Rigidity of Hilbert metrics

Bruno Colbois, Patrick Verovic

Abstract

We study the groups of isometries for Hilbert metrics on bounded open convex domains in $\mathbb{R}^n$ and show that if $\mathcal{C}$ is such a set with a *strictly* convex boundary, the Hilbert geometry is asymptotically Riemannian at infinity. As a consequence of this result, we prove there are no Hausdorff quotients of $\mathcal{C}$ by isometry subgroups with finite *volume* except when $\partial\mathcal{C}$ is an ellipsoid.

AMS classification: 58B20, 52A20

Introduction

Let $\mathcal{C}$ be a bounded open convex domain in $\mathbb{R}^n$ with boundary $\partial\mathcal{C}$ and $\|\cdot\|$ be the canonical Euclidean norm in $\mathbb{R}^n$. It is then possible to define a distance $d_{\mathcal{C}}$ on $\mathcal{C}$, the so-called Hilbert metric (discovered by D. Hilbert in 1894), as follows. Given two distinct points p and q in $\mathcal{C}$, let a and b be the intersection points of the straight line defined by p and q with $\partial\mathcal{C}$ so that $p = ta + (1-t)b$ and $q = sa + (1-s)b$ with $0 < s < t < 1$. Then $d_{\mathcal{C}}(p,p) = 0$ and $d_{\mathcal{C}}(p,q) = \ln[a,p,q,b]$, where $[a,p,q,b] = \frac{1-t}{t} \times \frac{s}{1-s} > 1$ is the cross ratio of the ordered collinear points $\{a,p,q,b\}$. The fact $d_{\mathcal{C}}$ is a distance comes from basic properties of the cross ratio and the metric space $(\mathcal{C}, d_{\mathcal{C}})$ thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of $\mathbb{R}^n$ and in which the affine open segments joining two points of the boundary are geodesics isometric to $(\mathbb{R}, |\cdot|)$. On the other hand, the distance $d_{\mathcal{C}}$ is associated to the Finsler metric $F_{\mathcal{C}}$ on $\mathcal{C}$ given for $p \in \mathcal{C}$ and $v \in T_p\mathcal{C} = \mathbb{R}^n$ by $F_{\mathcal{C}}(p,v) = \|v\|(\frac{1}{\|p-p^-\|} + \frac{1}{\|p-p^+\|})$, where p^- (resp. p^+) is the intersection point of the half line $p + \mathbb{R}^- v$ (resp. $p + \mathbb{R}^+ v$) with $\partial\mathcal{C}$. For further information, we refer to [3], [4], [6] and [9] for an introduction to the subject. In the present paper, we study the subgroups Γ of the group $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ of isometries of $(\mathcal{C}, d_{\mathcal{C}})$ with *proper* actions on $\mathcal{C}$ (*i.e.* such that the quotient topological space $\mathcal{C}/\Gamma$ is Hausdorff) and prove the following *rigidity* result:

Theorem. *Let $\mathcal{C}$ be a bounded open convex domain in $\mathbb{R}^n$ whose boundary $\partial\mathcal{C}$ is a hypersurface of class C^3 which is strictly convex (in the sense the Hessian is positive definite). Then if $\partial\mathcal{C}$ is* not *an ellipsoid, any subgroup of the isometry group* $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ *whose action on $\mathcal{C}$ is proper is* finite.

We then deduce an important consequence involving the volume measure associated to the Finsler metric $F_{\mathcal{C}}$ on $\mathcal{C}$ which is defined by $\mu(f) = \int_{\mathcal{C}} f(p)\sigma(p)dp$ for every continuous function $f : \mathcal{C} \to \mathbb{R}$ with compact support, where $\sigma(p)$ is the Finsler density for the metric $F_{\mathcal{C}}$. Recall here that for any Finsler manifold (F, M) the density function σ over M is defined as following: given an arbitrary

Riemannian metric g on M, for all $p \in M$ the value $\sigma(p)$ is equal to the square root of the ratio of the *Euclidean* volume of the ball $\{v \in T_pM : g(p)\cdot(v,v) \leq 1\}$ by the *Euclidean* volume of the ball $\{v \in T_pM : F(p,v) \leq 1\}$ in the Euclidean vector space $(T_pM, g(p))$. This definition does not depend on the choice of g and it generalizes the well known Riemannian density (indeed, if F is Riemannian, then σ is nothing else than the famous $\sqrt{\det(g_{ij})}$). For more information about the Finsler density and some problems related to this notion, one can have a look at [14].

Corollary. *Let $\mathcal{C}$ be as above. Then if $\partial\mathcal{C}$ is* not *an ellipsoid, $(\mathcal{C}, d_{\mathcal{C}})$ does* not *allow quotients of finite* volume *(and thus* compact *quotients) by subgroups of* $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ *whose actions on $\mathcal{C}$ are proper.*

Hilbert metrics are important objects people have been interested in for many reasons. First of all, they give a range of basic and rich enough examples of geodesic metric spaces (see the investigations of H. Busemann in [3] and [4]). They next generalize in Finsler geometry the Riemannian hyperbolic spaces which are obtained in the case $\partial\mathcal{C}$ is an ellipsoid and correspond to Klein's model (see [3], [4]). On the other hand, Hilbert geometries are considered in affine geometry from the 'projective transformation groups' point of view (see [9] and the recent preprint of Y. Benoist [1]). The question we are dealing with in the present paper has already been tackled in the literature for the *compact* case. Namely, the non-existence of compact quotients has been proved in the framework of affine geometry by J.-P. Benzécri in [2] (see also the illuminating Lecture Notes of W. M. Goldman [9]). Using a dynamical systems approach, a proof of this result has also been given by D. Egloff ([5], Theorem 3.59) in dimension two and by P. Foulon in higher dimensions as a consequence of a more general rigidity theorem (see [7], [8]). The situation when the boundary of $\mathcal{C}$ is no longer *strictly* convex has been investigated in [5] (Proposition 3.2), [9] and [10], and it is known that if $\mathcal{C}$ is for example a triangle, there *exist* co-compact subgroups of $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$.

