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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR LINEAR CONSERVATIONS LAWS WITH VELOCITY FIELDS IN L^∞.

OLIVIER BESSON * AND JÉRÔME POUSIN †

Abstract. A Space-Time Integrated Least Squares (STILS) method is derived for solving the linear conservation law with a velocity field in L^∞. An existence and uniqueness result is given for the solution of this equation. A maximum principle is established and finally a comparison with renormalized solution is presented.

Key words. Conservation laws, transport equation, space-time least squares methods.

1. Introduction. Many works are dedicated to linear conservation laws, and according to the regularity of datum, different points of view have been used. The semi-group approach first developed in [7] requires a C^1 regularity for the velocity field. Moreover this vector field has to be extendable by zero outside a neighborhood of the spatial domain Ω. The characteristic flow generated by the velocity u can be defined for less regular fields. In [18] for a velocity field in $L^1(0,T;W^{1,1}(\Omega))$ with $\mathrm{div}\ u \in L^1(0,T;L^\infty(\Omega))$ the notion of renormalized solution is introduced allowing to handle initial conditions with very low regularity. When the velocity field u belongs to $H^{1/2}$ with a divergence free existence and uniqueness has been proved in [14] and when the velocity field u belongs to BV, results of existence and uniqueness of solutions in L^∞ is provided in [3, 13], see also [9]. For domains Ω included in $\mathbb{R}^2$ and for a time independent velocity field u in L^2_{loc} with a divergence free a solution to the linear conservation laws is presented in [23] and compare to renormalized solutions. The question of uniqueness for weak solutions in L^∞ to linear conservation laws is discussed in [16] for a velocity field u in L^∞ with a divergence free and a domain included in $\mathbb{R}^2$.
In this paper, the question of existence and uniqueness is addressed for linear conservation laws on a domain Ω with Lipchitz boundary that satisfy the cone property. In our case the velocity field u is only bounded. i.e. $u \in L^\infty$ and $\mathrm{div}\ u \in L^\infty$. The proposed method does not deal with the characteristic flow generated by the velocity field, but uses the functional setting of anisotropic Sobolev's spaces in the same way as in [22] combined with a formulation of the problem in the time-space least squares sense in the same spirit as in [20] and [2]. In [18], the velocity field is required to be more regular than in our formulation ($u \in L^\infty$, and $\mathrm{div}\ u \in L^\infty$). This allow them to handle boundary conditions with very few regularity. In our method we must assume that the boundary conditions have some regularity.
The least squares method is widely used to solve partial differential equations, see [19] and [20] for elasticity and fluid mechanics problems. Few general mathematical results have been obtained for this method in the case of first order time dependent conservation laws. It seems that the STILS method (Space-Time Integrated Least Squares) is originated to [11] and [26]. In [11, 28], a least squares method is used to solve a 2D stationary first order conservation equation with regularity assumptions on the advection velocity. Other results have been obtained in [4, 5, 6]. In this paper, a general mathematical analysis of this method is given for the linear conservation law when the advection velocity u has low regularity, more precisely when $u \in L^\infty$, and $\mathrm{div}\ u \in L^\infty$. The solution obtained in this way is compared with renormalized solutions [18].
In section 2 a description of the problem is given. In section 3 a variational formulation of the problem is given. The section 4 is dedicated to the proof of the existence and uniqueness of

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solutions to the variational formulation described in section 3. Moreover a comparison with renormalized solution is given.

2. The problem description. Let $\Omega \subset \mathbb{R}^d$ be a domain with a Lipschitz boundary $\partial\Omega$ satisfying the cone property. If $T > 0$ is given, set $Q = \Omega\times]0, T[$. Consider an advection velocity $u : Q \to \mathbb{R}^d$ and $f : Q \to \mathbb{R}$ a given source term. In all this paper, the velocity u has the following regularity

$$u \in L^\infty(Q)^d \text{ and } \operatorname{div} u \in L^\infty(Q). \tag{2.1}$$

Let

$$\Gamma_- = \{x \in \partial\Omega : (\, u(x,t) \,|\, n(x) \,) < 0\}$$

where $n(x)$ is the outer normal to $\partial\Omega$ at point x. For the sake of the presentation, it is assumed that Γ_- do not depend on t.
The problem consists in finding a function $c : Q \to \mathbb{R}$ satisfying the following partial differential equation

$$\partial_t c + \operatorname{div}(c\, u) = f \quad \text{in } Q, \tag{2.2}$$

and the initial and inflow boundary conditions

$$c(x,0) = c_0(x) \qquad \text{for } x \text{ in } \Omega \tag{2.3}$$
$$c(x,t) = c_1(x,t) \qquad \text{for } x \text{ on } \Gamma_-. \tag{2.4}$$

As usual, when c_1, c_0, and u are sufficiently regular, changing the source term f if necessary, one can assume that $c_1 = 0$ on Γ_-, and $c_0 = 0$ on Ω. A similar result will be given later, using a suitable trace theorem.

3. Functional Setting. In this section the functional setting for a variational formulation of the problem (2.2-2.4) will be settled, (see also [4, 5, 6]). Moreover a trace operator is given in this context.

3.1. The Hilbert spaces. For $u \in L^\infty(Q)^d$, with $\operatorname{div}\ u \in L^\infty(Q)$, define $\widetilde{u}$ as

$$\widetilde{u} = (1, u_1, u_2, \dots, u_d)^t \in L^\infty(Q)^{d+1}$$

and for a sufficiently regular function φ defined on Q, set

$$\widetilde{\nabla}\varphi = \left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \dots, \frac{\partial\varphi}{\partial x_d}\right)^t,$$

and

$$\widetilde{\operatorname{div}}(\widetilde{u}\,\varphi) = \frac{\partial\varphi}{\partial t} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(u_i\,\varphi).$$

Finally $\widetilde{n}$ denotes the outward unit vector on ∂Q. The following theorem is proved in [12].

