Isometric group actions on Hilbert spaces: structure of orbits

Yves de Cornulier, Romain Tessera, Alain Valette

November 8, 2005

Abstract

Our main result is that a finitely generated nilpotent group has no isometric action on an infinite dimensional Hilbert space with dense orbits. In contrast, we construct such an action with a finitely generated metabelian group.

Mathematics Subject Classification: Primary 22D10; Secondary 43A35, 20F69.
Key words and Phrases: Affine actions, Hilbert spaces, minimal actions, nilpotent groups.

1 Introduction

The study of isometric actions of groups on affine Hilbert spaces has, in recent years, found applications ranging from the $K$-theory of $C^*$-algebras [HiKa], to rigidity theory [Sh2] and geometric group theory [Sh3, CTV]. This renewed interest motivates the following general problem: How can a given group act by isometries on an affine Hilbert space?

This paper is a sequel to [CTV], but can be read independently. In [CTV], we focused, given an an isometric action of a finitely generated group $G$ on a Hilbert space $\alpha : G \to \text{Isom}(\mathcal{H})$, on the growth of the function $g \mapsto \alpha(g)(0)$. Here the emphasis is on the structure of orbits.

In §2, we consider affine isometric actions of $\mathbb{Z}^n$ or $\mathbb{R}^n$. On finite-dimensional Euclidean spaces, the situation is clear-cut: such an action is an orthogonal sum of a bounded action and an action by translations. Even if the general case is more subtle, something remains from the finite-dimensional case. We say that a convex subset of a Hilbert space is locally bounded if its intersection with any finite dimensional subspace is bounded.
Theorem. (see Theorem 2.2) Let either \( \mathbb{Z}^n \) or \( \mathbb{R}^n \) act isometrically on a Hilbert space \( \mathcal{H} \), with linear part \( \pi \). Let \( O \) be an orbit under this action. Then there exist

- a subspace \( T \) of \( \mathcal{H} \) (the “translation part”), contained in the invariant vectors of \( \pi \), of finite dimension \( \leq n \), and
- a closed, locally bounded convex subset \( U \) of the orthogonal subspace \( T^\perp \),

such that \( O \) is contained in \( T \times U \).

In §3, we address a question due to A. Navas: which locally compact groups admit an affine isometric action with dense orbits (i.e. a minimal action) on an infinite-dimensional Hilbert space?

The main result of the paper is a negative answer in the case of finitely generated nilpotent groups.

Theorem. (see Theorem 3.15 and its corollaries) A compactly generated, nilpotent-by-compact group does not admit any affine isometric action with dense orbits on an infinite-dimensional Hilbert space.

Actually, for compactly generated nilpotent groups, one can describe all affine isometric actions with dense orbits; see Corollary 3.16.

In the course of our proof, we introduce the following new definitions: a unitary or orthogonal representation \( \pi \) of a group is strongly cohomological if it satisfies: for every nonzero subrepresentation \( \rho \leq \pi \), we have \( H^1(G, \rho) \neq 0 \). It is easy to observe that the linear part of a affine isometric action with dense orbits is strongly cohomological. The non-trivial step in the proof of the main theorem is the following result.

Proposition. (see Corollary 3.14) Let \( \pi \) be an orthogonal or unitary representation of a second countable, nilpotent group \( G \). Suppose that \( \pi \) is strongly cohomological. Then \( \pi \) is a trivial representation.

Another case for which we have a negative answer is the following.

Theorem. (see Theorem 3.18) Let \( G \) be a connected semisimple Lie group. Then \( G \) has no isometric action on a nonzero Hilbert space with dense orbits.

It is not clear how the main theorem can be generalized, in view of the following example.

Proposition. (see Proposition 3.2) There exists a finitely generated metabelian group admitting an affine isometric action with dense orbits on \( \ell_2^R(\mathbb{Z}) \).
Recall that an isometric action $\alpha : G \to \text{Isom}(\mathcal{H})$ almost has fixed points if for every $\varepsilon > 0$ and every compact subset $K \subset G$ there exists $v \in \mathcal{H}$ such that $\sup_{g \in K} \|v - \alpha(g)v\| \leq \varepsilon$. There is a link between this notion and strongly cohomological representations.

