Vanishing and non-vanishing for the first $L^p$-cohomology of groups

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Abstract

We prove two results on the first $L^p$-cohomology $\overline{\mathcal{H}}^1_{(p)}(\Gamma)$ of a finitely generated group $\Gamma$:

1) If $N < H \subset \Gamma$ is a chain of subgroups, with $N$ non-amenable and normal in $\Gamma$, then $\overline{\mathcal{H}}^1_{(p)}(\Gamma) = 0$ as soon as $\overline{\mathcal{H}}^1_{(p)}(H) = 0$. This allows for a short proof of a result of Lück [Lück7]: if $N \triangleleft \Gamma$, $N$ is infinite, finitely generated as a group, and $\Gamma/N$ contains an element of infinite order, then $\overline{\mathcal{H}}^1_{(2)}(\Gamma) = 0$.

2) If $\Gamma$ acts isometrically, properly discontinuously on a $CAT(-1)$ space $X$, with at least 3 limit points in $\partial X$, then for $p$ larger than the critical exponent $e(\Gamma)$ of $\Gamma$ in $X$, one has $\overline{\mathcal{H}}^1_{(p)}(\Gamma) \neq 0$. As a consequence we extend a result of Shalom [Sha00]: let $G$ be a cocompact lattice in a rank 1 simple Lie group; if $G$ is isomorphic to $\Gamma$, then $e(G) \leq e(\Gamma)$.

1 Introduction

Fix $p \in [1, \infty]$. Let $\Gamma$ be a countable group. Assume first that $\Gamma$ admits a $K(\Gamma, 1)$-space which is a simplicial complex $X$ finite in every dimension. Let $\tilde{X}$ be the universal cover of $X$. Denote by $\ell^p C^k$ the space of $p$-summable complex cochains on $\tilde{X}$, i.e. the $\ell^p$-functions on the set of $k$-simplices of $\tilde{X}$. The $L^p$-cohomology of $\Gamma$ is the reduced cohomology of the complex

$$d_k : \ell^p C^k \to \ell^p C^{k+1},$$

where $d_k$ is the simplicial coboundary operator; we denote it by

$$\overline{\mathcal{H}}^k_{(p)}(\Gamma) = \ker d_k/Tm d_{k-1}.$$
As explained at the beginning §8 of [Gro93], this definition only depends on $\Gamma$.

For $p = 2$, the space $\overline{H}^k_{(2)}(\Gamma)$ is a module over the von Neumann algebra of $\Gamma$, and its von Neumann dimension is the $k$-th $L^2$-Betti number of $\Gamma$, denoted by $b^k_{(2)}(\Gamma)$; recall that $b^k_{(2)}(\Gamma) = 0$ if and only if $\overline{H}^k_{(2)}(\Gamma) = 0$.

For $k = 1$, it is possible to define the first $L^p$-cohomology of $\Gamma$ under the mere assumption that $\Gamma$ is finitely generated. Denote by $\mathcal{F}(\Gamma)$ the space of all complex-valued functions on $\Gamma$, and by $\lambda_\Gamma$ the left regular representation of $\Gamma$ on $\mathcal{F}(\Gamma)$. Define then the space of $p$-Dirichlet finite functions on $\Gamma$:

$$D_p(\Gamma) = \{ f \in \mathcal{F}(\Gamma) : \lambda_\Gamma(g)f - f \in \ell^p(\Gamma) \text{ for every } g \in \Gamma \}.$$ 