THEOREM 3.1. *Under the assumption $u \in L^\infty(Q)^d$, and $\operatorname{div}\ u \in L^\infty(Q)$, the normal trace of u, $(\,\widetilde{u} \,|\, \widetilde{n}\,)$ is in $L^\infty(\partial Q)$.*

Let now

$$\begin{aligned}\partial Q_- &= \{(x,t)\in \partial Q, (\,\widetilde{u}\,|\,\widetilde{n}\,) < 0\}\\ &= \Gamma_- \times (0,T)\ \cup\ \Omega\times\{0\},\end{aligned}$$

and set

$$c_b(x,t) = \begin{cases} c_0(x) & \text{if} \quad (x,t)\in\Omega\times\{0\}\\ c_1(x,t) & \text{if} \quad (x,t)\in\Gamma_-\times(0,T).\end{cases} \tag{3.1}$$

We will assume that

$$c_b \in L^2(\partial Q_-).$$

For $\varphi\in\mathcal{D}(\overline{Q})$, consider the norm

$$\|\varphi\|_{H(u,Q)} = \left(\|\varphi\|^2_{L^2(Q)} + \left\|\widetilde{\operatorname{div}}(\widetilde{u}\,\varphi)\right\|^2_{L^2(Q)} + \int_{\partial Q_-} |\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,|\varphi^2\,d\widetilde{\sigma}\right)^{1/2},$$

(see also [4, 5, 6, 8]) and then define the space $H(u,Q)$ as the closure of $\mathcal{D}(\overline{Q})$ for this norm:

$$H(u,Q) = \overline{\mathcal{D}(\overline{Q})}^{H(u,Q)}$$

If u is regular enough, it can be seen that

$$H(u,Q) = \left\{\rho\in L^2(Q), \widetilde{\operatorname{div}}(\widetilde{u}\,\rho)\in L^2(Q), \rho|_{\partial Q_-}\in L^2(\partial Q_-, |\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,|\,d\widetilde{\sigma})\right\}$$

(see e.g. [25, 22]). We now give a trace result for functions belonging to $H(u,Q)$. Let us start with the well known normal trace operator γ defined from $H(div,Q)$ with values in $H^{-\frac{1}{2}}(\partial Q)$ (see [21, 10])

$$v\mapsto (\,\widetilde{n}\,|\,v\,)\,|_{\partial Q},$$

$\forall v\in H(div,Q)$, with the associated Green formula:

$$\int_Q \widetilde{\operatorname{div}}(v)\psi + \left(v\,|\,\widetilde{\nabla}\psi\right)\,dx\,dt = <(\,v\,|\,\widetilde{n}\,), \psi>_{H^{-\frac{1}{2}}(\partial Q);H^{\frac{1}{2}}(\partial Q)},$$

$\forall\psi\in H^1(Q)$. Plugging $v=\widetilde{u}\rho$ in the previous formula, we have:

$$\int_Q \widetilde{\operatorname{div}}(\widetilde{u}\rho)\psi + \left(\widetilde{u}\,|\,\widetilde{\nabla}\psi\right)\rho\,dx\,dt = <\rho\,(\,\widetilde{u}\,|\,\widetilde{n}\,), \psi>_{H^{-\frac{1}{2}}(\partial Q);H^{\frac{1}{2}}(\partial Q)},$$

$\forall\psi\in H^1(Q)$. Let us now consider the bilinear form $L:\ \mathcal{D}(\overline{Q})\times\mathcal{D}(\overline{Q})\subset H(u,Q)\times H(u,Q)\longrightarrow\mathbb{R}$ defined for all $\varphi,\psi\in\mathcal{D}(\overline{Q})$ by:

$$L(\varphi,\psi) = \int_Q \widetilde{\operatorname{div}}(\widetilde{u}\varphi)\psi + \left(\widetilde{u}\,|\,\widetilde{\nabla}\psi\right)\varphi\,dx\,dt + \int_{\partial Q_-} |\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,|\varphi\psi\,d\widetilde{\sigma}.$$

Accounting for theorem 3.1 we have

$$\begin{aligned}|L(\varphi,\psi)| \le{}& \left\|\widetilde{\operatorname{div}}(\widetilde{u}\varphi)\right\|_{L^2(Q)}\|\psi\|_{L^2(Q)}\\ &+ \left\|\widetilde{\operatorname{div}}(\widetilde{u}\psi) - \widetilde{\operatorname{div}}(\widetilde{u})\psi\right\|_{L^2(Q)}\|\varphi\|_{L^2(Q)}\\ &+ \|\varphi\|_{L^2(\partial Q_-,|(\,\widetilde{u}\,|\,\widetilde{n}\,)|d\widetilde{\sigma})}\|\psi\|_{L^2(\partial Q_-,|(\,\widetilde{u}\,|\,\widetilde{n}\,)|\,d\widetilde{\sigma})}.\end{aligned}$$

And the following estimate holds true

$$|L(\varphi,\psi)| \le (1 \ + \ \|\mathrm{div}(u)\|_{L^\infty(Q)})\, \|\varphi\|_{H(u,Q)}\, \|\psi\|_{H(u,Q)} \,.$$

Since it is straightforward to check that $L(\varphi,\varphi) = \|\varphi\|^2_{L^2(\partial Q_+, |(\,\widetilde{u}\,|\,\widetilde{n}\,)|d\widetilde{\sigma})}$, if we extend by continuity the bilinear form L to $H(u,Q)\times H(u,Q)$ we have:

PROPOSITION 3.2. *Under the assumption $u \in L^\infty(Q)^d$, and $\mathrm{div}\ u \in L^\infty(Q)$ there exists a linear continuous trace operator*

$$\begin{aligned} \gamma_{\widetilde{n}} :\ H(u,Q) &\longrightarrow L^2(\partial Q, |\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,|d\widetilde{\sigma}) \\ \varphi \ &\mapsto \ \gamma_{\widetilde{n}}\varphi = \varphi_{|\partial Q}, \end{aligned}$$

which can be localized as:

$$\begin{aligned} \gamma_{\widetilde{n}_\pm} :\ H(u,Q) &\longrightarrow L^2(\partial Q_\pm, |\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,|d\widetilde{\sigma}) \\ \varphi \ &\mapsto \ \gamma_{\widetilde{n}_\pm}\varphi = \varphi_{|\partial Q_\pm}. \end{aligned}$$

