HILBERT DOMAINS QUASI-ISOMETRIC TO NORMED VECTOR SPACES

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Abstract. We prove that a Hilbert domain which is quasi-isometric to a normed vector space is actually a convex polytope.

1. Introduction

A *Hilbert domain* in $\mathbf{R}^m$ is a metric space $(\mathcal{C}, d_{\mathcal{C}})$, where $\mathcal{C}$ is an *open bounded convex* set in $\mathbf{R}^m$ and $d_{\mathcal{C}}$ is the distance function on $\mathcal{C}$ — called the *Hilbert metric* — defined as follows.

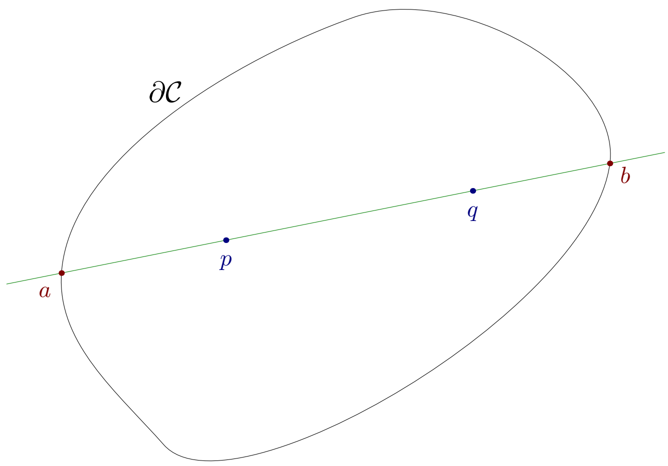
Given two distinct points p and q in $\mathcal{C}$, let a and b be the intersection points of the straight line defined by p and q with $\partial\mathcal{C}$ so that $p = (1-s)a + sb$ and $q = (1-t)a + tb$ with $0 < s < t < 1$. Then

$$d_{\mathcal{C}}(p,q) \coloneqq \frac{1}{2}\ln[a,p,q,b],$$

where

$$[a,p,q,b] \coloneqq \frac{1-s}{s} \times \frac{t}{1-t} > 1$$

is the cross ratio of the 4-tuple of ordered collinear points (a,p,q,b).
We complete the definition by setting $d_{\mathcal{C}}(p,p) \coloneqq 0$.



The metric space $(\mathcal{C}, d_{\mathcal{C}})$ thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of $\mathbf{R}^m$ and in which the affine open segments joining two points of the boundary $\partial\mathcal{C}$ are geodesics that are isometric to $(\mathbf{R}, |\cdot|)$.

For further information about Hilbert geometry, we refer to [3, 4, 8, 10] and the excellent introduction [14] by Socié-Méthou.

The two fundamental examples of Hilbert domains $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbf{R}^m$ correspond to the case when $\mathcal{C}$ is an ellipsoid, which gives the Klein model of m-dimensional hyperbolic geometry (see for

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example [14, first chapter]), and the case when $\overline{\mathcal{C}}$ is a m-simplex, for which there exists a norm $\|\cdot\|_{\mathcal{C}}$ on $\mathbf{R}^m$ such that $(\mathcal{C}, d_{\mathcal{C}})$ is isometric to the normed vector space $(\mathbf{R}^m, \|\cdot\|_{\mathcal{C}})$ (see [7, pages 110–113] or [13, pages 22–23]).

Much has been done to study the similarities between Hilbert and hyperbolic geometries (see for example [6], [15] or [1]), but little literature deals with the question of knowing to what extend a Hilbert geometry is close to that of a normed vector space. So let us mention two results in this latter direction which are relevant for our present work.

Theorem 1.1 ([9], Theorem 2)**.** *A Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbf{R}^m$ is isometric to a normed vector space if and only if $\mathcal{C}$ is the interior of a m-simplex.*

Theorem 1.2 ([5], Theorem 3.1)**.** *If $\mathcal{C}$ is an open bounded convex polygonal set in $\mathbf{R}^2$, then $(\mathcal{C}, d_{\mathcal{C}})$ is Lipschitz equivalent to Euclidean plane.*

In light of these two results, it is natural to ask whether the converse of Theorem 1.2 holds. In other words, if a Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbf{R}^m$ is quasi-isometric to a normed vector space, what can be said about $\mathcal{C}$? Here, by *quasi-isometric* we mean the following (see [2]):

Definition 1.1. Given real numbers $A \geqslant 1$ and $B \geqslant 0$, a metric space (S, d) is said to be (A, B)-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ if and only if there exists a map $f : S \longrightarrow V$ such that

$$\frac{1}{A} d(p, q) - B \leqslant \|f(p) - f(q)\| \leqslant A d(p, q) + B$$

for all $p, q \in S$.