If $S$ is a finite generating set of $\Gamma$, define a norm on $D_p(\Gamma)/\mathbb{C}$ by:

$$\|f\|_{D^p} = \sum_{s \in S} \|\lambda_\Gamma(s)f - f\|_p^p.$$ 

Denote by $i : \ell^p(\Gamma) \to D_p(\Gamma)$ the inclusion. The first $L^p$-cohomology of $\Gamma$ is

$$\overline{H}^1_{(p)}(\Gamma) = D_p(\Gamma)/i(\ell^p(\Gamma)) + \mathbb{C}.$$ 

Let us recall briefly why this definition is coherent with the previous one. If $\Gamma$ admits a finite $K(\Gamma, 1)$-space $X$, we can choose one such that the 1-skeleton of $\check{X}$ is a Cayley graph $\mathcal{G}(\Gamma, S)$ of $\Gamma$. This means that $S$ is some finite generating subset of $\Gamma$, that $C^0 = \Gamma$, and that $C^1$ is the set $\mathcal{E}_\Gamma$ of oriented edges:

$$\mathcal{E}_\Gamma = \{(x, sx) : x \in \Gamma, s \in S\}.$$ 

Then $d_0$ is the restriction to $\ell^p(\Gamma)$ of the coboundary operator

$$d_\Gamma : \mathcal{F}(\Gamma) \to \mathcal{F}(\mathcal{E}_\Gamma) : f \mapsto [(x, y) \mapsto f(y) - f(x)].$$ 

Since $\check{X}$ is contractible, by Poincaré’s lemma any closed cochain is exact, i.e. any element in $\text{Ker} \ d_1$ can be written as $d_\Gamma f$, for some $f \in D_p(\Gamma)$ defined up to an additive constant. This means that $d_\Gamma : D_p(\Gamma) \to \ell^p(\mathcal{E}_\Gamma)$ induces an isomorphism of Banach spaces $D_p(\Gamma)/\mathbb{C} \to \text{Ker} \ d_1$, which maps $i(\ell^p(\Gamma))$ to $\text{Im} \ d_0$. This shows the equivalence of both definitions of $\overline{H}^1_{(p)}(\Gamma)$.

Our first result is:

**Theorem 1.** Let $N \subset H \subset \Gamma$ be a chain of groups, with $H$ and $\Gamma$ finitely generated, $N$ infinite and normal in $\Gamma$. 

1) If \( H \) is non-amenable and \( \overline{H}^1_{(p)}(H) = 0 \), then \( \overline{H}^1_{(p)}(\Gamma) = 0 \).

2) If \( b_{(2)}^1(H) = 0 \), then \( b_{(2)}^1(\Gamma) = 0 \).

We do not know whether part (1) of Theorem 1 holds when \( H \) is amenable.

As an application of part (2) of Theorem 1, we will give a very short proof of the following result of W. Lück (Theorem 0.7 in [L97]):

**Corollary 1.** Let \( \Gamma \) be a finitely generated group. Assume that \( \Gamma \) contains an infinite, normal subgroup \( N \), which is finitely generated as a group, and such that \( \Gamma/N \) is not a torsion group. Then \( b_{(2)}^1(\Gamma) = 0 \).

Using his theory of \( L^2 \)-Betti numbers for equivalence relations and group actions, D. Gaboriau was able to improve the previous result by merely assuming that \( \Gamma/N \) is infinite (see [Gab02, Théorème 6.8]). It is a challenging, and vaguely irritating question, to find a purely group cohomological proof of Gaboriau’s result.

As shown by Gaboriau’s result, non-vanishing of \( \overline{\pi}^1_{(2)} \) is an obstruction for the existence of finitely generated normal subgroups. We now present a non-vanishing result. Its proof is based on an idea due to G. Elek (see [Ele97, Theorem 2]).

Let \( X \) be a CAT(-1)-space (see [BH99] for the definitions), and let \( \Gamma \) be an infinite, finitely generated, properly discontinuous subgroup of isometries of \( X \). Recall that the critical exponent of \( \Gamma \) is defined as

\[
\epsilon(\Gamma) = \inf \{ s > 0; \sum_{g \in \Gamma} e^{-d(o, o)} < +\infty \},
\]

where \( o \) is any origin in \( X \), and where \( |\cdot - \cdot| \) denotes the distance in \( X \). In many cases, \( \epsilon(\Gamma) < +\infty \); in particular, this happens when the isometry group of \( X \) is co-compact (see Proposition 1.7 in [BM96]).