**Proposition.** (see Proposition 3.10) Let $G$ be a topological group and $\alpha$ an isometric action on a Hilbert space that does not almost have fixed points. Then its linear part $\pi$ has a nonzero subrepresentation that is strongly cohomological.

However the converse is not true as shown by the following example.

**Proposition.** (see Proposition 3.4) There exists a countable group admitting an affine isometric action with dense orbits, almost having fixed points on $\ell_2^R(N)$ (more precisely, every finitely generated subgroup has a fixed point).

**Acknowledgements.** We thank A. Fathi and A. Navas for a useful indication concerning Theorem 2.2.

## 2 Actions of $\mathbb{Z}^n$ and $\mathbb{R}^n$

Let $\mathcal{H}$ be a Hilbert space.

**Definition 2.1.** A convex subset $K$ of $\mathcal{H}$ is said to be locally bounded if $K \cap F$ is bounded for every finite-dimensional subspace $F$ of $\mathcal{H}$.

**Theorem 2.2.** Let $G = \mathbb{Z}^n$ or $\mathbb{R}^n$ act isometrically on a Hilbert space $\mathcal{H}$, with linear part $\pi$. Let $\mathcal{O}$ be an orbit under this action. Then there exist

- a subspace $T$ of $\mathcal{H}$, contained in $\mathcal{H}^{\pi(G)}$, of finite dimension $\leq n$, and
- a closed, locally bounded convex subset $U$ of $T^\perp$,

such that $\mathcal{O}$ is contained in $T \times U$.

**Proof.** The case of $\mathbb{R}^n$ is reduced to the case of $\mathbb{Z}^n$ by taking a dense, free abelian subgroup of finite rank in $\mathbb{R}^n$.

Let $(\pi, \mathcal{H})$ be a unitary representation of $\mathbb{Z}^n$. Let $b \in Z^1(\mathbb{Z}^n, \pi)$ define an affine action of $\mathbb{Z}^n$ with linear part $\pi$, and let $\mathcal{O}$ be an orbit. We can suppose that $0 \in \mathcal{O}$, so that $\mathcal{O}$ is the range of $b$.

To emphasize the main idea of the proof, let us start with the case when $n = 1$. Write $\mathcal{H}_0 = \text{Ker}(\pi(1) - \text{Id}) = \mathcal{H}^{\pi(G)}$. The representation decomposes as $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Denote by $\pi_0$ and $\pi_1$ the corresponding subrepresentations of $\pi$. The cocycle $b$ decomposes as $b = b_0 + b_1$. Note that $b_0$ is an additive morphism:
$\mathbb{Z} \to \mathcal{H}_0$; define $T$ as the linear subspace generated by $b_0(1)$. On the other hand, let us show that the sequence $(b_1(k))_{k \in \mathbb{Z}}$ is contained in a locally bounded convex subset of $\mathcal{H}_1$. First, note that

$$\|(\pi(1) - \text{Id})b(k)\| \leq 2\|b(1)\|.$$ 

Indeed, since $b(k) = \sum_{j=0}^{k-1} \pi(1)^j b(1)$, we get

$$(\pi(1) - \text{Id})b(k) = (\pi(k) - \text{Id})b(1).$$

Moreover, since $\mu = \pi_1(1) - \text{Id}$ is injective, it follows that the closed convex set $U = \mu^{-1}(B(0, 2\|b(1)\|))$ is locally bounded, and $O$ is contained in $T \times U$.

Let us turn to the general case. Write $I = \{1, \ldots, n\}$. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{Z}^n$. Define, for every subset $J \subset I$, a closed subspace $\mathcal{H}_J$ of $\mathcal{H}$, as follows: $\mathcal{H}_J' = \{\xi \in \mathcal{H}, \forall i \in I - J, \pi(e_i)\xi = \xi\}$, and $\mathcal{H}_J$ is the orthogonal subspace in $\mathcal{H}_J'$ of $\sum_{K \subseteq J} \mathcal{H}_K'$. It is immediate that $\mathcal{H}$ is the direct sum of all $\mathcal{H}_J$'s ($J \subset I$), and that $\mathcal{H}_J$ is $\mathbb{Z}^n$-stable, defining a subrepresentation $\pi_J$ of $\pi$.