Recalling that a convex *polytope* in $\mathbf{R}^m$ is the convex hull of a finite set of points whose linear span is the whole space $\mathbf{R}^m$, we then show

Theorem 1.3. *If a Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbf{R}^m$ is (A, B)-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geqslant 1$ and $B \geqslant 0$, then $\mathcal{C}$ is the interior of a convex polytope.*

2. Proof of Theorem 1.3

The proof of Theorem 1.3 is based on an idea developed by Förtsch and Karlsson in their paper [9].

It needs the following fact due to Karlsson and Noskov:

Theorem 2.1 ([11], Theorem 5.2)**.** *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a Hilbert domain in $\mathbf{R}^m$ and $x, y \in \partial\mathcal{C}$ such that $[x, y] \nsubseteq \partial\mathcal{C}$. Then, given a point $p_0 \in \mathcal{C}$, there exists a constant $K = K(p_0, x, y) > 0$ satisfying*

$$d_{\mathcal{C}}(x_n, y_n) \geqslant d_{\mathcal{C}}(x_n, p_0) + d_{\mathcal{C}}(x_n, p_0) - K$$

for all sequences $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ in $\mathcal{C}$ that converge respectively to x and y in $\mathbf{R}^m$.

Now, here is the key result which gives the proof of Theorem 1.3:

Proposition 2.1. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be a Hilbert domain in $\mathbf{R}^m$ which is (A, B)-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geqslant 1$ and $B \geqslant 0$.*
Then, if $N = N(A, \|\cdot\|)$ denotes the maximum number of points in the ball $\{v \in V \mid \|v\| \leqslant 2A\}$ whose pairwise distances with respect to $\|\cdot\|$ are greater than or equal to $1/(2A)$, and if $X \subseteq \partial\mathcal{C}$ is such that $[x, y] \nsubseteq \partial\mathcal{C}$ for all $x, y \in X$ with $x \neq y$, we have

$$\mathrm{card}(X) \leqslant N.$$

Proof.
Let $f : \mathcal{C} \longrightarrow V$ such that

$$\frac{1}{A} d_{\mathcal{C}}(p, q) - B \leqslant \|f(p) - f(q)\| \leqslant A d_{\mathcal{C}}(p, q) + B \tag{2.1}$$

for all $p, q \in \mathcal{C}$.

First of all, up to translations, we may assume that $0 \in \mathcal{C}$ and $f(0) = 0$.

Then suppose that there exists a subset X of the boundary $\partial\mathcal{C}$ such that $[x, y] \nsubseteq \partial\mathcal{C}$ for all $x, y \in X$ with $x \neq y$ and $\mathrm{card}(X) \geqslant N + 1$. So, pick $N + 1$ distinct points $x_1, \ldots, x_{N+1}$ in X, and for each $k \in \{1, \ldots, N + 1\}$, let $\gamma_k : [0, +\infty) \longrightarrow \mathcal{C}$ be a geodesic of $(\mathcal{C}, d_{\mathcal{C}})$ that satisfies $\gamma_k(0) = 0$, $\lim\limits_{t \to +\infty} \gamma_k(t) = x_k$ in $\mathbf{R}^m$ and $d_{\mathcal{C}}(0, \gamma_k(t)) = t$ for all $t \geqslant 0$.
This implies that for all integers $n \geqslant 1$ and every $k \in \{1, \ldots, N + 1\}$, we have

$$\left\| \frac{f(\gamma_k(n))}{n} \right\| \leqslant A + \frac{B}{n} \tag{2.2}$$

from the second inequality in Equation 2.1 with $p = \gamma_k(n)$ and $q = 0$.
On the other hand, Theorem 2.1 yields

$$d_{\mathcal{C}}(\gamma_i(n), \gamma_j(n)) \geqslant 2n - K(0, x_i, x_j)$$

for all integers $n \geqslant 1$ and every $i, j \in \{1, \ldots, N + 1\}$ with $i \neq j$, and hence

$$\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geqslant \frac{2}{A} - \frac{1}{n}\left(\frac{K(0, x_i, x_j)}{A} - B \right) \tag{2.3}$$