**Theorem 2.** Assume that \( \epsilon(\Gamma) \) is finite. If the limit set of \( \Gamma \) in \( \partial X \) has at least 3 points, then for \( p > \max \{ 1, \epsilon(\Gamma) \} \) the Banach space \( \overline{H}^1_{(p)}(\Gamma) \) is non zero.

When \( \Gamma \) is in addition co-compact, Theorem 2 was already known to Pansu and Gromov (see [Pan89] and page 258 in [Gro93]).

Theorem 2 is optimal for the co-compact lattices in rank one semi-simple Lie group: for those \( p > \epsilon(\Gamma) \) if and only if \( \overline{H}^1_{(p)}(\Gamma) \neq 0 \), thanks to a
result of Pansu [Pan89]. Since $L^p$-cohomology of groups is an invariant of isomorphism, by combining Pansu’s result with Theorem 2, we obtain the following generalisation of a result of Shalom (Theorem 1.1 in [Sha00]):

**Corollary 2.** Let $G$ be a co-compact lattice in a rank one semi-simple Lie group. Assume that $G$ is isomorphic to a properly discontinuous subgroup $\Gamma$ of isometries of a CAT(-1) space $X$. Then $e(G) \leq e(\Gamma)$.

Shalom established this by different methods in the special case where $X$ is the symmetric space associated to $SO(n,1)$ or $SU(n,1)$; his result also holds for non-co-compact lattices (when the Lie group is different from $SO(2,1)$).

## 2 Group cohomology; proof of Theorem 1

Let $V$ be a topological $\Gamma$-module, i.e. a real or complex topological vector space endowed with a continuous, linear representation $\pi: \Gamma \times V \to V : (g,v) \mapsto \pi(g)v$. If $H$ is a subgroup of $\Gamma$, we denote by $V|_H$ the space $V$ viewed as an $H$-module for the restricted action, and by $V^H$ the set of $H$-fixed points:

$$V^H = \{ v \in V | \pi(h)v = v, \forall h \in H \}.$$

We now introduce the space of 1-cocycles and 1-coboundaries on $\Gamma$, and the 1-cohomology with coefficients in $V$:

- $Z^1(\Gamma, V) = \{ b: \Gamma \to V | b(gh) = b(g) + \pi(g)b(h), \forall g, h \in \Gamma \}$
- $B^1(\Gamma, V) = \{ b \in Z^1(\Gamma, V) | \exists v \in V : b(g) = \pi(g)v - v, \forall g \in \Gamma \}$
- $H^1(\Gamma, V) = Z^1(\Gamma, V)/B^1(\Gamma, V)$

Suppose that $V$ is a Banach space. The space $Z^1(\Gamma, V)$ of 1-cocycles is a Fréchet space when endowed with the topology of pointwise convergence on $\Gamma$. The 1-reduced cohomology space with coefficients in $V$ is

$$H^1(\Gamma, V) = \overline{Z^1(\Gamma, V)/B^1(\Gamma, V)}.$$

Recall that $V$ *almost has invariant vectors* if, for every finite subset $F$ in $\Gamma$, and every $\epsilon > 0$, there exists a vector $v$ of norm 1 in $V$, such that $\|\pi(g)v - v\| < \epsilon$ for all $g \in F$. This condition is crucial for applications, as it allows to control the growth of the cohomology in a group. In the case of $L^p$-cohomology, this property is automatically satisfied, as the cohomology groups are finite-dimensional in this case.
$v\| < \epsilon$ for every $g \in F$. The following result is due to Guichardet (Thm. 1 and Cor. 1 in [Gui72])\(^1\)

**Proposition 1.** Let $\Gamma$ be a countable group.

1) Let $V$ be a Banach $\Gamma$-module with $V^\Gamma = 0$. The map $H^1(\Gamma, V) \rightarrow \overline{H^1}(\Gamma, V)$ is an isomorphism if and only if $V$ does not almost have invariant vectors.