The cocycle $b$ decomposes as $b = \sum_J b_J$. Since $\pi_\emptyset$ is a trivial representation, $b_\emptyset$ is given by a morphism: $\mathbb{Z}^n \to \mathcal{H}_\emptyset$. Let $T_\pi$ denote the (finite-dimensional) subspace generated by $b_\emptyset(\mathbb{Z}^n)$.

Let $J$ be any nonempty subset of $I$, and fix $i \in J$. Then $\pi_J(e_i) - 1$ is injective. For all $j \notin J$, so that $\pi_J(e_j) = 1$, we have $b_J(e_j) = 0$. Indeed, expanding the relation $b_J(e_i + e_j) = b_J(e_j + e_i)$, we obtain $(\pi(e_i) - 1)b(e_j) = 0$. Thus, the affine action associated to $b_J$ is trivial on all $e_j$, $j \notin J$. Set $\mu_J = \prod_{j \notin J}(\pi_J(e_j) - 1)$. Then $\mu_J$ is injective on $\mathcal{H}_J$. Let $\Omega_J \subset \mathcal{H}_J$ be the range of $b_J$. We easily check that

$$\mu_J \left( b_J \left( \sum_j n_je_j \right) \right) \leq \sum_{j \in J} 2^n \|b_J(e_j)\|,$$

which is bounded. Thus, $\Omega_J$ is contained in $\mu_J^{-1}(B_J)$ for some ball $B_J$; since $\mu_J$ is injective, $\mu_J^{-1}(B_J)$ is a locally bounded convex subset. Write $U = \bigoplus_{j \neq \emptyset} \mu_J^{-1}(B_J)$: this is a closed locally bounded convex subset of $\mathcal{H}$, contained in the orthogonal of $\mathcal{H}_\emptyset$. By construction, the orbit $\Omega$ of zero for the action associated to $b$ is contained in $T_\pi \times U$.

3 Actions with dense orbits

We owe the following question to A. Navas.
Question 1 (Navas). Which finitely generated groups acts isometrically on an infinite-dimensional separable Hilbert space with a dense orbit?

More generally, the question makes sense for compactly generated groups. In the case of $\mathbb{Z}^n$ or $\mathbb{R}^n$, the answer is provided by Theorem 2.2.

**Corollary 3.1.** Any isometric action with dense orbits of either $\mathbb{Z}^n$ or $\mathbb{R}^n$ on a Hilbert space $H$, factors through an additive homomorphism with dense image to $H$ (so that $H$ is finite-dimensional).

### 3.1 Existence results

Here is a first positive result regarding Navas’ question.

**Proposition 3.2.** There exists an isometric action of a metabelian 3-generator group on a infinite-dimensional separable Hilbert space, all of whose orbits are dense.

**Proof.** Observe that $\mathbb{Z}[\sqrt{2}]$ acts by translations, with dense orbits, on $\mathbb{R}$; so the free abelian group of countable rank $\mathbb{Z}[\sqrt{2}]$ acts by translations, with dense orbits, on $\ell^2_\mathbb{R}(\mathbb{Z})$. Observe now that the latter action extends to the wreath product $\mathbb{Z}[\sqrt{2}] \wr \mathbb{Z} = \mathbb{Z}[\sqrt{2}] \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\ell^2_\mathbb{R}(\mathbb{Z})$ by the shift. That wreath product is metabelian, with 3 generators.

**Corollary 3.3.** There exists an isometric action of a free group of finite rank on a Hilbert space, with dense orbits.

In the example given by Proposition 3.2, the given isometric action clearly does not almost have fixed points, i.e. it defines a non-zero element in reduced 1-cohomology. The next result shows that this is not always the case.

**Proposition 3.4.** There exists a countable group $\Gamma$ with an affine isometric action $\alpha$ on a Hilbert space, such that $\alpha$ has dense orbits, and every finitely generated subgroup of $\Gamma$ has a fixed point. In particular, the action almost has fixed points.

**Proof.** We first construct an uncountable group $G$ and an affine isometric action having dense orbits and almost having fixed points.

In $H = \ell^2_\mathbb{R}(\mathbb{N})$, let $A_n$ be the affine subspace defined by the equations

$$x_0 = 1, \ x_1 = 1, ..., \ x_n = 1,$$

and let $G_n$ be the pointwise stabilizer of $A_n$ in the isometry group of $H$. Let $G$ be the union of the $G_n$’s. View $G$ as a discrete group.
It is clear that \( G \) almost has fixed points in \( \mathcal{H} \), since any finite subset of \( G \) has a fixed point. Let us prove that \( G \) has dense orbits.