Finally define the space

$$\begin{aligned} H_0 = H_0(u,Q,\partial Q_-) &= \{\rho \in H(u,Q),\ \rho = 0 \text{ on}\, \partial Q_-\} \\ &= H(u,Q) \cap \ \mathrm{Ker}\, \gamma_{\widetilde{n}_-}. \end{aligned}$$

3.2. Curved Poincaré inequality. We now give an extension of the *curved Poincaré inequality* obtained in [4, 5].

THEOREM 3.3. *If $u \in L^\infty(Q)^d$ and $\mathrm{div}\ u \in L^\infty(Q)$, the semi-norm on $H(u,Q)$ defined by*

$$|\rho|_{1,u} = \left(\int_Q (\widetilde{\mathrm{div}}(\widetilde{u}\rho))^2 dx\ dt + \int_{\partial Q_-} |\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,|\rho^2\ d\widetilde{\sigma}\right)^{1/2} \tag{3.2}$$

is a norm, equivalent to the norm given on $H(u,Q)$.

Proof. We have to show that there is a constant C such that

$$\|\varphi\|_{L^2(Q)} \le C \cdot |\varphi|_{1,u}$$

for all $\varphi \in \mathcal{D}(\overline{Q})$. We have

$$\int_Q \left[\widetilde{\mathrm{div}}(\widetilde{u}\,\varphi)\cdot\xi + \varphi\cdot\left(\widetilde{u}\,|\,\widetilde{\nabla}\xi\right)\right] dx\ dt - \int_{\partial Q_-} \xi\varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma} = \int_{\partial Q_+} \xi\varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma} \tag{3.3}$$

for all regular enough function ξ. For $\alpha : (0,T) \to \mathbb{R}$, choose $\xi = \alpha\cdot\varphi$, then

$$\frac{\partial\xi}{\partial t} + (\,u\,|\,\nabla\xi\,) = \alpha\cdot\left(\frac{\partial\varphi}{\partial t} + (\,u\,|\,\nabla\varphi\,)\right) + \alpha'\,\varphi = \alpha\cdot\left(\frac{\partial\varphi}{\partial t} + \mathrm{div}\,(u\,\varphi) - \varphi\ \mathrm{div}\ u\right) + \alpha'\,\varphi.$$

Let $v \in L^\infty(0,T)$ be defined by

$$v(t) = \sup_{x\in\Omega} |\,\mathrm{div}(u(t,x))|.$$

With the above choices, equation (3.3) has the form

$$\int_Q \left[(\alpha' + \alpha v - \alpha(v + \operatorname{div} u))\, \varphi^2 + 2\alpha\varphi \cdot \widetilde{\operatorname{div}}(\tilde{u}\, \varphi)\right] dx\, dt -$$

$$\int_{\partial Q_-} \alpha\varphi^2\, (\, \tilde{u}\, |\, \tilde{n}\,)\, d\tilde{\sigma} = \int_{\partial Q_+} \alpha\varphi^2\, (\, \tilde{u}\, |\, \tilde{n}\,)\, d\tilde{\sigma} \quad (3.4)$$

Let α be the solution of the differential equation

$$\alpha' + \alpha v = -2, \quad \alpha(T) = 0.$$

An easy computation gives

$$\alpha(t) = 2e^{-w(t)} \int_t^T e^{w(s)} ds \geq 0,$$

with $w(t) = \int_0^t e^{v(s)} ds$. Introducing this value in equation (3.4) we obtain

$$\int_Q \left[-2\varphi^2 - \alpha(v + \operatorname{div} u)\varphi^2 + 2\alpha\varphi\, \widetilde{\operatorname{div}}(\tilde{u}\, \varphi)\right] dx\, dt -$$

$$\int_{\partial Q_-} \alpha\varphi^2\, (\, \tilde{u}\, |\, \tilde{n}\,)\, d\tilde{\sigma} = \int_{\partial Q_+} \alpha\varphi^2\, (\, \tilde{u}\, |\, \tilde{n}\,)\, d\tilde{\sigma} \geq 0.$$

Hence

$$\int_Q \varphi^2 dx\, dt \leq \int_Q \alpha\varphi \cdot \widetilde{\operatorname{div}}(\tilde{u}\, \varphi)\, dx\, dt - \frac{1}{2} \int_{\partial Q_-} \alpha\varphi^2\, (\, \tilde{u}\, |\, \tilde{n}\,)\, d\tilde{\sigma} \leq$$

$$\frac{1}{2} \int_Q \varphi^2\, dx\, dt + \frac{1}{2} \int_Q \alpha^2 \widetilde{\operatorname{div}}(\tilde{u}\, \varphi)^2\, dx\, dt - \frac{1}{2} \int_{\partial Q_-} \alpha\varphi^2\, (\, \tilde{u}\, |\, \tilde{n}\,)\, d\tilde{\sigma}$$

so

$$\int_Q \varphi^2 dx\, dt \leq \int_Q \alpha^2 \cdot \widetilde{\operatorname{div}}(\tilde{u}\, \varphi)^2\, dx\, dt - \int_{\partial Q_-} \alpha\varphi^2\, (\, \tilde{u}\, |\, \tilde{n}\,)\, d\tilde{\sigma}.$$

If $A = max(\|\alpha\|^2_{L^\infty}, \|\alpha\|_{L^\infty})$, we get

$$\int_Q \varphi^2 dx\, dt \leq A \left(\int_Q \widetilde{\operatorname{div}}(\tilde{u}\, \varphi)^2\, dx\, dt - \int_{\partial Q_-} \varphi^2\, (\, \tilde{u}\, |\, \tilde{n}\,)\, d\tilde{\sigma} \right),$$

and the theorem is proved. ∎

Henceforth the space $H(u, Q)$ is equipped with the norm $|\varphi|_{1,u}$.