1. Let $p \in [1, \infty[$. Assume that $\Gamma$ is infinite. The map $H^1(\Gamma, \ell^p(\Gamma)) \rightarrow \overline{H^1}(\Gamma, \ell^p(\Gamma))$ is an isomorphism if and only if $\Gamma$ is non-amenable. \(\square\)

We will prove:

**Proposition 2.** Let $p \in [1, \infty[$. Let $N \subset H \subset \Gamma$ be a chain of groups, with $\Gamma$ finitely generated, $N$ infinite and normal in $\Gamma$. If $H^1(H, \ell^p(H)) = 0$, then $H^1(\Gamma, \ell^p(\Gamma)) = 0$.

The link between $\overline{H^1}(\Gamma)$ and $H^1(\Gamma, \ell^p(\Gamma))$ has been noticed by several people - see e.g. lemma 3 in [BV97] (for $p = 2$ and $\Gamma$ non-amenable), or §2 in [Pul03] (in general). We give the easy argument for completeness.

**Lemma 1.** For finitely generated $\Gamma$, there are isomorphisms

$$D_p(\Gamma)/(i(\ell^p(\Gamma)) + \mathbb{C}) \simeq H^1(\Gamma, \ell^p(\Gamma)) \text{ and } \overline{H^1}(\Gamma) \simeq \overline{H^1}(\Gamma, \ell^p(\Gamma)).$$

**Proof of lemma 1:** The map $D_p(\Gamma) \rightarrow Z^1(\Gamma, \ell^p(\Gamma)) : f \mapsto [g \mapsto \lambda_t(g) f - f]$ is continuous, with kernel the space $\mathbb{C}$ of constant functions, and the image of $i(\ell^p(\Gamma))$ is exactly $H^1(\Gamma, \ell^p(\Gamma))$. Moreover this map is onto because of the classical fact that $H^1(\Gamma, \mathcal{F}(\Gamma)) = 0$. \(\square\)

Before proving Proposition 2 (for which we will actually give two proofs), we explain how to deduce Theorem 1 from it.

**Proof of Theorem 1 from Proposition 2**

1) In view of lemma 1, the assumption of Theorem 1 reads $\overline{H^1}(H, \ell^p(H)) = 0$. Since $H$ is non-amenable, by Proposition 1 we have $H^1(H, \ell^p(H)) = 0$. By Proposition 2 we deduce $H^1(\Gamma, \ell^p(\Gamma)) = 0$. By lemma 1 again, we get the conclusion.

\(^1\)Strictly speaking, Guichardet proves this result for unitary $\Gamma$-modules; but his proof, only appealing to the Banach isomorphism theorem, carries over without change to Banach $\Gamma$-modules.
2) If $H$ is non-amenable, the result is a particular case of the first part. If $H$ is amenable, then so is $N$, and the result follows from the Cheeger-Gromov vanishing theorem [CG86]: if a group $\Gamma$ contains an infinite, amenable, normal subgroup, then all the $L^2$-Betti numbers of $\Gamma$ are zero. \hfill\Box

**Important remark:** Cheeger and Gromov [CG86] defined $L^2$-Betti numbers of a group $\Gamma$ without any assumption on $\Gamma$, in particular not assuming $\Gamma$ to be finitely generated. Using their definition, D. Gaboriau has shown us (private communication) a proof that $b^1_{(2)}(\Gamma) = 0$ always implies $\overline{\ell^2}(\Gamma, \ell^2(\Gamma) = 0$. As a consequence, part (2) of Theorem 1 holds without any assumption on the subgroup $H$.

Our first proof of Proposition 2 will require the following lemma, which is classical for $p = 2$.

**Lemma 2.** Let $p \in [1, \infty]$. Let $H$ be a countable group. Let $X$ be a countable set on which $H$ acts freely. The following statements are equivalent:

i) The permutation representation $\lambda_X$ of $H$ on $\ell^p(X)$, almost has invariant vectors;

ii) $H$ is amenable.

**Proof of Lemma 2:** We recall (see [Eym72]) that a group $\Gamma$ is amenable if and only if it satisfies Reiter’s condition $(P_p)$, i.e. for every finite subset $F \subset \Gamma$ and $\epsilon > 0$, there exists $f \in \ell^p(\Gamma)$ such that $f \geq 0$, $\|f\|_p = 1$, and $\|\lambda_G(g) f - f\|_p < \epsilon$ for $g \in F$. In particular $\ell^p(\Gamma)$ almost has invariant vectors.