**Claim 1.** For all \( x, y \in \mathcal{H} \), we have \( \lim_{n \to \infty} |d(x, A_n) - d(y, A_n)| = 0 \).

By density, it is enough to prove Claim 1 when \( x, y \) are finitely supported in \( \ell^2_\mathbb{R}(\mathbb{N}) \). Take \( x = (x_0, x_1, \ldots, x_k, 0, 0, \ldots) \) and choose \( n > k \). Then
\[
d(x, A_n)^2 = \sum_{j=0}^k (x_j - 1)^2 + \sum_{j=k+1}^n 1^2 = n + 1 - 2 \sum_{j=0}^k x_j + \sum_{j=0}^k x_j^2,
\]
so that \( d(x, A_n) = \sqrt{n + O(\frac{1}{\sqrt{n}})} \), which proves Claim 1.

**Claim 2.** \( G \) has dense orbits in \( \mathcal{H} \).

Observe that two points \( x, y \in \mathcal{H} \) are in the same \( G_n \)-orbit if and only if \( d(x, A_n) = d(y, A_n) \). Fix \( x_0, z \in \mathcal{H} \). We want to show that \( \lim_{n \to \infty} d(G_n x_0, z) = 0 \). So fix \( \varepsilon > 0 \); by the first claim, \( |d(x_0, A_n) - d(z, A_n)| < \varepsilon \) for \( n \) large enough.

So we find \( y \in \mathcal{H} \) such that \( ||y - z|| < \varepsilon \) and \( d(x_0, A_n) = d(y, A_n) \). By the previous observation, \( y \) is in \( G_n x_0 \), proving the claim.

Using separability of \( \mathcal{H} \), it is now easy to construct a countable subgroup \( \Gamma \) of \( G \) also having dense orbits on \( \mathcal{H} \).

**Question 2.** Does there exist an affine isometric action of a finitely generated group on a Hilbert space, having dense orbits and almost having fixed points?

### 3.2 Non-existence results

Let us show that locally compact, compactly generated nilpotent groups cannot act with dense orbits on an infinite-dimensional separable Hilbert space. We actually prove something slightly stronger.

**Definition 3.5.** We say that an isometric action of a group \( G \) on a metric space \((X, d)\) has **coarsely dense orbits** if there exists \( C \geq 0 \) such that, for every \( x, y \in X \),
\[
d(x, G.y) \leq C.
\]

Observe that, for an action of a topological group, having coarsely dense orbits is stable under passing to a cocompact subgroup.

**Definition 3.6.** If \( G \) is a topological group and \( \pi \) a unitary representation, we say that \( \pi \) is **strongly cohomological** if every nonzero subrepresentation of \( \pi \) has nonzero first cohomology.

**Lemma 3.7.** Let \( G \) be a topological group and \( \pi \) a unitary representation, admitting a 1-cocycle \( b \) with coarsely dense image. Then \( \pi \) is strongly cohomological.
Proof. If \( \sigma \) is a nonzero subrepresentation of \( \pi \), let \( b_\sigma \) be the orthogonal projection of \( b \) on \( H_\sigma \), so that \( b_\sigma \in Z^1(G, \sigma) \). Then \( b_\sigma(G) \) is coarsely dense in \( H_\sigma \), in particular \( b_\sigma \) is unbounded. So \( b_\sigma \) defines a non-zero class in \( H^1(G, \sigma) \). \( \square \)

The following Lemma is Proposition 3.1 in Chapitre III of [Gu2].

Lemma 3.8. Let \( \pi \) be a unitary representation of \( G \) that does not contain the trivial representation. Let \( z \) be a central element of \( G \). Suppose that \( 1 - \pi(z) \) has a bounded inverse (equivalently, \( 1 \) does not belong to the spectrum of \( \pi(z) \)). Then \( H^1(G, \pi) = 0 \).

Proof. If \( g \in G \), expanding the equality \( b(gz) = b(zg) \), we obtain that \( (1 - \pi(z))b(g) \) is bounded by \( 2\|b(z)\| \), so that \( b \) is bounded by \( 2\|1 - \pi(z)\|^{-1}\|b(z)\| \). \( \square \)

Lemma 3.9. Let \( G \) be a locally compact, second countable group, and \( \pi \) a strongly cohomological representation. Then \( \pi \) is trivial on the centre \( Z(G) \).