The starting point of the present paper was to know whether there are quotients of $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ with finite *volume* and the strategy we use here to prove that in general such quotients do *not* exist involves very basic tools in metric and Finsler geometry. Indeed, the key idea of the proof consists in showing that the closer to $\partial\mathcal{C}$ we are, the less the metric $d_{\mathcal{C}}$ on $\mathcal{C}$ is different from a Riemannian one (this is given by Proposition 1.3 below). Therefore, if an infinite group of isometries for $d_{\mathcal{C}}$ had a proper action on $\mathcal{C}$, then every point would be sent to the boundary $\partial\mathcal{C}$ and the Hilbert metric would be as close as desired to a Riemannian metric and thus Riemannian itself. Then we conclude $\partial\mathcal{C}$ is an ellipsoid by combining results due to different authors ([3], [11]) in the following general statement:

Theorem. *Let $\mathcal{C}$ be* any *bounded open convex domain in $\mathbb{R}^n$. If the metric $d_{\mathcal{C}}$ is Riemannian, then $\partial\mathcal{C}$ is an ellipsoid.*

To be complete let us point out that in December 1999 we discussed a first version of this work (see [15]) with Édith Socié-Méthou who proposed to use an approach for this problem that is different from ours and that she later developed in her Ph.D. thesis ([12]) and in [13].

1. Geometry at infinity

In this part, we give two independent technical lemmas before proving that for a bounded open convex domain $\mathcal{C} \subset \mathbb{R}^n$ with a strictly convex boundary of class C^3 the Hilbert metric $d_\mathcal{C}$ is asymptotically Riemannian at infinity (i.e. when moving to the boundary $\partial\mathcal{C}$). Our approach for doing this consists in a local approximation of $\partial\mathcal{C}$ by a parabola (Lemma 1.1) in order to show the geometry of $(\mathcal{C}, d_\mathcal{C})$ behaves near $\partial\mathcal{C}$ as if it were Klein's geometry in the Euclidean unit ball near its boundary (Lemma 1.2). We then give Proposition 1.3.

Throughout the section, the canonical Euclidean norms in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ will be both denoted by $\|\cdot\|$ and the open ball in $(\mathbb{R}^{n-1}, \|\cdot\|)$ centered at 0 with radius $r > 0$ by $B(0,r)$. On the other hand, $(e_1, \ldots, e_n)$ will be the canonical basis in $\mathbb{R}^n$ and $\langle \cdot | \cdot \rangle$ the canonical scalar product.

The first lemma shows that for each point $\omega \in \partial\mathcal{C}$ there is a ball in $(\mathbb{R}^n, \|\cdot\|)$ with center ω and radius *independent* of ω in which the boundary of $\mathcal{C}$ can be written as the graph of a function defined in $\mathbb{R}^{n-1}$ (after an appropriate coordinate change depending on ω) having the property that its value at x lies between $(1 - M\|x\|)\|x\|^2$ and $(1 + M\|x\|)\|x\|^2$, where $M > 0$ is a number which does *not* depend on ω but *only* on $\mathcal{C}$:

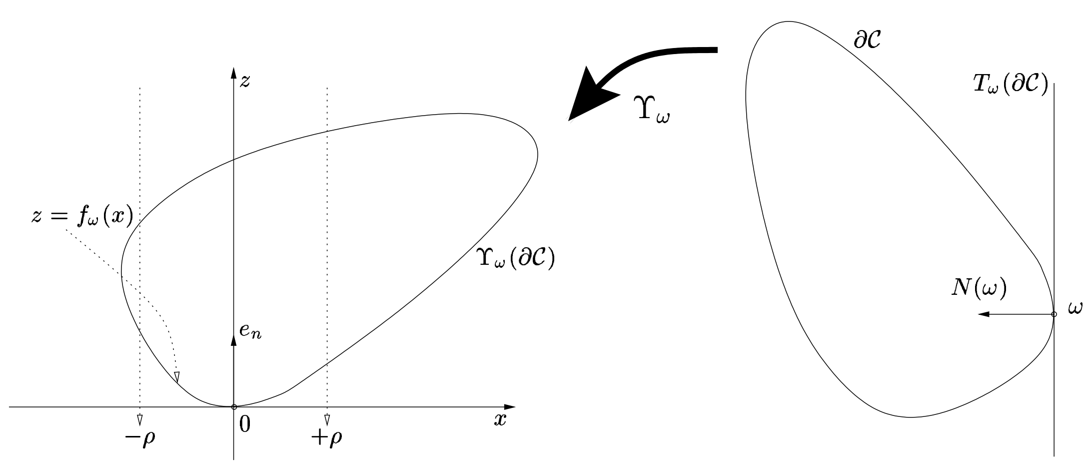
Lemma 1.1. *Let $\mathcal{C}$ be a bounded open convex domain in $\mathbb{R}^n$ whose boundary $\partial\mathcal{C}$ is a hypersurface of class C^3 that is strictly convex (in the sense the Hessian is positive definite) and denote $N : \mathcal{C} \to \mathbb{R}^n$ the normal vector field over $\partial\mathcal{C}$ pointing inwards.*

Then there are positive constants $0 < M$ and $0 < \rho \leq 1/M$ with a family $(\Upsilon_\omega)_{\omega \in \partial\mathcal{C}}$ of affine isometries in $(\mathbb{R}^n, \|\cdot\|)$ together with a family $(f_\omega)_{\omega \in \partial\mathcal{C}}$ of functions defined in $B(0,\rho) \subset \mathbb{R}^{n-1}$ such that for each $\omega \in \partial\mathcal{C}$ we have:

(i) $\Upsilon_\omega(\omega) = 0$ *and* $\overrightarrow{\Upsilon}_\omega \cdot N(\omega) = e_n$, *where* $\overrightarrow{\Upsilon}_\omega$ *is the linear part of* Υ_ω.

(ii) For all $(x, z) \in B(0,\rho) \times \mathbb{R}$, $\;z = f_\omega(x) \Longrightarrow (x, z) \in \Upsilon_\omega(\partial\mathcal{C})$.

(iii) For all $x \in B(0,\rho)$, $\;(1 - M\|x\|)\|x\|^2 \leq f_\omega(x) \leq (1 + M\|x\|)\|x\|^2$.