So if $H$ is amenable, then $\ell^p(X)$ almost has invariant vectors since it contains $\ell^p(H)$ as a sub-module. This proves $(i) \Rightarrow (ii)$.

To prove $(ii) \Rightarrow (i)$, we assume that $\ell^p(X)$ almost has invariant vectors and prove in 3 steps that $H$ satisfies Reiter’s property $(P_p)$, so is amenable. So fix a finite subset $F \subset H$, and $\epsilon > 0$; find $f \in \ell^p(X)$, $\|f\|_p = 1$, such that $\|\lambda_X(h) f - f\|_p < \frac{\epsilon}{2^p}$ for $h \in F$.

1) Replacing $f$ with $|f|$, we may assume that $f \geq 0$.

2) Set $g = f^p$, so that $g \in \ell^1(X)$, $\|g\|_1 = 1, g \geq 0$. For $h \in F$, we have:

$$\|\lambda_X(h) g - g\|_1 = \sum_{x \in X} |f(h^{-1}x)^p - f(x)^p|$$
\[
\leq p \sum_{x \in X} |f(h^{-1}x) - f(x)| (f(h^{-1}x)^{p-1} + f(x)^{p-1})
\]

\[
\leq p \left( \sum_{x \in X} |f(h^{-1}x) - f(x)|^p \right)^{\frac{1}{p}} \left( \sum_{x \in X} (f(h^{-1}x)^{p-1} + f(x)^{p-1})^{\frac{p-1}{p}} \right)^{\frac{p}{p-1}}
\]

\[
\leq p \|\lambda_X(h) f - f\|_p \left( 2 \frac{p-1}{p} \sum_{x \in X} (f(h^{-1}x)^p + f(x)^p) \right)^{\frac{p-1}{p}} = 2p \|\lambda_X(h) f - f\|_p < \epsilon
\]

where we have used consecutively \(^2\) the inequalities

- \(|a^p - b^p| \leq p|a - b|(a^{p-1} + b^{p-1})\) for \(a, b > 0\);
- Hölder’s inequality;
- \((a + b)\frac{p}{p-1} \leq 2^{\frac{p}{p-1}}(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}})\) for \(a, b > 0\); and the fact that \(\|f\|_p = 1\).

3) Let \((x_n)_{n \geq 1}\) be a set of representatives for the orbits of \(H\) in \(X\). Define a function \(g_n\) on \(H\) by \(g_n(h) = g(hx_n)\), and set \(G = \sum_{n=1}^{\infty} g_n\). Then \(G \geq 0\) and \(\|G\|_1 = \sum_{h \in H} \sum_{n=1}^{\infty} g(hx_n) = \sum_{x \in X} g(x) = 1\). Moreover, for \(h \in F:\)

\[
\|\lambda_H(h) G - G\|_1 = \sum_{\gamma \in H} \left| \sum_{n=1}^{\infty} (g(h^{-1}\gamma x_n) - g(\gamma x_n)) \right| \leq \|\lambda_X(h) g - g\| < \epsilon
\]

by the previous step. This establishes property \((P_1)\) for \(H\). \(\square\)

**First proof of Proposition 2 (homological algebra):**

**Claim:** \(H^1(H, \ell^p(G)|_H) = 0\). Choosing representatives for the right cosets of \(H\) in \(\Gamma\), we identify \(\ell^p(G)|_H\) in an \(H\)-equivariant way with the \(\ell^p\)-direct sum \(\oplus \ell^p(H)\) of \([\Gamma : H]\) copies of \(\ell^p(H)\). Since cohomology commutes with finite direct sums, the claim is clear if \([\Gamma : H] < \infty\). So assume that \([\Gamma, H] = \infty\). If \(b \in Z^1(H, \ell^p(G)|_H)\), write \(b = (b_k)_{k \geq 1}\), where \(b_k \in Z^1(H, \ell^p(H))\) for every \(k \geq 1\). By assumption, for each \(k\), there is a function \(f_k \in \ell^p(H)\) such that \(b_k(h) = \lambda_H(h) f_k - f_k\) for every \(h \in H\). Set

\[
B_N(h) = (\lambda_H f_1 - f_1, \ldots, \lambda_N(h) f_N - f_N, 0, 0, \ldots)
\]

