X$ generated by $\mathcal{R}$. If $\Gamma = \langle X \mid \mathcal{R}\rangle$, we call *generator* any element of X, *relator* any element of $\mathcal{R}$ and *relation* any nontrivial element of $N_{\mathcal{R}}$.

Let $w = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ be a reduced word in $\mathbb{F}_X$ with $\varepsilon_i \in \{\pm 1\}$. The integer n is called the *length* of w and denoted $\ell(w)$. The length of the shortest relation of Γ will be denoted g_Γ, for we observe it is the girth of the Cayley graph of Γ (with respect to the generating set X). In case $\mathcal{R} = \varnothing$, we set $g_\Gamma = +\infty$.

If $\Gamma = \langle X \mid \mathcal{R}\rangle$, given $\gamma \in \Gamma$ we define its *length* to be

$$\begin{aligned} \ell_\Gamma(\gamma) &:= \min\{n : \gamma = x_1 \dots x_n \text{ with } x_i \in X \sqcup X^{-1}\} \\ &= \min\{\ell(w) : w \in \mathbb{F}_X,\ w = \gamma \text{ in } \Gamma\} . \end{aligned}$$

HNN-extensions and Baumslag-Solitar groups. Suppose now that $H = \langle X \mid \mathcal{R}\rangle$, and that $\phi : A \to B$ is an isomorphism between subgroups of H. The *HNN extension* of H with respect to A, B and ϕ is given by the presentation

$$HNN(H, A, B, \phi) := \langle X \sqcup \{t\} \mid \mathcal{R},\ t^{-1}at = \phi(a)\ \forall a \in A\rangle .$$

Unless specified otherwise, we always refer to the above presentation while discussing of length of elements in a HNN-extension. An element $\gamma \in HNN(H, A, B, \phi)$ can always be written

$$\gamma = h_0 t^{\varepsilon_1} h_1 \dots t^{\varepsilon_n} h_n \text{ with } n \geqslant 0,\ \varepsilon_i \in \{\pm 1\},\ h_i \in H . \tag{1}$$

The decomposition of γ in (1) is called *reduced* if no subword of type $t^{-1}at$ (with $a \in A$) or tbt^{-1} (with $b \in B$) appears. We recall the following result, which is called *Britton's Lemma*

Lemma 1 *([LS77], Chapter IV.2.) Let $\gamma \in HNN(H, A, B, \phi)$ and write as in (1) $\gamma = h_0 t^{\varepsilon_1} h_1 \dots t^{\varepsilon_n} h_n$. If $n \geqslant 1$ and if the decomposition is reduced, then $\gamma \neq 1$ in $HNN(H, A, B, \phi)$.*

This shows in particular that the integer n appearing in a reduced decomposition is uniquely determined by γ.

Let us finally recall that *Baumslag-Solitar groups* are defined by the presentations $BS(p,q) = \langle a, b \mid ab^p a^{-1} = b^q \rangle$ for $p, q \in \mathbb{Z}^*$. We have $BS(p,q) = HNN(\mathbb{Z}, q\mathbb{Z}, p\mathbb{Z}, \phi)$ where ϕ is given by $\phi(qk) = pk$

Marked groups and their topology. Introductory expositions of these topics can be found in [Ch00] or [CG04]. We only recall some basics and what we need in following sections.

A *marked group on k generators* is a pair (Γ, S) where Γ is a group and $S = (s_1, \ldots, s_k)$ is a family which generates Γ. A marked group (Γ, S) comes always with a canonical epimorphism $\phi : \mathbb{F}_k \to \Gamma$ and this gives an isomorphism of marked groups between $\mathbb{F}_k / \ker \phi$ and Γ. Hence a class of marked groups can always be represented by a quotient of $\mathbb{F}_k$. In particular if a group is given by a presentation, this defines a marking on it.

Let $\mathcal{G}_k$ be the set of marked groups on k generators (up to marked isomorphism). Let us recall that the topology on $\mathcal{G}_k$ comes from the following ultrametric: for $(\Gamma_1, S_1) \neq (\Gamma_2, S_2) \in \mathcal{G}_k$ we set $d\big((\Gamma_1, S_1), (\Gamma_2, S_2)\big) := e^{-\lambda}$ where λ is the length of a shortest element of $\mathbb{F}_k$ which vanishes in one group and not in the other one. But what the reader has to keep in mind is the following characterization of convergent sequences.

Proposition 1 *Let (Γ_n, S_n) be a sequence of marked groups. The following are equivalent:*

(i) (Γ_n, S_n) is convergent in $\mathcal{G}_k$;

(ii) for all $w \in \mathbb{F}_k$ we have either $w = 1$ in Γ_n for n big enough, or $w \neq 1$ in Γ_n for n big enough.

Proof. (i)⇒(ii): Set $(\Gamma, S) = \lim\limits_{n\to\infty} (\Gamma_n, S_n)$ and take $w \in \mathbb{F}_k$. For n sufficiently large we have $d\big((\Gamma, S), (\Gamma_n, S_n)\big) < e^{-\ell(w)}$, which implies that we have $w = 1$ in Γ_n if and only if $w = 1$ in Γ.

