THE BASS CONJECTURE FOR GROUPS OF SUBEXPONENTIAL GROWTH

ANDERS KARLSSON AND MARKUS NEUHAUSER

Abstract. We connect Bass's conjecture on the vanishing of the Hattori–Stallings rank to growth inside groups and deduce that every group of subexponential growth satisfies this conjecture. The latter is however already known by more general results of Berrick–Chatterji–Mislin obtained from quite different considerations.

Let G be a finitely generated group and $\|\cdot\|$ the word length corresponding to a finite set of generators (we refer to [dlH00] for the definitions of word lengths, growth, etc.). For an element $g \in G$, the function

$$n \mapsto \|g^n\|$$

contains more information about g than just its order and is of basic importance in the case g is of infinite order. An element g is a *u-element* if it is of infinite order and there is a constant $C > 0$ such that

$$\|g^n\| \leq C \log n$$

for all $n > 1$ (cf. [LMR00]).

Lemma. *Assume that $g \in G$ is an infinite order element such that $g = ag^k a^{-1}$ for some $k > 1$ and $a \in G$. Then g is a u-element.*

Proof. Given $n > 1$, write it as $n = \sum_{i=0}^{m} b_i k^i$ where $0 \leq b_i \leq k-1$ and $m = \lfloor \log n / \log k \rfloor$. Now we have

$$\begin{aligned} \|g^n\| &= \left\| g^{b_0} g^{b_1 k} \dots g^{b_m k^m} \right\| = \left\| g^{b_0} a g^{b_1} a^{-1} \dots a^i g^{b_i} a^{-i} \dots a^m g^{b_m} a^{-m} \right\| \\ &\leq \|g\| \sum_{i=0}^{m} b_i + 2\|a\| m \leq \left(\frac{\log n}{\log k} + 2 \right) ((k-1)\|g\| + 2\|a\|) . \end{aligned}$$

□

On the other hand, any u-element has finite order in every quotient $G/G_{(n)}$, where $G_{(n)}$ are the subgroups in the lower central series of G. This is because a homomorphism always semidecreases the growth behavior of $\|g^n\|$ and nilpotent groups have no u-elements (see the proposition below). Note also that the property of being a u-element is a conjugacy invariant.

By the $\mathbb{C}G$-Bass conjecture we mean the assertion that the Hattori–Stallings rank r_P, which is a certain conjugacy invariant function $G \to \mathbb{C}$ associated to a finitely generated projective $\mathbb{C}G$-module P, vanishes on every infinite order element. We

1991 *Mathematics Subject Classification.* Primary 16S34, 20E07; Secondary 20F65, 22D15.
Key words and phrases. Bass conjecture, growth in groups, group algebra.
The authors gratefully acknowledge the financial support of research grant 20-65060.01 of the Swiss National Fund for Scientific Research.

refer the reader to [B 76], [E 01], and [BCM02] for more information. In [B 76] Bass established his conjecture for linear groups and showed that the conjecture implies the idempotent conjecture for torsion-free groups. Moreover, in that paper it is explained that if $r_P(g) \neq 0$ for an infinite order element g, then g is conjugate to g^{p^n} for some $n > 0$ and almost all primes p. (Similar divisibility conditions appear in an earlier work of Formanek [F 73] in the context of the idempotent conjecture.) In view of this and the lemma, we have the following sharpened formulation of Theorem 9 in Eckmann's paper [E 01]:

Theorem 1. *The $\mathbb{C}G$-Bass conjecture holds for every finitely generated group without u-elements. (If the group in addition is torsion-free the idempotent conjecture holds as well.)*

As is pointed out in [E 01], it is known that semihyperbolic groups (biautomatic groups: F_n, word hyperbolic groups, braid groups; CAT(0)-groups: $\mathbb{Z}^m$, Coxeter groups, etc.) have the property that $\|g^n\|$ is either bounded or of linear growth in n for each element g (due to Alonso and Bridson in this generality). Let us also mention that, automorphism groups of trees, groups of isometries of a Busemann non-positively curved space (e. g. the symmetric space associated to a C^*-algebra) for which every non-trivial element has positive displacement, mapping class groups ([FLM01]), and certain groups of symplectic diffeomorphisms ([P 02]), also do not contain u-elements.