Proof. For each $\omega \in \partial\mathcal{C}$ let us define the 'shape' $\mathcal{S}_\omega(\mathcal{C})$ of $\mathcal{C}$ with respect to ω as the orthogonal projection of $\mathcal{C}$ onto the affine tangent space $T_\omega(\partial\mathcal{C})$ and denote $e(\omega) \geq 0$ the Euclidean distance from ω to $\partial(\mathcal{S}_\omega(\mathcal{C}))$. If there were a sequence of points $(\omega_k)_{k\geq 0}$ in $\partial\mathcal{C}$ such that $\lim_{k\to+\infty} e(\omega_k) = 0$, by compactness of $\partial\mathcal{C}$ we could find ω in this boundary with $e(\omega) = 0$. But this is not possible because $\partial\mathcal{C}$ is differentiable at ω. So there is a constant $r > 0$ such that $e(\omega) \geq 2r$ for all $\omega \in \partial\mathcal{C}$.

Moreover, if we introduce the ball $\mathcal{B}_\omega = \{m \in T_\omega(\partial\mathcal{C}) : \|m - \omega\| < 2r\}$ in $T_\omega(\partial\mathcal{C})$, the intersection of the full open cylinder $\{m + sN(\omega) : m \in \mathcal{B}_\omega, s \in \mathbb{R}\}$ with $\partial\mathcal{C}$ has exactly two connected components one of which contains ω and that we will denote $\mathcal{U}_\omega$. We then immediately get from the convexity of $\mathcal{C}$ that for each $m \in \mathcal{B}_\omega$ there is a unique $s = \varphi_\omega(m) \in \mathbb{R}$ such that $m + sN(\omega) \in \mathcal{U}_\omega$ and so we get a map $\varphi_\omega : \mathcal{B}_\omega \to \mathbb{R}$ of class C^3.

Let us now fix an open set $\mathcal{W}$ in $\mathbb{R}^n$ such that $\Gamma = \mathcal{W} \cap \partial\mathcal{C}$ is non-empty and parallelizable.

We can then find a family $(\Upsilon_\omega)_{\omega\in\Gamma}$ of affine isometries in $(\mathbb{R}^n, \|\cdot\|)$ depending smoothly on $\omega \in \Gamma$ such that $\Upsilon_\omega(\omega) = 0$ and $\overrightarrow{\Upsilon}_\omega \cdot N(\omega) = e_n$.

Since $\mathcal{C}$ is a star-shaped set with respect to one (arbitrary) of its point and has a boundary which is a hypersurface of class C^3, there exists a function $F : \mathbb{R}^n \to \mathbb{R}$ with the same smoothness such that $\partial\mathcal{C} = F^{-1}(0)$ and the family $(F_\omega)_{\omega\in\Gamma}$ defined by $F_\omega = F \circ \Upsilon_\omega^{-1}$ thus satisfies $\Upsilon_\omega(\partial\mathcal{C}) = F_\omega^{-1}(0)$.

Next define for each $\omega \in \Gamma$ the function $g_\omega : B(0,r) \subset \mathbb{R}^{n-1} \to \mathbb{R}$ by $g_\omega(u) = \varphi_\omega(\Upsilon_\omega^{-1}(u,0))$; as we have $F_\omega(u, g_\omega(u)) = 0$ for all $u = (u_1,\ldots,u_{n-1}) \in B(0,2r)$, we get by differentiation that all the partial derivatives of third order for g_ω with respect to $u_1,\ldots,u_{n-1}$ are rational expressions of the partial derivatives for the function $F_\omega : (u,z) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \mapsto F_\omega(u,z)$ with respect to $u_1,\ldots,u_{n-1}$, z computed at the point $(u,z) = (u, g_\omega(u))$.

Therefore, by continuity of these partial derivatives with respect to both u and ω, if $\mathcal{O}$ is an open set in $\mathbb{R}^n$ whose closure is in $\mathcal{W}$, there is a constant $A > 0$ such that for all $\omega \in \mathcal{O} \cap \partial\mathcal{C}$ and all $u \in \bar{B}(0,r)$ (closed ball) we have from the Taylor expansion

$$\sum_{i=1}^{n-1} \lambda_i(\omega) u_i^2 - A\|u\|^3 \leq g_\omega(u) \leq \sum_{i=1}^{n-1} \lambda_i(\omega) u_i^2 + A\|u\|^3,$$

where $\lambda_1(\omega),\ldots,\lambda_{n-1}(\omega)$ are the principal curvatures of $\partial\mathcal{C}$ at the point $\omega \in \partial\mathcal{C}$.

The strict convexity and the compactness of $\partial\mathcal{C}$ together with the continuity of $\lambda_1,\ldots,\lambda_{n-1}$ over $\partial\mathcal{C}$ implies there exists $\alpha > 0$ such that $0 < \lambda_i(\omega) \leq 1/\alpha^2$ for all $\omega \in \partial\mathcal{C}$ and all $i \in \{1,\ldots,n-1\}$; so we can define $f_\omega(x) = g_\omega(\sqrt{\lambda_1(\omega)}x_1,\ldots,\sqrt{\lambda_{n-1}(\omega)}x_{n-1})$ for each $\omega \in \mathcal{O} \cap \partial\mathcal{C}$ and all $x = (x_1,\ldots,x_{n-1})$ in $B(0,\rho) \subset \mathbb{R}^{n-1}$, where $\rho = \min\{r\alpha, \alpha^3/A\} > 0$. We then get a family of functions $(f_\omega)_{\omega\in\mathcal{O}\cap\partial\mathcal{C}}$ that satisfies *(ii)* and *(iii)* of Lemma 1.1 with $M = A/\alpha^3 > 0$.

Finally, as $\partial\mathcal{C}$ is compact, it can be recovered by a finite number of open sets like $\mathcal{O}$ above and this proves Lemma 1.1. □

The aim of the second lemma is to give us an estimate of the value $F_\mathcal{C}(m_0, v)$ of the Finsler metric $F_\mathcal{C}$ at a point $m_0 \in \mathcal{C}$ for a vector $v \in \mathbb{R}^n$ in terms of the Euclidean distance from m_0 to $\partial\mathcal{C}$ and the direction of v.