(ii)⇒(i): Set $N = \{w \in \mathbb{F}_k : w = 1 \text{ in } \Gamma_n \text{ for } n \text{ big enough }\}$, $\Gamma = \mathbb{F}_k / N$, and fix $r \geqslant 1$. For n big enough, Γ_n and Γ have the same relations up to length r (for the balls in $\mathbb{F}_k$ are finite) and hence $d(\Gamma, \Gamma_n) < e^{-r}$ (we drop the markings since they are obvious). This implies $\Gamma_n \underset{n\to\infty}{\longrightarrow} \Gamma$. $\square$

Example 1 Let $\Gamma_n = \mathrm{Sym}(\mathbb{Z}/n\mathbb{Z})$, marked by the transposition $t = (01)$ and the n-cycle $s = (0 \dots n)$. Let Γ be the subgroup of $\mathrm{Sym}(\mathbb{Z})$ generated (and marked) by the transposition $t = (01)$ and the shift $s : k \mapsto k+1$. (We have $\Gamma \cong \mathbb{Z} \ltimes \mathrm{Sym}_0(\mathbb{Z})$ where $\mathrm{Sym}_0(\mathbb{Z})$ is the set of permutations of $\mathbb{Z}$ with finite support (we define the *support* of a permutation to be the complement of fixed points) and $\mathbb{Z}$ acts by conjugation by s.)

With the given markings, we have $\Gamma_n \to \Gamma$ in $\mathcal{G}_2$ when $n \to \infty$.

To see it, set $\mathbb{F}_2 = \langle t, s \mid \varnothing \rangle$. We take $w \in \mathbb{F}_2$ which we may write as

$$w = t^k t^{\alpha_1} s^{j_1} t^{-\alpha_1} \dots t^{\alpha_m} s^{j_m} t^{-\alpha_m} .$$

We set $f = t^{\alpha_1} s^{j_1} t^{-\alpha_1} \dots t^{\alpha_m} s^{j_m} t^{-\alpha_m}$ and $N = \max_{1 \leqslant i \leqslant m}(|\alpha_i| + 1)$. First we show that for $n \geqslant 3N$, one has $f = 1$ in Γ if and only if $f = 1$ in Γ_n. Indeed, the image of f in Γ is a product of transpositions, all with support in $\{-N+1, \dots N\}$. As $n \geqslant 3N$, the image of f in Γ_n is the same permutation of $\{-N+1, \dots N\}$, seen as a subset of $\mathbb{Z}/n\mathbb{Z}$.

To conclude, we now show that for $n \geqslant \max(3N, 2|k|)$, one has $w = 1$ in Γ if and only if $w = 1$ in Γ_n. Indeed, if $w = 1$ in Γ, then $k = 0$ and $f = 1$ in Γ. The former observation shows that $w = f = 1$ in Γ_n. On the other hand, if $w \neq 1$ in Γ, we distinguish two cases:

Case 1: $k \neq 0$

The image of f in Γ_n has support in $\{-N+1, \dots N\}$. Thus it stabilizes $N+1$ ($n \geqslant 3N$) so that the image of w sends $N+1$ to $N+1+k$. As $n \geqslant 2|k|$, we have $N+1+k \not\equiv N+1$ modulo n. Hence $w \neq 1$ in Γ_n.

Case 2: $k = 0$

We have necessarily $f \neq 1$ in Γ, which implies $w = f \neq 1$ in Γ_n. $\square$

3 Shortest relations in a HNN-extension and convergence of Baumslag-Solitar groups

Let $H = \langle X \mid \mathcal{R} \rangle$ and $\Gamma = HNN(H, A, B, \phi)$. In this section we give a lower estimate for g_Γ. As a higher estimate, we obviously get $g_\Gamma \leqslant g_H$, because a shortest relation in H is also a relation in Γ. Let us define:

$$\begin{aligned}\alpha &:= \min\left\{\ell_H(a) \ : \ a \in A\setminus\{1\}\right\} ; \\ \beta &:= \min\left\{\ell_H(b) \ : \ b \in B\setminus\{1\}\right\} .\end{aligned}$$

Theorem 5 *Let $H = \langle X \mid \mathcal{R}\rangle$ and $\Gamma = HNN(H, A, B, \phi)$. Let α and β be defined as above. Then we have*

$$\min\{g_H,\ \alpha+\beta+2,\ 2\alpha+6,\ 2\beta+6\} \leqslant g_\Gamma \leqslant g_H \ .$$

As the case of Baumslag-Solitar groups (treated below) will show, the lower bound given in Theorem 5 is in fact sharp. Before proving this Theorem, let us begin with a simpler observation.

Lemma 2 *Let $H = \langle X \mid \mathcal{R}\rangle$, $\Gamma = HNN(H, A, B, \phi)$ and r be a relation of Γ contained in $\mathbb{F}_X$. Then r is a relation of H. In particular, $\ell(r) \geqslant g_H$.*

Proof. Since $r = 1$ in Γ and since the canonical map $H \to \Gamma$ is injective, we get $r = 1$ in H. Hence the first assertion. The second one follows by definition of g_H. $\square$

Proof of Theorem 5. The second inequality has already been discussed. To establish the first one, let us take a relation r of Γ and show that $\ell(r) \geqslant m$, where we set $m := \min\{g_H,\ \alpha+\beta+2,\ 2\alpha+6,\ 2\beta+6\}$.