Proposition. *If G contains a u-element, then it is of exponential growth.*

Proof. Since a u-element g is of infinite order, all g^ns are distinct. Therefore the number of elements of length at most r is at least $e^{r/C}$, for some $C > 0$. ☐

We hence have:

Corollary. *Every group of subexponential growth satisfies the $\mathbb{C}G$-Bass conjecture (and the idempotent conjecture as well provided the group is torsion-free).*

Note that every group of subexponential growth is amenable and that it was already established by Berrick, Chatterji, and Mislin in their recent paper [BCM02] that any amenable group satisfies Bass's conjecture. Indeed, they prove that the Bost conjecture implies (a stronger version of) the Bass conjecture and then they rely on results of V. Lafforgue on the Bost conjecture. Somewhat earlier the class of all elementary amenable groups (which in particular excludes the groups of intermediate growth) was treated by Farrell and Linnell. To further put things into perspective we mention that $\mathrm{BS}(1,k) = \langle g, a : g = ag^k a^{-1} \rangle$ is an amenable group containing a u-element, and that groups of intermediate growth (first constructed by Grigorchuk in the early 1980s) are not linear.

Considerations of growth of individual elements are useful in other situations. For example, as it is known from [CK84] that $\mathrm{SL}(m,\mathbb{Z})$, for $m \geq 3$, is boundedly generated by u-elements (the elementary matrices) one can quickly deduce:

Theorem 2. *Any homomorphism from $\mathrm{SL}(m,\mathbb{Z})$, $m \geq 3$, into a group of subexponential growth (or a semihyperbolic group, or the mapping class group of a surface, or the automorphism group of a tree etc.) must have finite image.*

References

[B 76] Bass, H., *Euler characteristic and characters of discrete groups*, Invent. Math. **35** (1976), 155–196. MR **55:**5764

[BCM02] Berrick, A.J., Chatterji, I., Mislin, G., *From acyclic groups to the Bass conjecture for amenable groups*, Preprint, FIM, ETH Zürich, 2002.

[CK84] Carter, D., Keller, G., *Elementary expressions for unimodular matrices*, Comm. Algebra **12** (1984), 379–389. MR **86a:**11023

[E 01] Eckmann, B., *Idempotents in a complex group algebra, projective modules, and the von Neumann algebra*, Arch. Math. (Basel) **76** (2001), 241–249. MR **2002c:**16036

[FLM01] Farb, B., Lubotzky A., Minsky, Y., *Rank-1 phenomena for mapping class groups*, Duke Math. J. **106** (2001), 581–597. MR **2001k:**20076

[F 73] Formanek, E., *Idempotents in Noetherian group rings*, Canad. J. Math. **25** (1973), 366–369. MR **47:**5041

[dlH00] de la Harpe, P., *Topics in Geometric Group Theory*, Chicago Lectures in Math., University of Chicago Press, Chicago, 2000. MR **2001i:**20081

[LMR00] Lubotzky, A., Mozes, S., Raghunathan, M. S., *The word and Riemannian metrics on lattices of semisimple groups*, Publ. Math. Inst. Hautes Études Sci. **91** (2000), 5–53. MR **2002e:**22011

[P 02] Polterovich, L., *Growth of maps, distortion in groups and symplectic geometry*, Invent. math. **150** (2002), 655–686.

Institut de Mathématiques, Université de Neuchâtel, Rue Emile Argand 11, 2007 Neuchâtel, Switzerland

E-mail address: anders.karlsson@unine.ch

E-mail address: markus.neuhauser@unine.ch