Proof. Fix \( z \in Z(G) \). As \( G \) is second countable, we may write \( \pi = \int_G \rho d\mu(\rho) \), a disintegration of \( \pi \) as a direct integral of irreducible representations. Let \( \chi : \hat{G} \rightarrow S^1 : \rho \mapsto \rho(z) \) be the continuous map given by the value of the central character of \( \rho \) on \( z \). For \( \varepsilon > 0 \), set \( X_\varepsilon = \{ \rho \in \hat{G} : |\chi(\rho) - 1| > \varepsilon \} \) and \( \pi_\varepsilon = \int_{X_\varepsilon} \rho d\mu(\rho) \), so that \( \pi_\varepsilon \) is a subrepresentation of \( \pi \). Since \( |\rho(z) - 1|^{-1} < \varepsilon^{-1} \) for \( \rho \in X_\varepsilon \), the operator

\[
(\pi_\varepsilon(z) - 1)^{-1} = \int_{X_\varepsilon} (\rho(z) - 1)^{-1} d\mu(\rho)
\]

is bounded. We are now in position to apply Lemma 3.8, to conclude that \( H^1(G, \pi_\varepsilon) = 0 \). By definition, this means that \( \pi_\varepsilon \) is the zero subrepresentation, meaning that the measure \( \mu \) is supported in \( \hat{G} - X_\varepsilon \). As this holds for every \( \varepsilon > 0 \), we see that \( \mu \) is supported in \( \{ \rho \in \hat{G} : \rho(z) = 1 \} \), to the effect that \( \pi(z) = 1 \). \( \square \)

Proposition 3.10. Let \( G \) be a topological group, and \( \pi \) a unitary representation of \( G \). Suppose that \( \overline{H^1}(G, \pi) \neq 0 \). Then \( \pi \) has a nonzero subrepresentation that is strongly cohomological.

Proof. Suppose the contrary. Then, by an standard application of Zorn’s Lemma, \( \pi \) decomposes as a direct sum \( \pi = \bigoplus_{i \in I} \pi_i \), where \( H^1(G, \pi_i) = 0 \) for every \( i \in I \), so that \( \overline{H^1}(G, \pi) = 0 \) by Proposition 2.6 in Chapitre III of [Gu2]. \( \square \)

Remark 3.11. The converse is false, even for finitely generated groups: indeed, it is known (see [Gu1]) that every nonzero representation of the free group \( F_2 \) has non-vanishing \( H^1 \), so that every unitary representation of \( F_2 \) is strongly cohomological. But it turns out that \( F_2 \) has an irreducible representation \( \pi \) such that \( \overline{H^1}(F_2, \pi) = 0 \) (see Proposition 2.4 in [MaVa]).
Corollary 3.12. Let $G$ be a locally compact, second countable group, and let $\pi$ be a unitary representation of $G$ without invariant vectors. Write $\pi = \pi_0 \oplus \pi_1$, where $\pi_1$ consists of the $Z(G)$-invariant vectors. Then

1. $\pi_0$ does not contain any strongly cohomological subrepresentation (in particular, $H^1(G, \pi_0) = 0$);
2. every 1-cocycle of $\pi_1$ vanishes on $Z(G)$, so that $H^1(G, \pi_1) \simeq H^1(G/Z(G), \pi_1)$.

Proof. (1) follows by combining lemma 3.9 and Proposition 3.10. For (2), we use the idea of proof of Theorem 3.1 in [Sh2]: if $b \in Z^1(G, \pi_1)$, then for every $g \in G$, $z \in Z(G)$,

$$\pi_1(g)b(z) + b(g) = b(gz) = b(zg) = b(g) + b(z)$$

as $\pi_1(z) = 1$. So $\pi_1(g)b(z) = b(z)$; this forces $b(z) = 0$ as $\pi$ has no $G$-invariant vector. So $b$ factors through $G/Z(G)$.

Observe that Corollary 3.12 provides a new proof of Shalom’s Corollary 3.7 in [Sh2]: under the same assumptions, every cocycle in $Z^1(G, \pi)$ is almost cohomologous to a cocycle factoring through $G/Z(G)$ and taking values in a sub-representation factoring through $G/Z(G)$.