Write $r = h_0 t^{\varepsilon_1} h_1 \dots t^{\varepsilon_n} h_n$ with $\varepsilon_i \in \{\pm 1\}$, $h_i \in \mathbb{F}_X$ and $h_i \neq 1$ if $\varepsilon_i = -\varepsilon_{i+1}$. Up to replacement by a (shorter) conjugate, we may assume that r is cyclically reduced. If $n \neq 0$, we may also assume that $h_0 = 1$. Since $r = 1$ in Γ, one clearly has $\sum_{i=1}^n \varepsilon_i = 0$. In particular, n is even. Let us distinguish several cases and show $\ell(r) \geqslant m$ in each one:

Case $n = 0$**:** We get $r = h_0 \in \mathbb{F}_X$. Thus $\ell(r) \geqslant g_H \geqslant m$ by lemma 2.

Case $n = 2$**:** One gets $r = t^{\varepsilon} h_1 t^{-\varepsilon} h_2$. If we look at r in Γ, we have $r = 1$ and thus $h_1 \in A$ (if $\varepsilon = -1$) or $h_1 \in B$ (if $\varepsilon = 1$) by Britton's lemma. Suppose $\varepsilon = -1$ (in case $\varepsilon = 1$ the proof is similar and left to the reader). Looking at h_1 in $\mathbb{F}_X$, there are two possibilities (remember that we assumed $h_1 \neq 1$).

- If $h_1 = 1$ in Γ, lemma 2 implies $\ell(r) \geqslant \ell(h_1) \geqslant g_H \geqslant m$.
- If $h_1 \neq 1$ in Γ, then $\ell(h_1) \geqslant \alpha$. On the other hand $h_2^{-1} = t^{-1} h_1 t \in B$; thus $\ell(h_2) \geqslant \beta$ and $\ell(r) \geqslant \alpha + \beta + 2 \geqslant m$.

Case $n \geqslant 4$**:** We have $r = 1$ in Γ. By Britton's lemma, there is an index i such that either $\varepsilon_i = -1 = -\varepsilon_{i+1}$ and $h_i \in A$, or $\varepsilon_i = +1 = -\varepsilon_{i+1}$ and $h_i \in B$. Since cyclic conjugations preserve length, we may assume $i = 1$, so that $r = t^{\varepsilon_1} h_1 t^{-\varepsilon_1} h_2 t^{\varepsilon_3} h_3 \dots t^{\varepsilon_n} h_n$. Let us moreover assume $\varepsilon_1 = -1$ (again the case $\varepsilon_1 = 1$ is similar and left to the reader). Set $r' = w h_2 t^{\varepsilon_3} h_3 \dots t^{\varepsilon_n} h_n$ where $w \in F_X$ is such that $w = t^{-1} h_1 t$ in Γ (in fact this element is in B). Applying Britton's lemma to r', one sees there exists an index $j \geqslant 3$ such that either $\varepsilon_j = -1 = -\varepsilon_{j+1}$ and $h_j \in A$, or $\varepsilon_j = 1 = -\varepsilon_{j+1}$ and $h_j \in B$. There are three possibilities:

- If $h_1 = 1$ or $h_j = 1$ in Γ, lemma 2 implies $\ell(r) \geqslant g_H \geqslant m$ as above.
- If $h_1 \neq 1$ in Γ, $h_j \neq 1$ in Γ and $\varepsilon_j = 1$, then $\ell(h_1) \geqslant \alpha$ and $\ell(h_j) \geqslant \beta$. Thus $\ell(r) \geqslant \alpha + \beta + 4 > m$.
- If $h_1 \neq 1$ in Γ, $h_j \neq 1$ in Γ and $\varepsilon_j = -1$, then we can write $r = t^{-1} h_1 t w_1 t^{-1} h_j t w_2$ with $\ell(h_1) \geqslant \alpha$ and $\ell(h_j) \geqslant \alpha$. The subwords w_1, w_2 are not empty because r is cyclically reduced. Thus $\ell(r) \geqslant 2\alpha + 6 \geqslant m$ (Remark that $2\alpha + 6$ would be replaced by $2\beta + 6$ in the case $\varepsilon_1 = 1, \varepsilon_j = 1$).

The proof is complete. $\square$

We now turn to prove that for Baumslag-Solitar groups, the lower bound coming from Theorem 5 is in fact the length of the shortest relation. More precisely we have the following statement:

Proposition 2 *Let $p, q \in \mathbb{Z}^*$. We have*

$$g_{BS(p,q)} = \min\left\{|p| + |q| + 2,\ 2|p| + 6,\ 2|q| + 6\right\}.$$

Proof. Set $m := \min\{|p| + |q| + 2,\ 2|p| + 6,\ 2|q| + 6\}$ and $\Gamma = BS(p, q)$. We have $g_{\mathbb{Z}} = +\infty$, $\alpha = |q|$ and $\beta = |p|$. Thus, Theorem 5 implies $g_\Gamma \geqslant m$. To prove that $g_\Gamma \leqslant m$, we produce relations of length $|p| + |q| + 2$, $2|p| + 6$, and $2|q| + 6$. Namely:

- $a b^p a^{-1} b^{-q}$ has length $|p| + |q| + 2$;
- $a b^p a^{-1} b a b^{-p} a^{-1} b^{-1}$ has length $2|p| + 6$;

- $a^{-1}b^q aba^{-1}b^{-q}ab^{-1}$ has length $2|q|+6$. $\square$

Theorem 1 of introduction is now a consequence of Proposition 2, since a sequence of groups $\Gamma_n = \langle a, b \mid \mathcal{R}_n \rangle$ converges to the free group on two generators (marked by its canonical basis) if and only if g_{Γ_n} tends to ∞.