from the first inequality in Equation 2.1 with $p = \gamma_i(n)$ and $q = \gamma_j(n)$.
Now, fixing an integer $n \geqslant \max\{B/A, K(0, x_i, x_j) - B/A\}$, we get

$$\left\| \frac{f(\gamma_k(n))}{n} \right\| \leqslant 2A$$

for all $k \in \{1, \ldots, N + 1\}$ by Equation 2.2 together with

$$\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geqslant \frac{1}{2A}$$

for all $i, j \in \{1, \ldots, N + 1\}$ with $i \neq j$ by Equation 2.3.
But this contradicts the definition of $N = N(A, \|\cdot\|)$.
Therefore, Proposition 2.1 is proved. □

Remark. Given $v \in V$ such that $\|v\| = 2A$, we have $\|-v\| = 2A$ and $\|v - (-v)\| = 2\|v\| = 4A \geqslant 1/(2A)$, which proves that $N \geqslant 2$.

The second ingredient we will need for the proof of Theorem 1.3 is the following:

Proposition 2.2. *Let $\mathcal{C}$ be an open bounded convex set in $\mathbf{R}^2$.*
If there exists a nonempty finite subset Y of the boundary $\partial\mathcal{C}$ such that for every $x \in \partial\mathcal{C}$ one can find $y \in Y$ with $[x, y] \subseteq \partial\mathcal{C}$, then $\overline{\mathcal{C}}$ is a polygon.

Proof.
Assume $0 \in \mathcal{C}$ and let us consider the continuous map $\pi : \mathbf{R} \longrightarrow \partial\mathcal{C}$ which assigns to each $\theta \in \mathbf{R}$ the unique intersection point $\pi(\theta)$ of $\partial\mathcal{C}$ with the half-line $\mathbf{R}_+^*(\cos\theta, \sin\theta)$.

For each pair $(x_1, x_2) \in \partial\mathcal{C} \times \partial\mathcal{C}$, denote by $A(x_1, x_2) \subseteq \partial\mathcal{C}$ the arc segment defined by $A(x_1, x_2) \coloneqq \pi([\theta_1, \theta_2])$, where θ_1 and θ_2 are the unique real numbers such that $\pi(\theta_1) = x_1$ with $\theta_1 \in [0, 2\pi)$ and $\pi(\theta_2) = x_2$ with $\theta_1 \leqslant \theta_2 < \theta_1 + 2\pi$.

We shall prove Proposition 2.2 by induction on $n \coloneqq \mathrm{card}(Y) \geqslant 1$.

Before doing this, notice that adding a point of $\partial\mathcal{C}$ to Y does not change Y's property, and therefore we may assume that $n \geqslant 2$.

If $n = 2$, writting $Y = \{x_1, x_2\}$, we have that for all $x \in \partial\mathcal{C}$, $[x, x_1] \subseteq \partial\mathcal{C}$ or $[x, x_2] \subseteq \partial\mathcal{C}$.

Let $\theta_1 \in [0, 2\pi)$ and $\theta_2 \in [\theta_1, \theta_1 + 2\pi)$ such that $\pi(\theta_1) = x_1$ and $\pi(\theta_2) = x_2$, and define $\alpha_0 \coloneqq \max\{\theta \in [\theta_1, \theta_2] \mid [x_1, \pi(\theta)] \subseteq \partial\mathcal{C}\}$.

Then $A(x_1, x_2) = [x_1, \pi(\alpha_0)] \cup [\pi(\alpha_0), x_2]$. Hence, $A(x_1, x_2)$ is the union of two affine segments, and the same holds for $A(x_2, x_1)$.

Since $\partial\mathcal{C} = A(x_1, x_2) \cup A(x_2, x_1)$, this implies that $\partial\mathcal{C}$ is a union of four affine segments in $\mathbf{R}^2$, and therefore $\overline{\mathcal{C}}$ is a polygon (quadrilateral or triangle).

Now, assume Proposition 2.2 is true for some $n \geqslant 2$, and let Y such that $\mathrm{card}(Y) = n + 1$.

Write $Y = \{x_1, \ldots, x_{n+1}\}$ with $x_1 = \pi(\theta_1)$, $x_2 = \pi(\theta_2), \ldots, x_{n+1} = \pi(\theta_{n+1})$, where $\theta_1 \in [0, 2\pi)$ and $\theta_1 < \theta_2 < \cdots < \theta_{n+1} < \theta_{n+2} \coloneqq \theta_1 + 2\pi$.