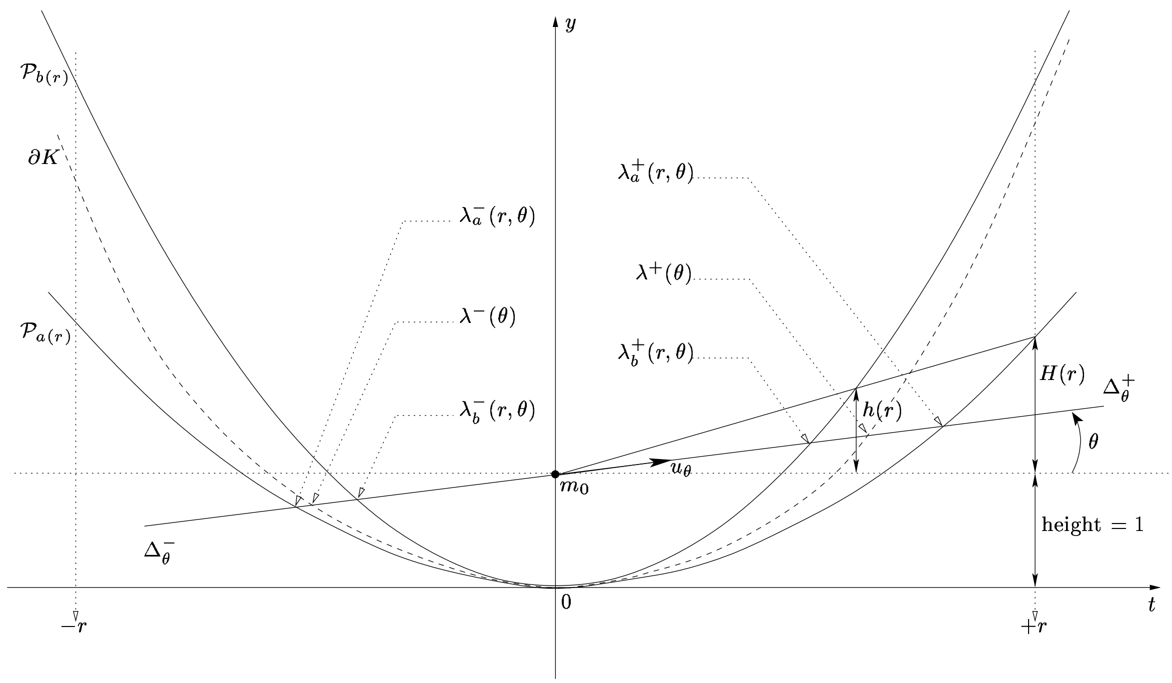
More precisely, let us consider a bounded open convex domain K in $\mathbb{R}^2$ such that there exist positive numbers $M > 0$, $r \geq M+2$ and a function $\xi : (-r, r) \to \mathbb{R}$ which satisfies $\mathrm{Graph}(\xi) \subset \partial K$ together with

$$\frac{1}{4}a(r)t^2 \leq \xi(t) \leq \frac{1}{4}b(r)t^2 \quad \text{for all} \quad |t| < r, \tag{1}$$

where $a(r) = 1 - M/r > 0$ and $b(r) = 1 + M/r > 0$. For each $c \in \mathbb{R}$ denote by $\mathcal{P}_c$ the parabola in $\mathbb{R}^2$ whose equation is $y = \frac{1}{4}ct^2$. Then define $H(r) \in \mathbb{R}$ so that the point $(r, H(r)+1)$ is in $\mathcal{P}_{a(r)}$ and let $h(r) \in \mathbb{R}$ so that $h(r)+1$ is the second component of the intersection point between $\mathcal{P}_{b(r)}$ and the straight line passing through $m_0 = (0,1)$ and $(r, H(r)+1)$. Hence,

$$H(r) = \frac{1}{4}a(r)r^2 - 1 \quad \text{and} \quad h(r) = \frac{2H(r)}{b(r)r^2}\left(H(r) + \sqrt{H(r)^2 + b(r)r^2}\right).$$

As $r \geq M+2$, we have $H(r) > 0$ and $h(r) > 0$, which implies the point m_0 is in K by (1) and the convexity of K.



For $\theta \in (-\pi/2, \pi/2)$ we can therefore consider $F_K(m_0, u_\theta)$, where $u_\theta = (\cos\theta, \sin\theta)$, and control this quantity in terms of $r \in [M+2, +\infty)$ *uniformly* in θ:

Lemma 1.2. *For all $\theta \in (-\pi/2, \pi/2)$ we have*

$$\varphi(r) \leq F_K(m_0, u_\theta) \leq \psi(r),$$

where φ and ψ are functions with the property that $\lim_{r\to+\infty} \varphi(r) = \lim_{r\to+\infty} \psi(r) = 1$.

Remark. This result asserts that the bigger r is (and then the closer is ∂K to the parabola with equation $y = \frac{1}{4}t^2$ within $(-r, r) \times \mathbb{R}$), the closer to the unit canonical Euclidean sphere is the Finsler sphere $\{v \in \mathbb{R}^2 : F_K(m_0, v) = 1\}$ at the point m_0.

Proof. For $\theta \in (-\pi/2, \pi/2)$ introduce the half lines $\Delta_\theta^+ = m_0 + \mathbb{R}^+ u_\theta$ and $\Delta_\theta^- = m_0 + \mathbb{R}^- u_\theta$ and let $\lambda_a^+(r,\theta)$ (respectively $\lambda_a^-(r,\theta)$) be the Euclidean distance between m_0 and the intersection point $\Delta_\theta^+ \cap \mathcal{P}_{a(r)}$ (respectively $\Delta_\theta^- \cap \mathcal{P}_{a(r)}$). Define $\lambda_b^+(r,\theta)$ (respectively $\lambda_b^-(r,\theta)$) in a similar way for $\mathcal{P}_{b(r)}$ and denote by $\lambda^+(\theta)$ (respectively $\lambda^-(\theta)$) the Euclidean distance between m_0 and the intersection point $\Delta_\theta^+ \cap \partial K$ (respectively $\Delta_\theta^- \cap \partial K$).
Using these notations, we then have $F_K(m_0, u_\theta) = \dfrac{1}{\lambda^-(\theta)} + \dfrac{1}{\lambda^+(\theta)}$ from the definition of F_K (see Introduction) and a straightforward computation gives

$$\lambda_a^+(r,\theta) = \frac{2}{\sqrt{\sin^2\theta + a(r)\cos^2\theta} - \sin\theta} \quad \text{and} \quad \lambda_a^-(r,\theta) = \frac{2}{\sqrt{\sin^2\theta + a(r)\cos^2\theta} + \sin\theta} \tag{2}$$

with the analogous formulas for $\lambda_b^+(r,\theta)$ and $\lambda_b^-(r,\theta)$ by changing $a(r)$ into $b(r)$.