REMARK 1. *a) Using the above result, if $c_b = 0$, the semi-norm*

$$|\rho|_{1,u} = \left(\int_Q (\mathcal{A}\rho)^2 dx\, dt \right)^{1/2}$$

in a norm on H_0 which is equivalent to the usual norm on $H(u, Q)$.
b) As an easy consequence of the above arguments, for any $\rho \in H(u, Q)$, the norm defined by:

$$|||\rho||| = \left(\|\rho\|^2_{L^2(Q)} + \frac{1}{2} \int_{\partial Q_+} (\, \tilde{u}\, |\, \tilde{n}\,)\, (T - t)\rho^2 d\tilde{\sigma} \right)^{1/2}$$

verifies

$$\|\rho\|_{L^2(Q)} \leq |||\rho||| \leq \sqrt{A}\,|\rho|_{1,u}.$$

3.3. A weak formulation. In $L^2(Q))$, a solution of equation (2.2) corresponds to a zero of the following convex, positive functional

$$J(c) = \frac{1}{2}\left(\int_Q \left(\widetilde{\mathrm{div}}(\widetilde{u}\,c) - f\right)^2 dx\,dt - \int_{\partial Q_-}(c-c_b)^2\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma}\right)$$

The Gâteau derivative of J is

$$DJ(c)\varphi = \int_Q \left(\widetilde{\mathrm{div}}(\widetilde{u}\,c) - f\right)\;\widetilde{\mathrm{div}}(\widetilde{u}\,\varphi)\,dx\,dt - \int_{\partial Q_-}(c-c_b)\varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma}$$

So a sufficient condition to get the least squares solution of (2.2 - 2.4) is the following *weak formulation*: Find $c \in H(u,Q)$ such that

$$\int_Q \widetilde{\mathrm{div}}(\widetilde{u}\,c)\cdot\widetilde{\mathrm{div}}(\widetilde{u}\,\varphi)\,dx\,dt - \int_{\partial Q_-} c\cdot\varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma} =$$
$$\int_Q f\cdot\widetilde{\mathrm{div}}(\widetilde{u}\,\varphi)\,dx\,dt - \int_{\partial Q_-} c_b\cdot\varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma} \quad (3.5)$$

for all $\varphi \in H(u,Q)$ (see [4, 5, 6, 8, 15, 17]).

3.4. Stampacchia's theorems. In this section, we assume that the domain Ω is bounded. Later we will use the following versions of Stampacchia's theorems (see [27, 24]).

THEOREM 3.4. *Let $\rho \in H(u,Q)$, then*

$$\widetilde{\mathrm{div}}(\widetilde{u}\,\rho) = 0 \text{ a.e. on the set } \{(x,t) \in Q; \rho(x.t) = 0\} \quad (3.6)$$

THEOREM 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function.*
a) If $\rho \in H(u,Q)$, then $f(\rho) \in H(u,Q)$.
b) If f is differentiable except at a finite number of points, say $\{z_1,\ldots,z_n\}$, then

$$\widetilde{\mathrm{div}}(\widetilde{u}\,f(\rho)) = \begin{cases} f'(\rho)\,\widetilde{\mathrm{div}}(\widetilde{u}\,\rho) \text{ if } \rho(x,t) \notin \{z_1,\ldots,z_n\} \\ 0 \text{ elsewere.} \end{cases} \quad (3.7)$$

For the proof of these theorems the following lemma is used.

LEMMA 3.6. *Let $\rho \in H(u,Q)$ then $|\rho| \in H(u,Q)$ and*

$$\widetilde{\mathrm{div}}(\widetilde{u}\,|\rho|) = sgn(\rho)\;\cdot\;\widetilde{\mathrm{div}}(\widetilde{u}\,\rho) \quad (3.8)$$

where

$$sgn(\rho(x,t)) = \begin{cases} +1 & \text{if} \quad \rho(x,t) > 0 \\ 0 & \text{if} \quad \rho(x,t) = 0 \\ -1 & \text{if} \quad \rho(x,t) < 0 \end{cases}$$

Proof. (see [24]). For $\varepsilon > 0$, let $f_\varepsilon(t) = \sqrt{t^2+\varepsilon}$. If $\rho \in H(u,Q)$, then $f_\varepsilon(\rho) \in H(u,Q)$ and

$$\widetilde{\mathrm{div}}(\widetilde{u}\, f_\varepsilon(\rho)) = \frac{\rho}{\sqrt{\rho^2+\varepsilon}}\, \widetilde{\mathrm{div}}(\widetilde{u}\, \rho).$$

We have

$$\int_Q |f_\varepsilon(\rho)|^2\, dx\, dt = \varepsilon T|\Omega| + \int_Q |\rho|^2 dx\, dt \to \|\rho\|^2_{L^2(Q)} \quad \text{if } \ \varepsilon \to 0$$

and

$$\begin{aligned}\int_Q \left(f'_\varepsilon(\rho)\, \widetilde{\mathrm{div}}(\widetilde{u}\, \rho)\right)^2\, dx\, dt \ &= \int_Q \frac{\rho^2}{\rho^2+\varepsilon}\, \widetilde{\mathrm{div}}(\widetilde{u}\, \rho)^2 dx\, dt \\ &\to \int_Q \widetilde{\mathrm{div}}(\widetilde{u}\, \rho)^2 dx\, dt \ \text{ if } \ \varepsilon \to 0\end{aligned}$$

So the set $\{f_\varepsilon(\rho)\}_{\varepsilon>0}$ is bounded in $H(u,Q)$ and there exists a sequence $(\varepsilon_n) \to 0$ such that $f_{\varepsilon_n}(\rho) \rightharpoonup \eta$ in $H(u,Q)$.
Since $\|f_{\varepsilon_n}(\rho)\|_{H(u,Q)} \to \|\rho\|_{H(u,Q)}$ if $n \to \infty$ and $f_\varepsilon(t) \to |\rho|$ if $\varepsilon \to 0$, we have $\eta = |\rho|$ and $|\rho| \in H(u,Q)$.
Let now $\varphi \in \mathcal{D}(Q)$, then

$$\begin{aligned}\int_Q f'_\varepsilon(\rho)\, \widetilde{\mathrm{div}}(\widetilde{u}\, \rho)\, \varphi\, dx\, dt \ &= \int_Q \widetilde{\mathrm{div}}(\widetilde{u}\, f_\varepsilon(\rho))\, \varphi\, dx\, dt \\ &\to \int_Q \widetilde{\mathrm{div}}(\widetilde{u}\, |\rho|)\, \varphi\, dx\, dt\end{aligned}$$