4 One parameter families of Baumslag-Solitar groups

In this section, the purpose is to prove Theorems 2, 3 and 4 of introduction. We begin with the case $p = \pm 1$, which is easier.

Proof of Theorem 2. We remark first that $BS(1,q) = BS(-1,-q)$ as marked groups. Thus we may assume $p = 1$. Hence, we let $\Gamma_q = BS(p,q)$ and $\Gamma = \mathbb{Z} \wr \mathbb{Z}$. In $\mathbb{Z} \wr \mathbb{Z}$, let us set $a = (1,0)$ and $b = (0, e_0)$. We have to show that for all $w \in \mathbb{F}_2$:

(1) if $w = 1$ in $\mathbb{Z} \wr \mathbb{Z}$, then $w = 1$ in $BS(1,q)$ for $|q|$ big enough;

(2) if $w \neq 1$ in $\mathbb{Z} \wr \mathbb{Z}$, then $w \neq 1$ in $BS(1,q)$ for $|q|$ big enough.

Let $w \in \mathbb{F}_2$. One can write $w = a^{\alpha} a^{\alpha_1} b^{\beta_1} a^{-\alpha_1} \ldots a^{\alpha_k} b^{\beta_k} a^{-\alpha_k}$. The image of w in Γ is $(\alpha, \sum_{i=1}^{k} \beta_i e_{\alpha_i})$, where $e_j \in \oplus_{h \in \mathbb{Z}} \mathbb{Z}$ is the Dirac mass at j. Let $m = \min_{1 \leqslant i \leqslant k} \alpha_i$. In $\Gamma_q = BS(1,q)$, we have

$$\begin{aligned} w &= a^{\alpha} a^{m} a^{\alpha_1 - m} b^{\beta_1} a^{m-\alpha_1} \ldots a^{\alpha_k - m} b^{\beta_k} a^{m-\alpha_k} a^{-m} \\ &= a^{\alpha} a^{m} b^{\beta_1 q^{\alpha_1 - m}} \ldots b^{\beta_k q^{\alpha_k - m}} a^{-m} \\ &= a^{\alpha} a^{m} b^{\sum_{h \in \mathbb{Z}} \left(\sum_{\alpha_i = h} \beta_i \right) q^{h-m}} a^{-m} \end{aligned}$$

(1) As $w \underset{\Gamma}{=} 1$, we have $\alpha = 0$ and $\forall h \in \mathbb{Z}$, $\sum_{\alpha_i = h} \beta_i = 0$. Hence

$$w \underset{\Gamma_q}{=} a^0 a^m b^{\sum_{h \in \mathbb{Z}} 0 \cdot q^{h-m}} a^{-m} = 1 \ \forall q \in \mathbb{Z}^* \ .$$

(2) As $w \underset{\Gamma}{\neq} 1$, either $\alpha \neq 0$ or $\exists h \in \mathbb{Z}$ such that $\sum_{\alpha_i = h} \beta_i \neq 0$. The image of w by the morphism $\Gamma_q \to \mathbb{Z}$ given by $a \mapsto 1, b \mapsto 0$ is α. Hence, if $\alpha \neq 0$,

then $w \underset{\Gamma_q}{\neq} 1$ $\forall q$. If $\alpha = 0$, we set h_0 to be the maximal value of h such that $\sum_{\alpha_i=h} \beta_i \neq 0$. For $|q|$ big enough, we have

$$\left| \sum_{\alpha_i=h_0} \beta_i q^{h_0-m} \right| > \left| \sum_{h<h_0} \sum_{\alpha_i=h} \beta_i q^{h-m} \right| .$$

For those values of q, we get

$$w \underset{\Gamma_q}{=} a^m b^{\left(\sum_{\alpha_i=h_0} \beta_i\right) q^{h_0-m} + \sum_{h<h_0} \left(\sum_{\alpha_i=h} \beta_i\right) q^{h-m}} a^{-m} \underset{\Gamma_q}{\neq} 1 .$$

The proof is complete. $\square$

We now treat the case $|p| \geqslant 2$. More precisely, we begin the proof of Theorem 3. We also have $BS(p,q) = BS(-p,-q)$ as marked groups. This equality will allow us to assume $p > 0$ in following proofs. We begin with a lemma which already shows that the sequence $(BS(p,q))_q$ is not itself convergent.