There are now three cases to be considered.

• <u>First case</u>: $|\tan\theta| \le H(r)/r$.

In this situation the half line Δ_θ^+ (respectively Δ_θ^-) cuts ∂K on the curve $\{(t, \xi(t)) : |t| < r\}$ and we have by (1)

$$\lambda_a^-(r,\theta) \le \lambda^-(\theta) \le \lambda_b^-(r,\theta) \quad \text{and} \quad \lambda_b^+(r,\theta) \le \lambda^+(\theta) \le \lambda_a^+(r,\theta)$$

which implies

$$\frac{1}{\lambda_a^+(r,\theta)} + \frac{1}{\lambda_b^-(r,\theta)} \le \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \le \frac{1}{\lambda_a^-(r,\theta)} + \frac{1}{\lambda_b^+(r,\theta)} \ . \tag{3}$$

Since $0 < a(r) \le b(r)$, we then have by (2) the inequality

$$\frac{1}{\lambda_a^+(r,\theta)} + \frac{1}{\lambda_b^-(r,\theta)} \ge \sqrt{\sin^2\theta + a(r)\cos^2\theta} = \sqrt{1 + (a(r)-1)\cos^2\theta} = \sqrt{1 - (M\cos^2\theta)/r}$$

and thus

$$\frac{1}{\lambda_a^+(r,\theta)} + \frac{1}{\lambda_b^-(r,\theta)} \ge \sqrt{a(r)} \ . \tag{4}$$

On the other hand,

$$\sqrt{\sin^2\theta + b(r)\cos^2\theta} = \sqrt{b(r) + (1 - b(r))\sin^2\theta} = \sqrt{b(r) - (M\cos^2\theta)/r} \le \sqrt{b(r)} \ ,$$

which leads to

$$\frac{1}{\lambda_a^-(r,\theta)} + \frac{1}{\lambda_b^+(r,\theta)} \le \sqrt{b(r)} \ . \tag{5}$$

Finally, from (3), (4) and (5) we get

$$\sqrt{a(r)} \le \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \le \sqrt{b(r)} \ .$$

• Second case: $\tan\theta > H(r)/r$.

The convexity of K and (1) imply the intersection point $\Delta_\theta^+ \cap \partial K$ lies in $[0,+\infty)\times[h(r)+1,+\infty)$ and therefore $\lambda^+(\theta) \geq h(r)$. As the point $\Delta_\theta^- \cap \partial K$ is still on the curve $\{(t,\xi(t)) : |t| < r\}$, we can write by condition (1) that $\lambda_a^-(r,\theta) \leq \lambda^-(\theta) \leq \lambda_b^-(r,\theta)$ in order to get

$$\frac{1}{\lambda_b^-(r,\theta)} \leq \frac{1}{\lambda^-(\theta)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{h(r)} \leq \frac{1}{\lambda_a^-(r,\theta)} + \frac{1}{h(r)}. \tag{6}$$

From $\tan\theta > H(r)/r$ (together with $\theta \in (-\pi/2,\pi/2)$), we have $\sin\theta > \dfrac{1}{\sqrt{1+H(r)^2/r^2}}$ and then

$$\frac{1}{\lambda_b^-(r,\theta)} = \frac{1}{2}\left(\sin\theta + \sqrt{\sin^2\theta + b(r)\cos^2\theta}\right) \geq \sin\theta \geq \frac{1}{\sqrt{1+H(r)^2/r^2}}\ . \tag{7}$$

On the other hand,

$$\begin{aligned}\frac{1}{\lambda_a^-(r,\theta)} &= \frac{1}{2}\left(\sin\theta + \sqrt{\sin^2\theta + a(r)\cos^2\theta}\right)\\ &\leq \frac{1}{2}\left(1 + \sqrt{1 + (a(r)-1)\cos^2\theta}\right) = \frac{1}{2}\left(1 + \sqrt{1 - (M\cos^2\theta)/r}\right) \leq 1.\end{aligned} \tag{8}$$

Hence, (6), (7) and (8) imply

$$\frac{1}{\sqrt{1+H(r)^2/r^2}} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq 1 + \frac{1}{h(r)}\ .$$

• Third case: $\tan\theta < -H(r)/r$.

Applying the same arguments as previously (where the roles of Δ_θ^- and Δ_θ^+ have been exchanged), we also get here

$$\frac{1}{\sqrt{1+H(r)^2/r^2}} \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} \leq 1 + \frac{1}{h(r)}\ .$$

Conclusion:

If we define $\varphi(r) = \min\left\{\dfrac{1}{\sqrt{1+H(r)^2/r^2}}, \sqrt{a(r)}\right\}$ and $\psi(r) = \max\left\{1 + \dfrac{1}{h(r)}, \sqrt{b(r)}\right\}$, the three cases above say that for all $\theta \in (-\pi/2,\pi/2)$ we have

$$\varphi(r) \leq \frac{1}{\lambda^-(\theta)} + \frac{1}{\lambda^+(\theta)} = F_K(m_0,u_\theta) \leq \psi(r),$$

which proves Lemma 1.2 since $\lim_{r\to+\infty}\varphi(r) = \lim_{r\to+\infty}\psi(r) = 1$. □