But

$$f'_\varepsilon(\rho)\, \widetilde{\mathrm{div}}(\widetilde{u}\, \rho)\, \varphi \to sgn(\rho)\, \widetilde{\mathrm{div}}(\widetilde{u}\, \rho)\, \varphi \quad \text{a.e.}$$

and

$$|f'_\varepsilon(\rho)\, \widetilde{\mathrm{div}}(\widetilde{u}\, \rho)\, \varphi| \le sgn(\rho)\, \widetilde{\mathrm{div}}(\widetilde{u}\, \rho)\, \varphi$$

and we get the second result. ∎

Proof of theorem 3.4. This is a consequence of lemma 3.6. Indeed when $\rho \ge 0$ then $|\rho| = \rho$ and

$$\widetilde{\mathrm{div}}(\widetilde{u}\, |\rho|) = \widetilde{\mathrm{div}}(\widetilde{u}\, \rho) = sgn(\rho)\, \widetilde{\mathrm{div}}(\widetilde{u}\, \rho).$$

If $\rho = 0$, then $sgn(\rho) = 0$, so $\widetilde{\mathrm{div}}(\widetilde{u}\, \rho) = 0$ a.e. on the subset $\{(x,t) \in Q, \rho(x,t) = 0\}$.
Let now $\rho \in H(u,Q)$, then $\rho^+ = \frac{1}{2}(|\rho|+\rho) \in H(u,Q)$, $\rho^- = \frac{1}{2}(|\rho|-\rho) \in H(u,Q)$, and $\rho = \rho^+ - \rho^-$. But

$$\{(x,t) \in Q, \rho(x,t) = 0\} = \left\{(x,t) \in Q, \rho^+(x,t) = 0\right\} \cap \left\{(x,t) \in Q, \rho^-(x,t) = 0\right\},$$

so

$$\widetilde{\mathrm{div}}(\widetilde{u}\, \rho^+) = 0 \quad \text{on } \left\{(x,t) \in Q, \rho^+(x,t) = 0\right\},$$

and

$$\widetilde{\operatorname{div}}(\widetilde{u}\,\rho^-) = 0 \quad \text{on } \left\{(x,t) \in Q, \rho^-(x,t) = 0\right\},$$

and we get

$$\widetilde{\operatorname{div}}(\widetilde{u}\,\rho) = 0 \quad \text{on } \{(x,t) \in Q, \rho(x,t) = 0\},$$

and the theorem is proved. ∎

The proof of theorem 3.5 is similar to the proof given in [24].

4. Study of the least squares formulation . This section is devoted to the study of equation (3.5). More precisely, an existence and uniqueness theorem for the solution of equation (3.5) is given. Then a maximum principle is deduced from the Stampacchia's theorems. With the notations and hypothesis of section 3 we have

THEOREM 4.1. *For a fixed $u \in L^\infty(Q)^d$ with $\operatorname{div} u \in L^\infty(Q)$, the problem (3.5):*

$$\int_Q \widetilde{\operatorname{div}}(\widetilde{u}\,c) \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,\varphi)\,dx\,dt - \int_{\partial Q_-} c \cdot \varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma} = \int_Q f \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,\varphi)\,dx\,dt - \int_{\partial Q_-} c_b \cdot \varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma}$$

for all $\varphi \in H(u,Q)$, has a unique solution. Moreover

$$|c|_{1,u} = \left(\left\| \widetilde{\operatorname{div}}(\widetilde{u}\,c) \right\|^2_{L^2(Q)} - \int_{\partial Q_-} c^2\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma} \right)^{1/2} \leq \|f\|_{L^2(Q)}\,.$$

This solution is the space-time least squares solution *of (2.2).*

Proof. This assertion is a consequence of the Curved Poincaré inequality (theorem 3.3) and of the Lax-Milgram theorem (see also [4, 5]). ∎

REMARK 2. *For the numerical solution of equation (3.5), a time marching approach can be used to avoid the consideration of the all of Q (see e.g. [8, 15, 17]).*

COROLLARY 4.2. *The solution c of equation (3.5) belongs to the space*

$$X = L^2(Q) \cap L^2(\partial Q_+, (\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma})$$

equipped with the norm $|||c|||$.

The following theorem is a maximum principle for the solution of problem (3.5).

THEOREM 4.3. *Assume that the domain Ω is bounded, that the function $f = 0$ in equation (3.5) and that the function $c_b \in L^\infty(\partial Q_-)$. If $\operatorname{div}\, u = 0$, the solution of equation (3.5) satisfies*

$$\inf c_b \leq c \leq \sup c_b.$$

Proof. The solution c verifies

$$\int_Q \widetilde{\operatorname{div}}(\widetilde{u}\,c) \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,\varphi)\,dx\,dt - \int_{\partial Q_-} c \cdot \varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma} = -\int_{\partial Q_-} c_b \cdot \varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\,d\widetilde{\sigma}$$

for all $\varphi \in H(u,Q)$. Let

$$M = \sup_{\partial Q_-} c_b$$

and put

$$\varphi = (c-M)^+,\; Q_1 = \left\{(x,t) \in \overline{Q}, c - M > 0\right\},\; \Sigma_1 = \partial Q_- \cap Q_1$$

Then, using the Stampacchia's lemma,

$$\int_{Q_1} \widetilde{\operatorname{div}}(\widetilde{u}\, c) \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,(c-M))\, dx\, dt - \int_{\Sigma_1} c \cdot (c-M)\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma} = \\ -\int_{\Sigma_1} c_b \cdot (c-M)\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma}.$$