\(^2\)The expert will recognize here the argument to pass from property \((P_p)\) to property \((P_1)\), as in [Eym72].
so that \( B_N \in B^1(H, \ell^p(\Gamma)|_H) \) and \( B_N \) converges to \( b \) pointwise on \( H \), for \( N \to \infty \). This already shows that \( \overline{H}^1(H, \ell^p(\Gamma)|_H) = 0 \). Notice now that, by Proposition 1(2), the assumption \( H^1(H, \ell^p(H)) = 0 \) implies that \( H \) is non-amenable. By lemma 2 applied to \( X = \Gamma \), this means that \( \ell^p(\Gamma)|_H \) does not almost have invariant vectors. By Proposition 1(1), we get \( H^1(H, \ell^p(\Gamma)|_H) = 0 \), proving the Claim.

Recall from group cohomology (see e.g. § 8.1 in [Gui80]) that, for any \( \Gamma \)-module \( V \), there is an exact sequence

\[
0 \to H^1(\Gamma/N, V^N) \to H^1(\Gamma, V) \to \overline{H}^1(N, V|_N) \to H^1(\Gamma/N, V^N),
\]

where \( i : V^N \to V \) denotes the inclusion. In particular, if \( V^N = 0 \), then the restriction map

\[ Rest^N_{\Gamma} : H^1(\Gamma, V) \to H^1(N, V|_N) \]

is injective. We apply this with \( V = \ell^p(\Gamma) \) (noticing that \( V^N = 0 \) as \( N \) is infinite).

Consider then the composition of restriction maps

\[ H^1(\Gamma, \ell^p(\Gamma)) \to H^1(H, \ell^p(\Gamma)|_H) \to H^1(N, \ell^p(\Gamma)|_N); \]

this composition is \( Rest^N_{\Gamma} \), which is injective as we just saw. On the other hand, by the claim this composition is also the zero map. So \( H^1(\Gamma, \ell^p(\Gamma)) = 0 \), as was to be established.

**Second proof of Proposition 2 (geometry):** This proof works under the extra assumption that \( H \) is finitely generated. Fix finite generating sets \( T \) for \( H \), \( S \) for \( \Gamma \), with \( T \subset S \), and consider the Cayley graph \( G(\Gamma, S) \) and its coboundary operator \( d_T : \mathcal{F}(\Gamma) \to \mathcal{F}(\mathcal{E}_T) \). Then \( D_p(\Gamma) = \{ f \in \mathcal{F}(\Gamma) : d_T f \in \ell^p(\mathcal{E}_T) \} \). Similarly, let \( d_H \) be the coboundary operator associated with the Cayley graph \( G(H, T) \).

Fix \( f \in D_p(\Gamma) \); the goal is to show that \( f \in \ell^p(\Gamma) + \mathbb{C} \). Let \( \{ g_i \}_{i \in I} \) be a set of representatives for the right cosets of \( H \) in \( \Gamma \), so that \( \Gamma = \bigsqcup_{i \in I} Hg_i \). For \( i \in I \), set \( f_i(x) = f(xg_i) \) (\( x \in H \).)

Then

\[
\|d_H(f_i)\|_{\ell^p} = \sum_{x \in H} \sum_{s \in T} |f(xg_i) - f(xg_i)|^p \leq \sum_{x \in \Gamma} \sum_{s \in S} |f(sx) - f(x)|^p = \|d_T f\|_{\ell^p} < \infty,
\]

i.e. \( f_i \in D_p(H) \). Using our assumption and lemma 1, we may write

\[
f_i = h_i + u_i
\]
where $h_i \in \ell^p(H)$ and $u_i \in \mathbb{C}$. Define functions $h$ and $u$ on $\Gamma$ by $h(xg_i) = h_i(x)$ and $u(xg_i) = u_i$ ($x \in H$).