Considering a convex set $\mathcal{C}$ as described at the beginning of the present section, we are now able to prove the key idea of this paper which states that the closer to $\partial\mathcal{C}$ we are, the less the metric $d_\mathcal{C}$ on $\mathcal{C}$ is different from a Riemannian one. In other words, we show the closer to $\partial\mathcal{C}$ a point $p \in \mathcal{C}$ is, the closer to an ellipsoid centered at p is the unit sphere $\{v \in \mathbb{R}^n : F_\mathcal{C}(p,v) = 1\}$ of the norm $F_\mathcal{C}(p,\cdot)$ in $T_p(\mathcal{C}) = \mathbb{R}^n$:

Proposition 1.3. *Let $\mathcal{C}$ be a bounded open convex domain in $\mathbb{R}^n$ whose boundary $\partial\mathcal{C}$ is a hypersurface of class C^3 that is strictly convex. For any $p \in \mathcal{C}$ let $\delta(p) > 0$ be the Euclidean distance from p to $\partial\mathcal{C}$. Then there exists a family $(\vec{\ell}_p)_{p\in\mathcal{C}}$ of linear transformations in $\mathbb{R}^n$ such that*

$$\lim_{\delta(p)\to 0} \frac{F_{\mathcal{C}}(p,v)}{\|\vec{\ell}_p(v)\|} = 1 \text{ uniformly in } v \in \mathbb{R}^n\backslash\{0\}.$$

Remark. The proposition means that the unit sphere of the norm $F_{\mathcal{C}}(p,\cdot)$ approaches the ellipsoid defined by the unit sphere of the *Euclidean* norm $\|\cdot\| \circ \vec{\ell}_p$ in $\mathbb{R}^n$ as $\delta(p)$ goes to zero.

Proof. Let $p \in \mathcal{C}$ sufficiently close to $\partial\mathcal{C}$ such that $\delta = \delta(p) < \rho$ (see Lemma 1.1) and that there is a unique $\omega \in \partial\mathcal{C}$ satisfying $\|p - \omega\| = \delta$.
Define $\Phi_p \in \mathrm{GL}(\mathbb{R}^n)$ by $\Phi_p(x,z) = (X,Z) = (2x/\sqrt{\delta}, z/\delta)$ from $\mathbb{R}^{n-1}\times\mathbb{R}$ to $\mathbb{R}^{n-1}\times\mathbb{R}$; it sends $\Upsilon_\omega(p) = \delta e_n$ to $m_0 = (0,\dots,0,1)$ and changes inequality *(iii)* in Lemma 1.1 applied to ω into

$$\frac{1}{4}(1 - M\sqrt{\delta}\|X\|/2)\|X\|^2 \le f_\omega(\sqrt{\delta}X)/\delta \le \frac{1}{4}(1 + M\sqrt{\delta}\|X\|/2)\|X\|^2 \quad \text{for all} \quad X \in B(0, 1/\delta^{\frac{1}{4}}).$$

Hence we deduce

$$\frac{1}{4}(1 - M\delta^{\frac{1}{4}})\|X\|^2 \le f_\omega(\sqrt{\delta}X)/\delta \le \frac{1}{4}(1 + M\delta^{\frac{1}{4}})\|X\|^2 \quad \text{for all} \quad X \in B(0, 1/\delta^{\frac{1}{4}}). \tag{9}$$

Now introduce $r = 1/\delta^{\frac{1}{4}}$, fix $k \in \{1,\dots,n-1\}$ and consider the function $\xi(t) = f_\omega(\sqrt{\delta}te_k)/\delta$ for $t \in (-r,r)$; in restriction to the 2-plane $\Pi_k = \{te_k + Ze_n : (t,Z) \in \mathbb{R}^2\}$, condition (9) means

$$\frac{1}{4}a(r)t^2 \le \xi(t) \le \frac{1}{4}b(r)t^2 \quad \text{for all} \quad |t| < r, \tag{10}$$

where $a(r) = 1 - M/r$ and $b(r) = 1 + M/r$.
If we then define $\ell_p \in \mathrm{Aff}(\mathbb{R}^n)$ by $\ell_p = \Phi_p \circ \Upsilon_\omega$ and focus on the convex set $K = \partial(\ell_p(\mathcal{C})) \cap \Pi_k$, we are exactly in the situation of Lemma 1.2 (where condition (1) is given by (10) above) which implies

$$\varphi(r) \le \mathcal{F}_p(m_0, \cos\theta e_k + \sin\theta e_n) \le \psi(r) \quad \text{for all} \quad \theta \in (-\pi/2, \pi/2),$$

where $\mathcal{F}_p$ is the Finsler metric associated to $\ell_p(\mathcal{C})$ and φ, ψ are functions such that $\lim\limits_{r\to+\infty}\varphi(r) = \lim\limits_{r\to+\infty}\psi(r) = 1$.
As $\mathcal{F}_p(m_0,\cdot)$ is a symmetric, continuous and positive homogeneous function over $\mathbb{R}^n$, the last inequalities mean

$$\varphi(r)\|v\| \le \mathcal{F}_p(m_0, v) \le \psi(r)\|v\| \quad \text{for all} \quad v \in \Pi_k.$$

But $k \in \{1,\dots,n-1\}$ has been chosen arbitrarily and we actually get

$$\varphi(r)\|v\| \le \mathcal{F}_p(m_0, v) \le \psi(r)\|v\| \quad \text{for all} \quad v \in \mathbb{R}^n \tag{11}$$

because $\varphi(r)$ and $\psi(r)$ do not depend on k.
Since ℓ_p is an affine transformation, it is an isometry from $(\mathcal{C}, d_{\mathcal{C}})$ to $(\ell_p(\mathcal{C}), d_{\ell_p(\mathcal{C})})$ or equivalently an isometry from $(\mathcal{C}, F_{\mathcal{C}})$ to $(\ell_p(\mathcal{C}), \mathcal{F}_p)$, that is $\mathcal{F}_p(\ell_p(m), \vec{\ell}_p(v)) = F_{\mathcal{C}}(m,v)$ for all $m \in \mathcal{C}$ and all $v \in \mathbb{R}^n$, where $\vec{\ell}_p$ is the linear part of ℓ_p.