Since $\operatorname{div}\ u = 0$, we get

$$\int_{Q_1} (\widetilde{\operatorname{div}}(\widetilde{u}\,(c-M)))^2\, dx\, dt - \int_{\Sigma_1} ((c-M))^2\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma} = \\ -\int_{\Sigma_1} (c_b - M) \cdot (c-M)\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma}\ \le 0.$$

Hence, using theorem 3.3, the set Q_1 has a zero measure, so $c \le M$. We show in the same way that $c \ge \inf\ c_b$. ∎

Fianlly, let us do the following
REMARK 3. *The reduction of problem (2.2-2.4) to an homogeneous Dirichlet problem on ∂Q_-, i.e. assume that $c_b = 0$ can be done as follows.*
Let c be the solution of (2.2):

$$\int_{Q} \widetilde{\operatorname{div}}(\widetilde{u}\, c) \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,\varphi)\, dx\, dt - \int_{\partial Q_-} c \cdot \varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma} = \\ \int_Q f \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,\varphi)\, dx\, dt - \int_{\partial Q_-} c_b \cdot \varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma},$$

for all $\varphi \in H(u,Q)$, and let η the solution of

$$\int_{Q} \widetilde{\operatorname{div}}(\widetilde{u}\, \eta) \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,\varphi)\, dx\, dt - \int_{\partial Q_-} \eta \cdot \varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma} = -\int_{\partial Q_-} c_b \cdot \varphi\,(\,\widetilde{u}\,|\,\widetilde{n}\,)\, d\widetilde{\sigma}$$

for all $\varphi \in H(u,Q)$. Then $\rho = c - \eta$ is the unique solution of

$$\int_{Q} \widetilde{\operatorname{div}}(\widetilde{u}\, \rho) \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,\psi)\, dx\, dt = \int_Q \Big(f - \widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\Big) \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,\psi)\, dx\, dt \qquad (4.1)$$

for all $\psi \in H_0 = H_0(u,Q,\partial Q_-)$. Moreover the solution of problem (4.1) is equivalent to the solution of (2.2).
Therefore, modifying the source term if necessary, it is sufficient to only deal with homogeneous Dirichlet boundary conditions on ∂Q_-.

5. Comparison with renormalized solutions.. This section is devoted to the comparison between the least squares solution of equation (2.2-2.4) and the renormalized solution of these equations in the sense of [18].
More precisely, let $u \in L^\infty(Q)^d$, with $\operatorname{div} u = 0$, and $(\, u \,|\, n\,) = 0$ on $\partial\Omega$. Let $\phi \in L^\infty(Q)$ be the space-time least square solution of

$$\partial_t \phi + \operatorname{div}(\phi\, u) = 0 \quad \text{in } Q,$$

with the boundary condition

$$\phi = \phi_b \quad \text{on } \partial Q_-,$$

and $\phi_b \in L^\infty(\partial Q_-)$.
Now let $\psi = -\widetilde{\operatorname{div}}(\widetilde{u}\, \phi) \in L^2(Q)$; as we have seen in remark 3, the space time least squares solution $\rho \in L^\infty(Q)$ of

$$\partial_t \rho + \operatorname{div}(\rho\, u) = \psi \quad \text{in } Q, \tag{5.1}$$

$$\rho = 0 \quad \text{on } \partial Q_-, \tag{5.2}$$

gives an equivalent solution to the previous problem.

DEFINITION 5.1. [18] *For $u \in L^\infty(Q)^d$, $\operatorname{div}\ u \in L^\infty(Q)$, $c_b \in L^\infty(\partial Q_-)$, and $f \in L^2(Q)$, the function $c \in L^\infty(Q)$ is a renormalized solution of*

$$\widetilde{\operatorname{div}}(\widetilde{u}\; c) = f \quad \text{with} \quad c = c_b \quad \text{on } \partial Q_-$$

if for any $\beta \in \mathcal{C}^1(\mathbb{R}), \beta(0) = 0$, $\beta(c)$ is a weak solution of

$$\widetilde{\operatorname{div}}(\widetilde{u}\; \beta(c)) = \beta'(c) f,$$

and

$$\beta(c) = \beta(c_b) \quad \text{on } \partial Q_-.$$

where the equations are understood in distributions sense.
In this section we will show that ρ is a renormalized solution of equations (5.1)- (5.2).
Let us first prove the following result which is a sort of Meyers-Serrin theorem.

THEOREM 5.2. *Let $u \in H^1_0(Q)^d$ with $\operatorname{div} u = 0$, and let $\rho \in L^\infty(Q)$ with $\widetilde{\operatorname{div}}(\widetilde{u}\, \rho) \in L^2(Q)$, and $\rho(x, 0) = 0$. Then $\rho \in H_0(u, Q)$.*

Proof. Let $\epsilon > 0$ be given and sufficiently small. Set

$$Q_\epsilon = \Omega \times (\epsilon, T) \quad D_\epsilon = \Omega \times (0, \epsilon),$$

and define $\rho_\epsilon = \rho|_{Q_\epsilon} * \omega_\epsilon$, where ω_ϵ is the usual mollifier. Then $\rho_\epsilon \in \mathcal{D}(\overline{Q})$, and $\rho_\epsilon(x, 0) = 0$.
Let us show that

$$\left\| \widetilde{\operatorname{div}}(\widetilde{u}\, \rho) - \widetilde{\operatorname{div}}(\widetilde{u}\, \rho_\epsilon) \right\|_{L^2(Q)} \overset{\epsilon \to 0}{\longrightarrow} 0.$$

We have

$$\begin{aligned} \widetilde{\operatorname{div}}(\widetilde{u}\, \rho) - \widetilde{\operatorname{div}}(\widetilde{u}\, \rho_\epsilon) = {} & \widetilde{\operatorname{div}}(\widetilde{u}\, \rho) - \widetilde{\operatorname{div}}(\widetilde{u}\, \rho) * \omega_\epsilon \\ & + \widetilde{\operatorname{div}}(\widetilde{u}\, \rho) * \omega_\epsilon - \widetilde{\operatorname{div}}(\widetilde{u}\, \rho * \omega_\epsilon) \\ & + \widetilde{\operatorname{div}}(\widetilde{u}\, \rho * \omega_\epsilon) - \widetilde{\operatorname{div}}(\widetilde{u}\, \rho_\epsilon). \end{aligned}$$