**First claim:** $h \in \ell^p(\Gamma)$.

Indeed, since $H$ is non-amenable (by Proposition 1), there exists a constant $C > 0$ (depending only on $p, H, T$) such that for every $i \in I$:

$$
\|h_i\|_p \leq C \|d_H(h_i)\|_p.
$$

Then summing over $i$:

$$
\|h\|_p = \sum_{i \in I} \|h_i\|_p \leq C^p \sum_{i \in I} \|d_H(h_i)\|_p = C^p \sum_{i \in I} \sum_{x \in H} \sum_{s \in T} |h_i(sx) - h_i(x)|^p
$$

$$
= C^p \sum_{i \in I} \sum_{x \in H} \sum_{s \in T} |f_i(sx) - f_i(x)|^p = C^p \sum_{x \in \Gamma} \sum_{s \in S} |f(sx) - f(x)|^p
$$

$$
\leq C^p \sum_{x \in \Gamma} \sum_{s \in S} |f(sx) - f(x)|^p = C^p \|d_\Gamma(f)\|_p' < \infty.
$$

**Second claim:** $u$ is constant.

Indeed, since $f = h + u$, and $d_\Gamma(f), d_\Gamma(h) \in \ell^p(\mathbb{E}_\Gamma)$, we have $d_\Gamma(u) \in \ell^p(\mathbb{E}_\Gamma)$. In particular this implies, for fixed indices $i, j \in I$:

$$
\infty > \sum_{x \in N} |u((g_jg_i^{-1})xg_i) - u(xg_i)|^p = \sum_{x \in N} |u((g_jg_i^{-1})xg_i) - u_i|^p = \sum_{x \in N} |u(x(g_jg_i^{-1})g_i) - u_i|^p
$$

since $N$ is normal in $\Gamma$. The latter sum is equal to

$$
\sum_{x \in N} |u_j - u_i|^p < \infty.
$$

Since $N$ is infinite, this forces $u_i = u_j$, i.e. $u$ is constant.

The first and the second claim together prove Proposition 2. \hfill \Box

### 3 Some results of W. Lück

The following result was obtained by Lück in [L94], Theorem 2.1. We recall his short, elegant argument.

**Lemma 3.** Let $N$ be a finitely generated group, and let $\alpha$ be an automorphism of $N$. Let $H = N \rtimes \alpha \mathbb{Z}$ be the corresponding semi-direct product. Then $b_1^1(\Gamma)(H) = 0$. 

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Proof: The proof depends on two classical properties of the $L^2$-Betti numbers for a finitely generated group $\Gamma$:

- $b^1_2(\Gamma) \leq d(\Gamma)$, where $d(\Gamma)$ denotes the minimal number of generators of $\Gamma$;
- if $\Lambda$ is a subgroup of finite index $d$ in $\Gamma$, then $b^k_{(2)}(\Lambda) = d \cdot b^k_{(2)}(\Gamma)$.

Let then $p : H \to \mathbb{Z}$ denote the quotient map; for $n \geq 1$, set $H_n = p^{-1}(n\mathbb{Z})$, a subgroup of index $n$ in $H$. Then:

$$n \cdot b^1_{(2)}(H) = b^1_{(2)}(H_n) \leq d(H_n) \leq d(N) + 1.$$  

Since this holds for every $n \geq 1$, the lemma follows. \hfill \Box

Proof of Corollary 1: Since $\Gamma/N$ is not a torsion group, we find a subgroup $H$ of $\Gamma$, containing $N$, such that $H/N$ is infinite cyclic. Since $N$ is finitely generated, we have $b^1_{(2)}(H) = 0$, by lemma 3. The result follows then immediately from Theorem 1. \hfill \Box