Therefore, with $\ell_p(p) = m_0$, we finally obtain

$$\varphi(r)\|\vec{\ell}_p(v)\| \leq F_{\mathcal{C}}(p,v) \leq \psi(r)\|\vec{\ell}_p(v)\| \quad \text{for all} \quad v \in \mathbb{R}^n$$

and Proposition 1.3 follows since $r = 1/\delta^{\frac{1}{4}} \to +\infty$ as $\delta \to 0$. $\square$

2. Proof of the theorem

We give in this section the proof of the central *rigidity* result announced in the Introduction as a consequence of Proposition 1.3 for a bounded open convex domain $\mathcal{C}$ in $\mathbb{R}^n$ with a strictly convex boundary of class C^3:

Theorem 2.1. *If $\partial\mathcal{C}$ is not an ellipsoid, then any subgroup Γ of the isometry group* $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ *with a proper action on $\mathcal{C}$ is finite.*

Before showing this result, we establish the following lemma:

Lemma 2.2. *Let $\mathcal{C}$ be any bounded open* strictly *convex domain in $\mathbb{R}^n$ and Γ any subgroup of* $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ *with a proper action on $\mathcal{C}$. Then if there is a point in $\mathcal{C}$ with a finite orbit, Γ has to be finite too.*

Proof of Lemma 2.2. We will consider $\mathcal{C}$ in $P(\mathbb{R}^{n+1})$ via the classical imbedding $\mathbb{R}^n \hookrightarrow P(\mathbb{R}^{n+1})$ (see for example [9], section 2). Then the strict convexity of $\mathcal{C}$ implies that $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}}) < \mathrm{PGL}(\mathbb{R}^{n+1})$ according to [10], Proposition 3. Let $p_0 \in \mathcal{C}$ with a finite orbit $\Gamma\cdot p_0$ and pick $n+1$ points $p_1, \ldots, p_{n+1}$ in $\mathcal{C}$ such that $(p_0, p_1, \ldots, p_{n+1})$ is a projective frame. Next define the sequence $(\Gamma_j)_{0\leq j\leq n+1}$ of subgroups in Γ by

$$\Gamma_0 = \mathrm{Stab}_{\Gamma}(p_0) \text{ and } \Gamma_j = \mathrm{Stab}_{\Gamma_{j-1}}(p_j) \text{ for all } j \in \{1, \ldots, n+1\},$$

where $\mathrm{Stab}_G(x)$ denotes the stabilizer of the point x under the action of the group G. We therefore have $\Gamma_{n+1} < \cdots < \Gamma_1 < \Gamma_0 < \Gamma$ with $\Gamma_{n+1} = \{Id_{\mathcal{C}}\}$ because $\Gamma_{n+1} < \mathrm{PGL}(\mathbb{R}^{n+1})$ fixes the projective frame $(p_0, p_1, \ldots, p_{n+1})$.
As for each $j \in \{0, \ldots, n\}$ we have $d_{\mathcal{C}}(p_j, \gamma\cdot p_{j+1}) = d_{\mathcal{C}}(p_j, p_{j+1})$ for all $\gamma \in \Gamma_j$, the orbit $\Gamma_j\cdot p_{j+1}$ lies in a compact set and thus if it were infinite, it would have an accumulation point; but this is not possible because the action is proper. Hence, $\Gamma_j\cdot p_{j+1}$ is finite for all $j \in \{0, \ldots, n\}$.
Since $\Gamma/\Gamma_0 \cong \Gamma\cdot p_0$ and $\Gamma_j/\Gamma_{j+1} \cong \Gamma_j\cdot p_{j+1}$ (as sets) for all $j \in \{0, \ldots, n\}$, we finally get that all the indexes $[\Gamma : \Gamma_0]$, $[\Gamma_0 : \Gamma_1], \ldots, [\Gamma_n : \Gamma_{n+1}]$ are finite. So, Γ is finite. $\square$

Proof of Theorem 2.1. Consider an infinite subgroup Γ of the isometry group $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ whose action on $\mathcal{C}$ is proper and choose a point p in $\mathcal{C}$. As Γ is not finite, the orbit $\Gamma\cdot p$ is infinite by Lemma 2.2 and thus if it where contained in a compact set, it would have an accumulation point which is not possible since the action is proper. So the Euclidean distance between $\Gamma\cdot p$ and $\partial\mathcal{C}$ is zero, which means there exists a sequence $(\gamma_k)_{k\geq 0}$ in Γ such that the limit of the Euclidean distance between $p_k = \gamma_k \cdot p$ and $\partial\mathcal{C}$ is zero as k goes to infinity. Then, from Proposition 1.3, there is a sequence $(\vec{\ell}_k)_{k\geq 0} \in \mathrm{GL}(\mathbb{R}^n)$ which satisfies

$$\lim_{k\to+\infty} \frac{F_{\mathcal{C}}(p_k, T_{p_k}\gamma_k(v))}{\|\vec{\ell}_k(T_{p_k}\gamma_k(v))\|} = 1 \text{ uniformly in } v \in \mathbb{R}^n\backslash\{0\},$$

where $\|\cdot\|$ still denotes the canonical Euclidean norm in $\mathbb{R}^n$.

But since $\gamma_k \in \mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$, we have $F_{\mathcal{C}}(p_k, T_{p_k}\gamma_k(v)) = F_{\mathcal{C}}(p, v)$ and the equality above writes

$$\lim_{k\to+\infty} \|\mathcal{L}_k(v))\| = F_{\mathcal{C}}(p, v) \text{ uniformly in } v \in \mathbb{R}^n\backslash\{0\}$$

with $\mathcal{L}_k = \vec{\ell}_k \circ T_{p_k}\gamma_k$.

As this limit is uniform in v, the sequence $(\mathcal{L}_k)_{k\geq 0}$ is bounded in $\mathrm{L}(\mathbb{R}^n)$ (the space of linear endomorphisms of $\mathbb{R}^n$ endowed with the operator norm associated to $\|\cdot\|$) and we can therefore assume it converges to a limit $\mathcal{L} \in \mathrm{L}(\mathbb{R}^n)$. We then obtain $F_{\mathcal{C}}(p, v) = \|\mathcal{L}(v)\|$ for all $v \in \mathbb{R}^n$ (hence $\mathcal{L} \in \mathrm{GL}(\mathbb{R}^n)$), which means the Finsler metric $F_{\mathcal{C}}$ is Riemannian at the point $p \in \mathcal{C}$ (the norm $\|\cdot\| \circ \mathcal{L}$ is indeed Euclidean in $\mathbb{R}^n$) and since this is true for every choice of p, we get that $F_{\mathcal{C}}$ (or equivalently the corresponding Hilbert metric $d_{\mathcal{C}}$) is a Riemannian metric on $\mathcal{C}$.