Lemma 3 *Let $\tilde{p}, \tilde{q} \in \mathbb{Z}^*$, $d = gcd(\tilde{p}, \tilde{q})$. We write $\tilde{p} = dp$, $\tilde{q} = dq$. Let $k \in \mathbb{Z}$, $h \geqslant 1$ and*

$$w = a^{h+1} b^{\tilde{p}} a^{-1} b^{-k} a^{-h} b a^{h+1} b^{-\tilde{p}} a^{-1} b^{k} a^{-h} b^{-1} .$$

If $|\tilde{q}| \geqslant 2$, we have $w = 1$ in $BS(\tilde{p}, \tilde{q})$ if and only if $\tilde{q} \equiv k$ (mod $p^h d$).

The congruence modulo $p^h d$ (instead of $\tilde{p}^h$) is the reason for the hypothesis "p relatively prime to q" appearing in Theorem 3.

Proof. Let $\Gamma_{\tilde{q}} = BS(\tilde{p}, \tilde{q})$. We have

$$w \underset{\Gamma_{\tilde{q}}}{=} a^h b^{\tilde{q}-k} a^{-h} b a^h b^{k-\tilde{q}} a^{-h} b^{-1} .$$

Let us now distinguish three cases:

Case 1: $\tilde{q} \not\equiv k$ (mod $\tilde{p}$).

We have $w \underset{\Gamma_{\tilde{q}}}{\neq} 1$ by Britton's lemma, since $|\tilde{q}| \geqslant 2$.

Case 2: $\tilde{q} \not\equiv k$ (mod $p^h d$), but $\tilde{q} \equiv k$ (mod $\tilde{p}$).

We write $\tilde{q} - k = np^g d$ with $g < h$ and n not a multiple of p. Hence $nq^g d$ is not a multiple of $\tilde{p} = pd$, for p is relatively prime to q. We have

$$w \underset{\Gamma_{\tilde{q}}}{=} a^{h-g} b^{nq^g d} a^{g-h} b a^{h-g} b^{-nq^g d} a^{g-h} b^{-1} \underset{\Gamma_{\tilde{q}}}{\neq} 1$$

by Britton's lemma (again because $|\tilde{q}| \geqslant 2$).

Case 3: $\tilde{q} \equiv k \pmod{p^h d}$.

Let us write $\tilde{q} - k = np^h d$. Then $w \underset{\Gamma_{\tilde{q}}}{=} b^{nq^h d} b b^{-nq^h d} b^{-1} \underset{\Gamma_{\tilde{q}}}{=} 1$. □

Proof of Theorem 3. The "if" is a particular case of Theorem 6 below.

We prove now the "only if" part. Let $\Gamma_n = BS(p, q_n)$. We assume the sequence $(\Gamma_n)_n$ to converge and condition (a) not to hold. We have to show that condition (b) holds.

Fix $h \geqslant 1$. For $k \in \mathbb{Z}$ we set

$$w_k = a^{h+1} b^p a^{-1} b^{-k} a^{-h} b a^{h+1} b^p a^{-1} b^k a^{-h} b^{-1} \ .$$

As $(\Gamma_n)_n$ converges, we have (for each $k \in \mathbb{Z}$) either $w_k \underset{\Gamma_n}{=} 1$ for n big enough, or $w_k \underset{\Gamma_n}{\neq} 1$ for n big enough. As q_n is relatively prime to p for all n, lemma 3 ensures that (for each $k \in \mathbb{Z}$) either $q_n \equiv k \pmod{p^h}$ for n big enough, or $q_n \not\equiv k \pmod{p^h}$ for n big enough. This implies that q_n is eventually constant modulo p^h ($\forall h \geqslant 1$).

It remains to show that $|q_n| \to \infty$. Assume by contradiction that there exists some $\ell \in \mathbb{Z}$ such that $q_n = \ell$ for infinitely many n. As (a) does not hold (i.e $(q_n)_n$ is not eventually constant), it is sufficient to treat the two following cases:

Case 1: $\exists \ell' \neq \ell$ such that $q_n = \ell'$ for infinitely many n.

Take h big enough so that $p^h > |\ell - \ell'|$. The sequence q_n cannot be eventually constant modulo p^h, in contradiction with the first part of the proof.

Case 2: $\exists$ a subsequence $(q_{n_j})_j$ of $(q_n)_n$ such that $|q_{n_j}| \to \infty$.

We set $w = ab^p a^{-1} b^{-\ell}$. For infinitely many n (those values for which $q_n = \ell$), we have $w \underset{\Gamma_n}{=} 1$. On the other hand $|q_{n_j}| > \ell$ for j big enough. For these values of j, we have

$$w \underset{\Gamma_{n_j}}{=} b^{q_{n_j} - \ell} \underset{\Gamma_{n_j}}{\neq} 1 \ .$$

This contradicts the assumption on the sequence $(\Gamma_n)_n$ to converge. □

What remains now to do is to prove the following Theorem, which is a little bit more general than the "if" part of Theorem 3. The proof will need some preliminary lemmas.