It is clear that

$$\left\| \widetilde{\operatorname{div}}(\widetilde{u}\,\rho) - \widetilde{\operatorname{div}}(\widetilde{u}\,\rho) * \omega_\epsilon \right\|_{L^2(Q)} \stackrel{\epsilon\to 0}{\longrightarrow} 0;$$

moreover, from [18, 9], we have

$$\left\| \widetilde{\operatorname{div}}(\widetilde{u}\,\rho) * \omega_\epsilon - \widetilde{\operatorname{div}}(\widetilde{u}\,\rho * \omega_\epsilon) \right\|_{L^2(Q)} \stackrel{\epsilon\to 0}{\longrightarrow} 0.$$

Finally, let

$$\begin{aligned} h_\epsilon &= \widetilde{\operatorname{div}}(\widetilde{u}\,\rho * \omega_\epsilon) - \widetilde{\operatorname{div}}(\widetilde{u}\,\rho_\epsilon) \\ &= \widetilde{\operatorname{div}}(\widetilde{u}\,(\rho - \rho|_{Q_\epsilon}) * \omega_\epsilon). \end{aligned}$$

If B_ϵ is the ball of center (x,t) and radius ϵ in $\mathbb{R}^{d+1}$, then

$$h_\epsilon(x,t) = \int_{B_\epsilon} \rho(y,s) 1_{D_\epsilon}(y,s) \left(\widetilde{u}(x,t) \,|\, \widetilde{\nabla}\omega_\epsilon(x-y,t-s) \right) \, dy\, ds.$$

Since

$$\int_{B_\epsilon} \rho(y,s) 1_{D_\epsilon}(y,s) \left(\widetilde{u}(y,s) \,|\, \widetilde{\nabla}\omega_\epsilon(x-y,t-s) \right) \, dy\, ds = 0,$$

we get

$$h_\epsilon(x,t) = \int_{B_\epsilon} \rho(y,s) 1_{D_\epsilon}(y,s) \left((\widetilde{u}(x,t) - \widetilde{u}(y,s))/\epsilon \,|\, \epsilon\widetilde{\nabla}\omega_\epsilon(x-y,t-s) \right) \, dy\, ds.$$

Hence, using the Cauchy-Schwartz inequality we get

$$\begin{aligned} \|h_\epsilon\|^2_{L^2(Q)} \le \|\rho\|^2_{L^2(Q)} \cdot \\ \int_Q \int_{B_\epsilon} 1_{D_\epsilon}(y,s) \, \|(\widetilde{u}(x,t) - \widetilde{u}(y,s))/\epsilon\|^2 \left\| \epsilon\widetilde{\nabla}\omega_\epsilon(x-y,t-s) \right\|^2 dy\, ds\, dx\, dt, \end{aligned}$$

since $\epsilon\widetilde{\nabla}\omega_\epsilon$ is bounded, we get, as in [18, 9]

$$\|h_\epsilon\|^2_{L^2(Q)} \le M^2\,\epsilon\,|\Omega| \;\|\rho\|^2_{L^2(Q)} \;\|\nabla u\|^2_{L^2(Q)^{d^2}} .$$

□

Let u^m be the following regularization of the velocity field u given by the unique solution of the following space-time stationary Stokes problem.

$$\begin{aligned} -\frac{1}{m}\widetilde{\Delta} u^m + u^m + \nabla p = u \quad & \text{in}\, Q \\ \widetilde{\operatorname{div}}\, u^m = 0 \quad & \text{in}\, Q \\ u^m = 0 \quad & \text{on}\, \partial Q \end{aligned}$$

with

$$\widetilde{\Delta} = \Delta + \frac{\partial^2}{\partial t^2}.$$

Then $u^m \in H_0^1(Q) \cap L^\infty(Q)$, and $u^m \to u$ in $L^2(Q)$ when $m \to \infty$.

Let now η^m be the renormalized solution of

$$\partial_t \eta^m + \operatorname{div}(\eta^m\, u^m) = \psi \quad \text{in } Q,$$

with the initial condition

$$\eta^m(x,0) = 0.$$

Since $\psi \in L^2(Q)$ we have $\widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m) \in L^2(Q)$. As $\eta^m \in L^\infty(Q)$, we deduce from theorem 5.2 that $\eta^m \in H_0(u^m, Q)$. Therefore η^m is also the space-time least squares solution of $\widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m) = \psi$ in $H_0(u^m, Q)$, and we have

$$\left\|\widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m)\right\|_{L^2(Q)} = \|\psi\|_{L^2(Q)}$$

$$\|\eta^m\|_{L^2(Q)} \le C\,\|\psi\|_{L^2(Q)}$$

Hence, extracting a subsequence if necessary, one can assume that

$$\eta^m \rightharpoonup \eta \quad \text{in } L^2(Q),$$

$$\widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m) \rightharpoonup \kappa \quad \text{in } L^2(Q).$$

Moreover, since $u^m \to u$ in $L^2(Q)$, we obtain

$$\widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m) \to \widetilde{\operatorname{div}}(\widetilde{u}\,\eta) \quad \text{in } H^{-1}(Q),$$

and so $\kappa = \widetilde{\operatorname{div}}(\widetilde{u}\,\eta) \in L^2(Q)$.
Let us now prove that

$$\widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m) \to \widetilde{\operatorname{div}}(\widetilde{u}\,\eta) \quad \text{in } L^2(Q).$$