Then, defining $x_{n+2} \coloneqq \pi(\theta_{n+2}) = x_1$, we shall consider two cases.

• Case 1: Suppose for every $k \in \{1, \ldots, n + 1\}$ and all $x \in A(x_k, x_{k+1})$, we have $[x, x_k] \subseteq \partial\mathcal{C}$ or $[x, x_{k+1}] \subseteq \partial\mathcal{C}$.

Considering $\alpha_k \coloneqq \max\{\theta \in [\theta_k, \theta_{k+1}] \mid [x_k, \pi(\theta)] \subseteq \partial\mathcal{C}\}$, then we get $A(x_k, x_{k+1}) = [x_k, \pi(\alpha_k)] \cup [\pi(\alpha_k), x_{k+1}]$.

Since $\partial\mathcal{C} = \displaystyle\bigcup_{k=1}^{n+1} A(x_k, x_{k+1})$, this implies that $\partial\mathcal{C}$ is a finite union of affine segments in $\mathbf{R}^2$, and thus $\overline{\mathcal{C}}$ is a polygon.

• Case 2: Suppose there exist $k \in \{1, \ldots, n + 1\}$ — we may assume to be equal to $n + 1$ up to permutations — and $x_0 \in A(x_k, x_{k+1}) = A(x_{n+1}, x_1)$ such that $x_0 \notin \{x_1, x_{n+1}\}$ and $[x_0, x_i] \subseteq \partial\mathcal{C}$ for some $i \in \{2, \ldots, n\}$.

Then $\mathcal{C}$ is contained in one of the two open half-planes in $\mathbf{R}^2$ bounded by the line passing through the points x_0 and x_i, and hence either $A(x_0, x_i) = [x_0, x_i]$, or $A(x_i, x_0) = [x_0, x_i]$.

But $x_1 \in A(x_0, x_i)$ and $x_{n+1} \in A(x_i, x_0)$, so we get either $x_1 \in [x_0, x_i] \subseteq \partial\mathcal{C}$, or $x_{n+1} \in [x_0, x_i] \subseteq \partial\mathcal{C}$. And this implies that Y can be replaced by $Y \smallsetminus \{x_1\}$ or $Y \smallsetminus \{x_{n+1}\}$ in the hypothesis of Proposition 2.2.

Since $\mathrm{card}(Y \smallsetminus \{x_1\}) = \mathrm{card}(Y \smallsetminus \{x_{n+1}\}) = n$, we get that Proposition 2.2 is true for $n + 1$.

This finishes the proof of Proposition 2.2 by induction. □

Before proving Theorem 1.3, let us recall the following useful result, where a convex *polyhedron* in $\mathbf{R}^m$ is the intersection of a finite number of closed half-spaces:

Theorem 2.2 ([12], Theorem 4.7)**.** *Let K be a convex set in $\mathbf{R}^m$ and $p \in \mathring{K}$.*
Then K is a convex polyhedron if and only if all its plane sections containing p are convex polyhedra.

Proof of Theorem 1.3.
Let $(\mathcal{C}, d_{\mathcal{C}})$ be a nonempty Hilbert domain in $\mathbf{R}^m$ that is (A, B)-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geqslant 1$ and $B \geqslant 0$.

According to Theorem 2.2, it suffices to prove Theorem 1.3 for $m = 2$ since any plane section of $\mathcal{C}$ gives rise to a 2-dimensional Hilbert domain which is also (A, B)-quasi-isometric to $(V, \|\cdot\|)$.

So, let $m = 2$, and consider the set $\mathcal{E} \coloneqq \{X \subseteq \partial\mathcal{C} \mid [x, y] \nsubseteq \partial\mathcal{C} \text{ for all } x, y \in X \text{ with } x \neq y\}$.

It is not empty since $\{x, y\} \in \mathcal{E}$ for some $x, y \in \partial\mathcal{C}$ with $x \neq y$ (indeed, $\mathcal{C}$ is a nonempty open set in $\mathbf{R}^2$), which implies together with Proposition 2.1 that $n \coloneqq \max\{\mathrm{card}(X) \mid X \in \mathcal{E}\}$ does exist and satisfies $2 \leqslant n \leqslant N$ (recall that $N \geqslant 2$).