From Corollary 3.12 we immediately deduce

Corollary 3.13. Let $G$ be a locally compact, second countable, nilpotent group, and let $\pi$ be a representation of $G$ without invariant vectors. Let $(Z_i)$ be the ascending central series of $G$ ($Z_0 = \{1\}$, and $Z_i$ is the centre modulo $Z_{i-1}$). Let $\sigma_i$ denote the subrepresentation of $G$ on the space of $Z_i$-invariant vectors, and finally let $\pi_i$ be the orthogonal of $\sigma_{i+1}$ in $\sigma_i$, so that $\pi = \bigoplus \pi_i$.

Then $H^1(G, \pi_i) \simeq H^1(G/Z_i, \pi_i)$ for all $i$, and $\pi$ is not a strongly cohomological subrepresentation. In particular, $\overline{H^1}(G, \pi) = 0$.

Note that the latter statement is a result of Guichardet [Gu1, Théorème 7], which can be stated as: $G$ has Property $H_T$ (i.e. every unitary representation with non-vanishing reduced cohomology contains the trivial representation). If we define Property $H_{CT}$ to be: every strongly cohomological representation is trivial, then, as a corollary of Proposition 3.10, Property $H_{CT}$ implies Property $H_T$; we have actually proved that locally compact, second countable nilpotent groups have Property $H_{CT}$.

Corollary 3.14. If $G$ is a locally compact, second countable nilpotent group, and $\pi$ is a strongly cohomological representation, then $\pi$ is a trivial representation.
**Theorem 3.15.** Let $G$ be a locally compact, second countable nilpotent group. Then $G$ has a isometric action on a (real) Hilbert space $\mathcal{H}$ with coarsely dense orbits if and only there exists a continuous morphism: $u : G \to (\mathcal{H}, +)$ with coarsely dense image.

**Proof.** Suppose that such an action exists, and let $\pi$ be its linear part. By lemma 3.7, $\pi$ is strongly cohomological, hence trivial by Corollary 3.14. So the action is given by a morphism $u : G \to (\mathcal{H}, +)$ with coarsely dense image. The converse is obvious. \[ \square \]

The following generalizes Corollary 3.1.

**Corollary 3.16.** Let $G$ be a locally compact, compactly generated nilpotent group, and let $\mathcal{H}$ be a (real) Hilbert space. Then

- $G$ has a isometric action on $\mathcal{H}$ with coarsely dense orbits if and only $\mathcal{H}$ has finite dimension $k$, and $G$ has a quotient isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m$, with $n + m \geq k$.

- $G$ has a isometric action on $\mathcal{H}$ with dense orbits if and only $\mathcal{H}$ has finite dimension $k$, and $G$ has a quotient isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m$, with $\max(n + m - 1, n) \geq k$.

**Proof.** Since $G$ is $\sigma$-compact, by [Com, Theorem 3.7] there exists a compact normal subgroup $N$ such that $G/N$ is second countable.

Let $\alpha$ be an affine isometric action of $G$ with coarsely dense orbits. Then $G/N$ has an isometric action with coarsely dense orbits on the set of $\alpha(N)$-fixed points (which is nonempty as $N$ is compact). So we can assume that $G$ is second countable.

Let $u$ be the morphism $G \to \mathcal{H}$ as in Theorem 3.15. Let $W$ be its kernel, so that $A = G/W$ is a locally compact, abelian group, which embeds continuously, coarsely densely in a Hilbert space. By standard structural results, $A$ has an open subgroup, containing a compact subgroup $K$, such that $A/K$ is a Lie group. Since $K$ embeds in a Hilbert space, it is necessarily trivial, so that $A$ is an abelian Lie group without compact subgroup. So $A$ is isomorphic to $\mathbb{R}^n \times \mathbb{Z}^m$ for some integers $n, m$. Since $A$ embeds coarsely densely in $\mathcal{H}$, the latter must have finite dimension $k \leq n + m$.