For this it is sufficient to show that

$$\left\|\widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m)\right\|_{L^2(Q)} \to \left\|\widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\right\|_{L^2(Q)},$$

but

$$\begin{aligned}
\left\|\widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\right\|^2_{L^2(Q)} &= \int_Q \left(\widetilde{\operatorname{div}}(\widetilde{u}\,\eta) - \widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m)\right)\widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\,dx\,dt + \\
&\quad \int_Q \widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m)\widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\,dx\,dt \\
&= \int_Q \left(\widetilde{\operatorname{div}}(\widetilde{u}\,\eta) - \widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m)\right)\widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\,dx\,dt + \\
&\quad \int_Q \psi\,\widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\,dx\,dt \\
&= \int_Q \left(\widetilde{\operatorname{div}}(\widetilde{u}\,\eta) - \widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m)\right)\widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\,dx\,dt + \\
&\quad \lim_{m\to\infty}\int_Q \widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m)^2\,dx\,dt,
\end{aligned}$$

and so

$$\left\|\widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m)\right\|_{L^2(Q)} \to \left\|\widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\right\|_{L^2(Q)}.$$

As a consequence of the above results, η is the least squares solution of $\widetilde{\operatorname{div}}(\widetilde{u}\,\eta) = \psi$, so $\eta \in H_0(u,Q)$. Indeed, let $w \in \mathcal{D}(\overline{Q})$ with $w = 0$ on ∂Q_-, then for all m, $w \in H_0(u^m, Q)$, and we have

$$\int_Q \widetilde{\operatorname{div}}(\widetilde{u^m}\,\eta^m) \cdot \widetilde{\operatorname{div}}(\widetilde{u^m}\,w)\,dx\,dt = \int_Q \psi \cdot \widetilde{\operatorname{div}}(\widetilde{u^m}\,w)\,dx\,dt.$$

So, passing to the limit

$$\int_Q \widetilde{\operatorname{div}}(\widetilde{u}\,\eta) \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,w)\,dx\,dt = \int_Q \psi \cdot \widetilde{\operatorname{div}}(\widetilde{u}\,w)\,dx\,dt,$$

and this equation remains true for all $w \in H_0(u,Q)$. Therefore $\eta \in H_0(u,Q)$, and is the least squares solution of $\widetilde{\operatorname{div}}(\widetilde{u}\,\eta) = \psi$, and so $\eta = \rho \in L^\infty(Q)$.

Now we can show that the sequence η^m strongly converges to η. Since $\operatorname{div}\ u^m = \operatorname{div}\ u = 0$ and $\eta^m \in H_0(u^m, Q)$, with the same idea as in theorem 3.3, we have

$$\int_Q \left[\widetilde{\operatorname{div}}(\widetilde{u^m}\ \eta^m) \cdot \xi + \eta^m \cdot \widetilde{\operatorname{div}}(\widetilde{u^m}\ \xi)\right] dx\,dt = \int_{\partial Q_+} \xi\eta^m\,(\,\widetilde{u} \,|\, \widetilde{n}\,)\,d\widetilde{\sigma}$$

for all regular enough function ξ. If we choose $\xi = (T-t)\eta^m$, then

$$\int_Q (\eta^m)^2\ dx\ dt = 2\int_Q \widetilde{\operatorname{div}}(\widetilde{u^m}\ \eta^m)\ \eta^m (T-t) dx\ dt. \tag{5.3}$$

But

$$\int_Q \widetilde{\operatorname{div}}(\widetilde{u^m}\ \eta^m)\ \eta^m(T-t)dx\ dt \to \int_Q \widetilde{\operatorname{div}}(\widetilde{u}\,\eta)\ \eta(T-t)dx\ dt = \int_Q \eta^2\ dx\ dt,$$

when $m \to \infty$, since $(\,u \,|\, n\,) = 0$ on $\partial\Omega$; so $\eta^m \to \eta$ when $m \to \infty$.

Finally, let us show that η is a renormalized solution. Let $\beta \in \mathcal{C}^1(\mathbb{R}), \beta(0) = 0$, then $\beta(\eta^m) \to \beta(\eta)$. Moreover

$$\widetilde{\operatorname{div}}(\widetilde{u^m}\,\beta(\eta^m)) = \beta'(\eta^m) \cdot \left(\,\widetilde{u^m} \,|\, \widetilde{\nabla}\eta^m\,\right),$$

but

$$\beta'(\eta^m) \to \beta'(\eta) \quad \text{in } L^p(Q)$$

and

$$\left(\,\widetilde{u^m} \,|\, \widetilde{\nabla}\eta^m\,\right) \to \left(\,\widetilde{u} \,|\, \widetilde{\nabla}\eta\,\right) \quad \text{in } L^2(Q),$$

so

$$\widetilde{\operatorname{div}}(\widetilde{u^m}\,\beta(\eta^m)) \to \widetilde{\operatorname{div}}(\widetilde{u}\,\beta(\eta)) \quad \text{in } L^1(Q),$$

thus η is a renormalized solution. We have proved the following

THEOREM 5.3. *If $u \in L^\infty(Q)^d$, $\operatorname{div}\ u = 0$, and $c_b \in L^\infty(\partial Q_-)$, the least squares solution of*

$$\widetilde{\operatorname{div}}(\widetilde{u}\,c) = 0 \quad \text{with} \quad c = c_b \quad \text{on } \partial Q_- \tag{5.4}$$

is a renormalized solution.

6. Conclusions and remarks . We have shown in this paper that the conservation law

$$\partial_t c + \mathrm{div}\,(c\,u) = f \quad \text{in } Q, \tag{6.1}$$

can be solved for irregular vector fields, using a very simple approach compared to the methods like e.g. [18, 3, 9].
Our method leads to some numerical schemes which are much simpler to use than the usual one (like the stream-line diffusion method, the characteristics method, the discontinuous finite element method with flux limiter, etc . . .). Some numerical examples are presented in [15, 8, 17].
In [7], it is proved that the solution of equation 6.1 gives an isomorphism on $L^2(Q)$ when $u \in \mathcal{C}^1(\Omega)$ is independent of t. Our method still gives an isomorphism, but in some unusual spaces. It allows to solve equations 6.1 with an irregular velocity field u. This situation where not known until now.

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