Theorem 6 *Let $p \in \mathbb{Z}^*$ and let $(q_n)_n$ be a sequence in $\mathbb{Z}^*$. If $|q_n| \to \infty$ and if $\forall h \geqslant 1$ the sequence $(q_n)_n$ is eventually constant modulo p^h, then the sequence $(BS(p, q_n))_n$ is convergent in $\mathcal{G}_2$.*

Lemma 4 *Let $p, q, q' \in \mathbb{Z}^*$ and $h \geqslant 1$. If $q \equiv q' \pmod{p^h}$, there exists $s_0, \ldots, s_h$; $s'_0, \ldots, s'_h$; $r_1, \ldots, r_h$, which are unique, such that:*

(i) $0 \leqslant r_i < p \ \forall i$; $s_0 = 1 = s'_0$;

(ii) $s_{i-1}q = s_i p + r_i$ and $s'_{i-1}q' = s'_i p + r_i \ \forall\, 1 \leqslant i \leqslant h$;

(iii) $s_i \equiv s'_i \pmod{p^{h-i}} \ \forall\, 0 \leqslant i \leqslant h$.

Proof. Given the congruence $q \equiv q' \pmod{p^h}$, we obtain (by Euclidean division) $s_0 q = q = s_1 p + r_1$ and $s'_0 q' = q' = s'_1 p + r_1$ with $0 \leqslant r_1 \leqslant p$ and $s_1 \equiv s'_1 \pmod{p^{h-1}}$. Hence we have $s_1 q \equiv s'_1 q' \pmod{p^{h-1}}$. (Let us emphasize that we do not necessary have $s_1 q \equiv s'_1 q' \pmod{p^h}$.)

Now, it just remains to iterate the above and uniqueness follows from construction. $\square$

Given a word w in $\mathbb{F}_2$, we may use Britton's lemma to reduce it in $BS(p, q)$ or $BS(p, q')$. But w could be reducible in one of these groups and not in the other one. Even if it is reducible in both groups the result is not the same word in general. The purpose of next statement is, under some assumptions, to control the parallel process of reduction in both groups. This will be useful to ensure that w is a relation in $BS(p, q)$ if and only if it is one in $BS(p, q')$ (under some assumptions).

Lemma 5 *Let $p, q, q' \in \mathbb{Z}^*$ and $h \geqslant m \geqslant 1$. Assume that $q \equiv q' \pmod{p^h}$ and let*

$$\begin{aligned} \alpha &= k_0 + k_1 q + k_2 s_1 q + \ldots + k_m s_{m-1} q \\ \alpha' &= k_0 + k_1 q' + k_2 s'_1 q' + \ldots + k_m s'_{m-1} q' \end{aligned}$$

where $|k_0| < \min(|q|, |q'|)$ and $s_0, \ldots, s_h$; $s'_0, \ldots, s'_h$; $r_1, \ldots, r_h$ are given by lemma 4.

(i) We have $\alpha \equiv 0 \pmod{p}$ *if and only if* $\alpha' \equiv 0 \pmod{p}$*. If this happens we get* $ab^{\alpha}a^{-1} \underset{BS(p,q)}{=} b^{\beta}$ *and* $ab^{\alpha'}a^{-1} \underset{BS(p,q')}{=} b^{\beta'}$ *with*

$$\begin{aligned} \beta &= \ell_1 q + \ell_2 s_1 q + \ldots + \ell_{m+1} s_m q \\ \beta' &= \ell_1 q' + \ell_2 s_1' q' + \ldots + \ell_{m+1} s_m' q' \ . \end{aligned}$$

(ii) We have $\alpha \equiv 0 \pmod{q}$ *if and only if* $\alpha' \equiv 0 \pmod{q'}$*. If this happens we get* $a^{-1}b^{\alpha}a \underset{BS(p,q)}{=} b^{\beta}$ *and* $a^{-1}b^{\alpha'}a \underset{BS(p,q')}{=} b^{\beta'}$ *with*

$$\begin{aligned} \beta &= \ell_0 + \ell_1 q + \ell_2 s_1 q + \ldots + \ell_{m-1} s_{m-2} q \\ \beta' &= \ell_0 + \ell_1 q' + \ell_2 s_1' q' + \ldots + \ell_{m-1} s_{m-2}' q' \ . \end{aligned}$$

Proof. (i) We have $\alpha \equiv \alpha' \pmod{p}$ by construction. Assume now that $\alpha \equiv 0 \equiv \alpha' \pmod{p}$. We have

$$\alpha = k_0 + k_1 r_1 + \ldots + k_m r_m + k_1 s_1 p + \ldots + k_m s_m p \ .$$

As $\alpha \equiv 0 \pmod{p}$, we obtain $ab^{\alpha}a^{-1} \underset{BS(p,q)}{=} b^{\beta}$ with

$$\beta = \frac{q}{p}(k_0 + k_1 r_1 + \ldots + k_m r_m) + k_1 s_1 q + \ldots + k_m s_m q$$