If the action has dense orbits, then either $m = 0$ and $n \geq k$, or $m \geq 1$ and $m \geq k - n + 1$; this means that $k \leq \max(n + m - 1, n)$. Conversely, if $k \leq n + m - 1$, then, since $\mathbb{Z}$ has a dense embedding in the torus $\mathbb{R}^k/\mathbb{Z}^k$, $\mathbb{Z}^{k+1}$ has a dense embedding in $\mathbb{R}^k$, and this embedding can be extended to $\mathbb{R}^n \times \mathbb{Z}^m$. \[ \square \]
From Corollary 3.16, we immediately deduce

**Corollary 3.17.** A compactly generated, nilpotent-by-compact group does not admit any isometric action with coarsely dense orbits on an infinite-dimensional Hilbert space.

Proposition 3.2 on the one hand, and Corollary 3.17 on the other, isolate the first test-case for Navas’question:

**Question 3.** Can a polycyclic group admit an affine isometric action with dense orbits on an infinite-dimensional Hilbert space?

Let us prove a related result for semisimple groups.

**Theorem 3.18.** Let $G$ be a connected, semisimple Lie group. Then $G$ cannot act on a Hilbert space $H \neq 0$ with coarsely dense orbits.

*Proof.* Suppose by contradiction the existence of such an action $\alpha$, and let $\pi$ denote its linear part. Then $\pi$ is trivial on the centre of $G$. Thus the centre acts by translations, generating a finite-dimensional subspace $V$ of $H$. The action induces a map $p : G \to O(V) \ltimes V$. Since $G$ is semisimple, the kernel of $p$ contains the sum $G_{nc}$ of all noncompact factors of $G$, and thus factors though the compact group $G/G_{nc}$. Thus $H^1(G, V) = 0$, and since $\pi$ is strongly cohomological, this implies that $V = 0$.

It follows that $\alpha$ is trivial on the centre of $G$, so that we can suppose that $G$ has trivial centre. Then $G$ is a direct product of simple Lie groups with trivial centre. We can write $G = H \times K$ where $K$ denotes the sum of all simple factors $S$ of $G$ such that $\alpha(S)(0)$ is bounded (in other words, $H^1(S, \pi|_S) = 0$). Then the restriction of $\alpha$ to $H$ also has coarsely dense orbits. Moreover, every simple factor of $H$ acts in an unbounded way, so that, by a result of Shalom [Sh1, Theorem 3.4],[1] the action of $H$ is proper. That is, the map $i : H \to \mathcal{H}$ given by $i(h) = \alpha(h)(0)$ is metrically proper and its image is coarsely dense. By metric properness, the subset $X = i(H) \subset \mathcal{H}$ satisfies: $X$ is coarsely dense, and every ball in $X$ (for the metric induced by $\mathcal{H}$) is compact.

Suppose that $\mathcal{H}$ is infinite dimensional and let us deduce a contradiction. For some $d > 0$, we have $d(x, X) \leq d$ for every $x \in \mathcal{H}$. If $\mathcal{H}$ is infinite dimensional, there exists, in a fixed ball of radius $7d$, infinitely many pairwise disjoint balls $B(x_n, 3d)$ of radius $3d$. Taking a point in $X \cap B(x_n, 2d)$ for every $n$, we obtain a closed, infinite and bounded discrete subset of $X$, a contradiction.

---

[1] Shalom only states the result for a simple group, but the proof generalizes immediately. See for instance [CLTV] for another proof, based on the Howe-Moore Property.
Thus $H$ is finite dimensional; since every simple factor of $H$ is non-compact, it has no non-trivial finite dimensional orthogonal representation, so that the action is by translations, and hence is trivial, so that finally $H = \{0\}$. 

**Remark 3.19.** The same argument shows that a semisimple, linear algebraic group over any local field, cannot act with coarsely dense orbits on a Hilbert space.

**References**


Yves de Cornulier
École Polytechnique Fédérale de Lausanne (EPFL)
Institut de Géométrie, Algèbre et Topologie (IGAT)
CH-1015 Lausanne, Switzerland
E-mail: decorul@clipper.ens.fr

Romain Tessera
Équipe Analyse, Géométrie et Modélisation
Université de Cergy-Pontoise, Site de Saint-Martin
2, rue Adolphe Chauvin F 95302 Cergy-Pontoise Cedex, France
E-mail: tessera@clipper.ens.fr

Alain Valette
Institut de Mathématiques - Université de Neuchâtel
Rue Emile Argand 11, CH-2007 Neuchâtel - Switzerland
E-mail: alain.valette@unine.ch