Then pick $Y \in \mathcal{E}$ such that $\mathrm{card}(Y) = n$, write $Y = \{x_1, \ldots, x_n\}$, and prove that for every $x \in \partial\mathcal{C}$ one can find $k \in \{1, \ldots, n\}$ such that $[x, x_k] \subseteq \partial\mathcal{C}$.

Owing to Proposition 2.2, this will show that $\overline{\mathcal{C}}$ is a polygon.

So, suppose that there exists $x_0 \in \partial\mathcal{C}$ satisfying $[x_0, x_k] \nsubseteq \partial\mathcal{C}$ for all $k \in \{1, \ldots, n\}$, and let us find a contradiction by considering $Z \coloneqq Y \cup \{x_0\}$.

First, since $x_0 \neq x_k$ for all $k \in \{1, \ldots, n\}$ (if not, we would get an index $k \in \{1, \ldots, n\}$ such that $[x_0, x_k] = \{x_0\} \subseteq \partial\mathcal{C}$, which is false), we have $x_0 \notin Y$. Hence $\mathrm{card}(Z) = n + 1$.

Next, since $Y \in \mathcal{E}$ and $[x_0, x_k] \nsubseteq \partial\mathcal{C}$ for all $k \in \{1, \ldots, n\}$, we have $Z \in \mathcal{E}$.

Therefore, the assumption of the existence of x_0 yields a set $Z \in \mathcal{E}$ whose cardinality is greater than that of Y, which contradicts the very definition of Y.

Conclusion: $\overline{\mathcal{C}}$ is a polygon, and this proves Theorem 1.3. □

Let us conclude by a natural question: Considering Theorem 1.1 by Förtsch and Karlsson, we may conjecture that if the quasi-isometry in Theorem 1.3 is close to an isometry, then $\mathcal{C}$ is the interior of a m-simplex. More precisely, does there exist a sufficiently small constant $A_0 = A_0(m) \geqslant 1$ such that if a Hilbert domain $(\mathcal{C}, d_{\mathcal{C}})$ in $\mathbf{R}^m$ is (A, B)-quasi-isometric to a normed vector space for some constants $A = A(m, \mathcal{C}) \in [1, A_0]$ and $B = B(m, \mathcal{C}) \geqslant 0$, then $\mathcal{C}$ is the interior of a m-simplex?

References

[1] Benoist, Y. A survey on convex divisible sets. Lecture notes, International Conference and Instructional Workshop on Discrete Groups in Beijing, 2006.
[2] Burago, D., Burago, Y., and Ivanov, S. *A course in metric geometry.* AMS, 2001.
[3] Busemann, H. *The geometry of geodesics.* Academic Press, 1955.
[4] Busemann, H., and Kelly, P. *Projective geometry and projective metrics.* Academic Press, 1953.
[5] Colbois, B., Vernicos, C., and Verovic, P. Hilbert geometry for convex polygonal domains. Tech. rep., University of Neuchâtel, 2008.
[6] Colbois, B., and Verovic, P. Hilbert geometry for strictly convex domains. *Geom. Dedicata 105* (2004), 29–42.
[7] de la Harpe, P. On Hilbert's metric for simplices. *Lond. Math. Soc. Lect. Note Ser. 1*, 181 (1993), 97–119.
[8] Egloff, D. Uniform Finsler Hadamard manifolds. *Ann. Inst. Henri Poincaré, Phys. Théor. 66* (1997), 323–357.
[9] Förtsch, T., and Karlsson, A. Hilbert metrics and Minkowski norms. *J. Geom. 83*, 1-2 (2005), 22–31.
[10] Goldman, W. Projective geometry on manifolds. Lecture notes, University of Maryland, 1988.
[11] Karlsson, A., and Noskov, G. Some groups having only elementary actions on metric spaces with hyperbolic boundaries. *Geom. Dedicata 104* (2004), 119–137.
[12] Klee, V. Some characterizations of convex polyhedra. *Acta Math. 102* (1959), 79–107.
[13] Nussbaum, R. Hilbert's projective metric and iterated nonlinear maps. *Mem. Am. Math. Soc. 75*, 391 (1988).

[14] Socié-Méthou, É. *Comportements asymptotiques et rigidités des géométries de Hilbert.* PhD thesis, University of Strasbourg, 2000.
[15] Vernicos, C. Introduction aux géométries de Hilbert. *Sémin. Théor. Spectr. Géom. 23* (2005), 145–168.

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