Thus we set $\ell_1 = \frac{1}{p}(k_0 + k_1 r_1 + \ldots + k_m r_m)$ and $\ell_i = k_{i-1}$ for $2 \leqslant i \leqslant m+1$, and doing the same calculation with α' in $BS(p,q')$, we obtain also

$$\beta' = \ell_1 q' + \ell_2 s_1' q' + \ldots + \ell_{m+1} s_m' q' \ .$$

(ii) As $|q| > |k_0|$ and $|q'| > |k_0|$, we have $\alpha \equiv 0 \pmod{q}$ if and only if $k_0 = 0$ if and only if $\alpha' \equiv 0 \pmod{q'}$. Suppose now that it is the case. We have $ab^{\alpha}a^{-1} \underset{BS(p,q)}{=} b^{\beta}$ with

$$\begin{aligned} \beta &= k_1 p + k_2 s_1 p + \ldots + k_m s_{m-1} p \\ &= k_1 p - k_2 r_1 - \ldots - k_m r_{m-1} + k_2 q + k_3 s_1 q + \ldots + k_m s_{m-2} q \ . \end{aligned}$$

Hence we set $\ell_0 = k_1 p - k_2 r_1 - \ldots - k_m r_{m-1}$ and $\ell_i = k_{i+1}$ for $1 \leqslant i \leqslant m-1$. Again, doing the same calculation with α' in $BS(p,q')$, we obtain also

$$\beta' = \ell_0 + \ell_1 q' + \ell_2 s_1' q' + \ldots + \ell_{m-1} s_{m-2}' q' \ .$$

This completes the proof. $\square$

Lemma 6 *Let $p \in \mathbb{Z}^*$ and let $(q_n)_n$ be a sequence in $\mathbb{Z}^*$ such that $|q_n| \to \infty$ and $\forall h \geqslant 1$ $(q_n)_n$ is eventually constant modulo p^h. Let $w \in \mathbb{F}_2$, which we consider as an element of $BS(p,q)$. We have either $w \neq 1$ for n big enough, or w is in the subgroup generated by b for n big enough.*

Proof. We define $\Gamma_n = BS(p, q_n)$. Let us write $w = b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \dots a^{\varepsilon_m} b^{\alpha_m}$ with $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathbb{Z}$ and assume the first term of the alternative not to hold, i.e. $w = 1$ in Γ_n for infinitely many n. Then the sum $\varepsilon_1 + \dots + \varepsilon_m$ has clearly to be zero (in particular m is even). We have to show that $w = b^{\lambda_n}$ for n big enough.

For n big enough, we may assume that, $|q_n| > |\alpha_j|$ for all $1 \leqslant j \leqslant m$ and the q_n's are all congruent modulo p^m. We take a value of n such that moreover $w = 1$ in Γ_n (there are infinitely many ones) and apply Britton's lemma. This ensures the existence of an index j such that $\varepsilon_j = 1 = -\varepsilon_{j+1}$ and $\alpha_j \equiv 0 \pmod p$ (since $|q_n| > |\alpha_j|$ for all j). By lemma 5, for all n big enough

$$w \underset{\Gamma_n}{=} b^{\alpha_0} \dots a^{\varepsilon_{j-1}} b^{\alpha_{j-1}+\beta_j+\alpha_{j+1}} a^{\varepsilon_{j+2}} \dots b^{\alpha_m}$$

with $\beta_j = \ell_1 q_n$ (depending on n). Hence we are allowed to write

$$w \underset{\Gamma_n}{=} b^{\alpha'_{0,n}} a^{\varepsilon'_1} b^{\alpha'_{1,n}} \dots a^{\varepsilon'_{m-2,n}} b^{\alpha'_{m-2,n}}$$

for n big enough, with $\varepsilon'_i = \pm 1$ and $\alpha'_{j,n} = k'_{0,j} + k'_{1,j} q_n$, where the $k'_{i,j}$'s do not depend on n.

Now, for n big enough, we may assume that, $|q_n| > |k'_{0,j}|$ for all $1 \leqslant j \leqslant m-1$ (and the q_n's are all congruent modulo p^m). Again we take a value of n such that moreover $w = 1$ in Γ_n and apply Britton's lemma. This ensures the existence of an index j such that either $\varepsilon'_j = 1 = -\varepsilon'_{j+1}$ and $\alpha'_{j,n} \equiv 0 \pmod p$, or $\varepsilon'_j = -1 = -\varepsilon'_{j+1}$ and $\alpha'_{j,n} \equiv 0 \pmod{q_n}$. In both cases, while applying lemma 5, we obtain

$$w \underset{\Gamma_n}{=} b^{\alpha''_{0,n}} a^{\varepsilon''_1} b^{\alpha''_{1,n}} \dots a^{\varepsilon''_{m-4,n}} b^{\alpha''_{m-4,n}}$$

for n big enough, with $\varepsilon''_i = \pm 1$ and $\alpha''_{j,n} = k''_{0,j} + k''_{1,j} q_n + k''_{2,j} s_{1,n} q_n$, where the $k''_{i,j}$'s do not depend on n.

And so on, and so forth, setting $m' = \frac{m}{2}$, we get finally $w = b^{\alpha_{0,n}^{(m')}}$ in Γ_n for n big enough, with

$$\alpha_{0,n}^{(m')} = k_{0,0}^{(m')} + k_{1,0}^{(m')} q_n + k_{2,0}^{(m')} s_{1,n} q_n + \ldots + k_{m',0}^{(m')} s_{m'-1,n} q_n$$

where the $k_{i,0}^{m'}$'s do not depend on n. It only remains to set $\lambda_n = \alpha_{0,n}^{(m')}$. $\square$

Let us now introduce the homomorphisms $\psi_q : BS(p,q) \to \mathrm{Aff}(\mathbb{R})$ (for $q \in \mathbb{Z}^*$) given by $\psi_q(a)(x) = \frac{q}{p}x$ and $\psi_q(b)(x) = x + 1$.