At this stage, recall the following fact (see for example [3], page 85):

Theorem 2.3. (E. Beltrami, 1866) *Let a connected open set X of the projective space $P(\mathbb{R}^{n+1})$ be metrized so that the metric is Riemannian and the geodesics lie on projective lines. Then the sectional curvature of this Riemannian metric is constant.*

Using this, we deduce the very useful result:

Theorem 2.4. *Let $\mathcal{C}$ be* any *bounded open convex domain in $\mathbb{R}^n$. If the metric $d_{\mathcal{C}}$ is Riemannian, then $\partial\mathcal{C}$ is an ellipsoid.*

Proof of Theorem 2.4. From Theorem 2.3 with $X = \mathcal{C}$, the sectional curvature of the Riemannian metric $d_{\mathcal{C}}$ is constant and thus non-positive since the space $(\mathcal{C}, d_{\mathcal{C}})$ is not compact. Then, by a theorem of H. Busemann ([3], p. 269, Theorem 41.6), the metric space $(\mathcal{C}, d_{\mathcal{C}})$ has non-positive curvature in the sense of Busemann (see [3], p. 237 for the definition) and this finally implies $\partial\mathcal{C}$ is an ellipsoid from a result due to P. Kelly and E. Straus ([11]). $\square$

This ends the proof of Theorem 2.1. $\square$

Remark. Although Theorem 2.4 seems generally to have been accepted, we are unaware of any proof in the literature.

We now determine whether there are quotients of $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ with finite *volume*:

Corollary 2.5. *Let $\mathcal{C}$ be as in Theorem 2.1. Then if $\partial\mathcal{C}$ is* not *an ellipsoid, $(\mathcal{C}, d_{\mathcal{C}})$ does* not *allow quotients of finite volume by subgroups of* $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ *whose actions on $\mathcal{C}$ are proper.*

Proof. The volume here is the one μ associated to the Finsler metric $F_{\mathcal{C}}$ on $\mathcal{C}$ and described in the Introduction. Now if a subgroup Γ of $\mathrm{Isom}(\mathcal{C}, d_{\mathcal{C}})$ has a proper action on $\mathcal{C}$, it is finite since $\partial\mathcal{C}$ is not an ellipsoid according to Theorem 2.1. So, if there where a fundamental domain $D \subset \mathcal{C}$ with $\mu(D)$ finite, we would get that $\mu(\mathcal{C}) = \sum_{\gamma\in\Gamma} \mu(\gamma{\cdot}D)$ is finite too; but this is not true. (If we indeed use Lemma 1.1 and Lemma 1.2 over the intersection U of $\mathcal{C}$ with a small enough ball in $(\mathbb{R}^n, \|\cdot\|)$ centered at any given point in $\partial\mathcal{C}$, we can get an estimate of the Euclidean volume of the ball $\{v \in \mathbb{R}^n : F_{\mathcal{C}}(p,v) \leq 1\}$ and compute that $\mu(U) = +\infty$.) □

Acknowledgment. The first author would like to thank Y. Benoist and W. M. Goldman for their interesting remarks about the problem from the affine point of view.

References

[1] Y. Benoist, *Automorphismes des cônes convexes*, Invent. Math. **141** (2000), 149-193.

[2] J.-P. Benzécri, *Sur les variétés localement affines et projectives*, Bull. Soc. Math. France **88** (1960), 229-332.

[3] H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955.

[4] H. Busemann, P. Kelly, *Projective geometry and projective metrics*, Academic Press, New York, 1953.

[5] D. Egloff, *Some new developments in Finsler geometry*, Ph.D. thesis, University of Freiburg, 1995.

[6] D. Egloff, *Uniform Finsler Hadamard manifolds*, Ann. Inst. H. Poincaré Phys. Théor. **66** (1997), 323-357.

[7] P. Foulon, *Géométrie des équations différentielles du second ordre*, Ann. Inst. H. Poincaré Phys. Théor. **45** (1986), 1-28.

[8] P. Foulon, *Locally symmetric Finsler spaces in negative curvature*, C. R. Acad. Sci. Paris **324** (1997), 1127-1132.

[9] W. M. Goldman, *Projective geometry on manifolds*, Lecture Notes, University of Maryland, 1988.

[10] P. de la Harpe, *On Hilbert's metric for simplices*, in *Geometric group theory* (vol. I, p. 97-119). Cambridge University Press, 1993.

[11] P. Kelly, E. Straus, *Curvature in Hilbert geometry*, Pacific J. Math. **8** (1958), 119-125.

[12] É. Socié-Méthou, *Comportements asymptotiques et rigidités en géométrie de Hilbert*, Ph.D. thesis, University of Strasbourg, 2000.

[13] É. Socié-Méthou, *Caractérisation des ellipsoïdes de $\mathbb{R}^n$ par leur groupe d'automorphismes*, to appear in Ann. Sci. École Norm. Sup.

[14] P. Verovic, *Problème de l'entropie minimale pour les métriques de Finsler*, Ergodic Theory Dynam. Systems **19** (1999), 1637-1654.

[15] P. Verovic (joint work with B. Colbois), *Un résultat de rigidité pour les métriques de Hilbert*, Séminaire de Théorie Spectrale et Géométrie, Institut Fourier (Grenoble), **18** (1999-2000), 171-173.

Bruno Colbois
Université de Neuchâtel
Institut de mathématiques
Rue Émile Argand 13
CH-2007 Neuchâtel
Switzerland

E-mail address: bruno.colbois@unine.ch

Patrick Verovic
Université de Savoie
Campus scientifique
Laboratoire de mathématiques
73376 Le Bourget-du-Lac Cedex
France

E-mail address: verovic@univ-savoie.fr