Lemma 7 *Let $w \in \mathbb{F}_2$. We have either $\psi_q(w) = 1$ for $|q|$ big enough or $\psi_q(w) \neq 1$ for $|q|$ big enough.*

Proof. Let us write $w = b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \ldots a^{\varepsilon_k} b^{\alpha_k}$ with $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathbb{Z}$. Set next $\sigma_0 = 0$, $\sigma_i = \varepsilon_1 + \ldots + \varepsilon_i$ for $1 \leqslant i \leqslant k$ and $m = \min_{0 \leqslant i \leqslant k} \sigma_i$. We get by calculation that

$$\psi_q(w)(x) = \left(\frac{q}{p}\right)^{\sigma_k} x + \left(\frac{q}{p}\right)^{m} P_w\left(\frac{q}{p}\right)$$

where P_w is the polynomial defined by $P_w(y) = \sum_{i=0}^{k} \alpha_i y^{\sigma_i - m}$. Let us assume the second term of alternative not to hold, i.e. $\psi_q(w) = 1$ for infinitely many values of q. Hence we have $\sigma_k = 0$ and for all those values of q, $P_w(\frac{q}{p}) = 0$. As P_w is a polynomial with infinitely many roots, it is the zero polynomial. This shows that $\psi_q(w) = 1$ for all q. $\square$

Proof of Theorem 6. It is easy to show that a word w is equal to 1 in $BS(p,q)$ if and only if it is in the subgroup generated by b and $\psi_q(w) = 1$. It is also a consequence of (the proof of) Theorem 1 in [GJ03]. Let $w \in \mathbb{F}_2$. Lemmas 6 and 7 immediately imply that either $w = 1$ in $BS(p, q_n)$ for n big enough or $w \neq 1$ in $BS(p, q_n)$ for n big enough. $\square$

Theorem 3 is now completely established. To end the paper, we prove Theorem 4. But we before recall that $\mathbb{Z}_p$ (for $p \in \mathbb{Z}$, $|p| \geqslant 2$) is the completion of $\mathbb{Z}$ with respect to the ultrametric given by the following absolute value:

$$|a \cdot p^m|_p := \left(\frac{1}{|p|}\right)^{-m} \quad \text{for } a \text{ relatively prime to } p \text{ and } m \geqslant 0\ .$$

Let us also mention that we have $\mathbb{Z}_p^{\times} = \mathbb{Z}_p \setminus p\mathbb{Z}_p$.

Proof of Theorem 4. We show first that the map Ψ is uniformly continuous (endowing $\mathbb{Z}$ with the above ultrametric). In view of the distances we put on $\mathcal{G}_2$ and $\mathbb{Z}$, it is equivalent to show that for any $h \geqslant 1$ there exists $r \geqslant 1$ such that we have $q \equiv q' \pmod{p^h}$ whenever $BS(p,q)$ and $BS(p,q')$ have the same relations up to length r.

Fix $h \geqslant 1$. Our candidate is $r = 4h + 4p + 6$. Assume that $BS(p,q)$ and $BS(p,q')$ have the same relations up to length r. For $0 \leqslant k \leqslant p-1$ let

$$w_k = a^{h+1}b^pa^{-1}b^{-k}a^{-h}ba^{h+1}b^pa^{-1}b^ka^{-h}b^{-1} .$$

(Remark that these words are exactly those which appear in the proof of the "only if" part of theorem 3. We are in fact improving this proof in order to get the uniform continuity.) We have $\ell(w_k) \leqslant r$ for all k. Having by assumption $w = 1$ in $BS(p,q)$ if and only if $w = 1$ in $BS(p,q')$, lemma 3 implies $q \equiv q' \pmod{p^h}$.

The space $\mathbb{Z}_p^\times$ being complete and the uniform continuity of Ψ being now proved, the existence of the uniformly continuous extension $\overline{\Psi}$ is a standard fact (see [Dug70], Chapter XIV, Theorem 5.2. for instance).

Let us now show that $\overline{\Psi}$ is surjective. The space $\overline{X_p}$ being compact, $\mathrm{im}(\overline{\Psi})$ is closed in $\mathbb{Z}_p^\times$. Moreover it is dense since it contains the set of q's relatively prime to p and such that $|q| \geqslant 2$.

Finally, we consider the sequence $(BS(p, 1+p+p^n))_n$, which is convergent by theorem 3 and we call the limit Γ. We have $\overline{\Psi}(\Gamma) = 1+p = \overline{\Psi}(BS(p,1+p))$. On the other hand, we have $\Gamma \neq BS(1+p)$, since $ab^pa^{-1}b^{-(p+1)} = 1$ in $BS(p,1+p)$ while $ab^pa^{-1}b^{-(p+1)} \neq 1$ in $BS(p,1+p+p^n)$ for all n. This is the non-injectivity of $\overline{\Psi}$. $\square$

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