Example: We point out that lemma 3 has no analogue in $L^p$-cohomology, with $p \neq 2$. To see it, let $M$ be a 3-dimensional, compact, hyperbolic manifold which fibers over the circle. Denote by $\Sigma_g$ the fiber of that fibration: this is a closed Riemann surface of genus $g \geq 2$. Then the fundamental group $\Gamma = \pi_1(M)$ admits a semi-direct product decomposition $\Gamma = \pi_1(\Sigma_g) \times \mathbb{Z}$, so that $\overline{b}^1_{(2)}(\Gamma) = 0$ by lemma 2. However

$$\inf\{p \geq 1 : \overline{b}^1_{(p)}(\Gamma) \neq 0\} = 2,$$

as was proven by Pansu [Pan89].

4 Proof of Theorem 2

Denote by $\partial X$ the (Gromov) boundary of $X$. Let $\Lambda = \overline{\Gamma o} \cap \partial X$ be the limit set of $\Gamma$ in $\partial X$ (the closure of $\Gamma o$ is taken in the compact set $X \cup \partial X$).

Since $X$ is a CAT($-1$) space, its boundary carries a natural metric $d$ (called a visual metric) which can be defined as follows (see [Bou95]; Théorème 2.5.1); for every $\xi$ and $\eta$ in $\partial X$:

$$d(\xi, \eta) = e^{-d(\xi, \eta)}.$$  

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where \((.,.)\) denotes the Gromov product on \(\partial X\) based on \(o\), namely

\[
(\xi|\eta) = \lim_{(x,y)\to (\xi,\eta)} \frac{1}{2}(|o - x| + |o - y| - |x - y|).
\]

Observe that there exists a constant \(B\) such that for every \(g \in \Gamma\) there is a point \(\xi\) in \(\partial X\) with \(d(g o, [o, \xi]) \leq B\). Indeed this property does not depend on the choice of the origin \(o\). So we choose \(o\) on a bi-infinite geodesic \((\eta_1, \eta_2)\). Then \(g o\) belongs to \((g\eta_1, g\eta_2)\). Now since \(X\) is Gromov-hyperbolic, one of the two points \(g\eta_1\) or \(g\eta_2\) satisfies the claim.

Let \(u\) be a Lipschitz function of \((\partial X, d)\) which is non-constant on \(\Lambda\); such functions do exist since \(\Lambda\) is not reduced to a point. Following G. Elek [Ele97], let \(f\) be the function on \(\Gamma\) defined by \(f(g) = u(\xi_\theta)\), where \(\xi_\theta\) is a point in \(\partial X\) such that \(d(g^{-1} o, [o, \xi_\theta]) \leq B\).

**Claim:** \(f \in D_p(\Gamma)\) for \(p > \max\{1, e(\Gamma)\}\). Indeed we have

\[
\|f\|^p_{D_p} = \sum_{s \in S} \sum_{g \in \Gamma} |f(s g) - f(g)|^p = \sum_{s \in S} \sum_{g \in \Gamma} |u(\xi_{sg}) - u(\xi_g)|^p \\
\leq C \sum_{s \in S} \sum_{g \in \Gamma} [d(\xi_{sg}, \xi_g)]^p \leq D \sum_{g \in \Gamma} \sum_{s \in S} e^{-p(\xi_{sg}|\xi_g)} \\
\leq D \sum_{g \in \Gamma} e^{-p|g^{-1} o - o|} < +\infty,
\]

where \(C, D\) are constants depending only on \(u, B\) and \(S\). The details for the first inequality in the last line are the following. Observe that \(|(sg)^{-1} o - g^{-1} o| = |s^{-1} o - o|\) is bounded above by an absolute constant. This implies that if \(x_g\) and \(x_{sg}\) denote respectively the points on \([o, \xi_\theta]\) and \([o, \xi_{sg}]\) whose distance from \(o\) is equal to \(|g^{-1} o - o|\), then \(|x_g - x_{sg}|\) is bounded above by an absolute constant. Now with the triangle inequality

\[
|x - y| \leq |x - x_{sg}| + |x_{sg} - x_g| + |x_g - y|,
\]

and from the definition of the Gromov product, it follows that

\[
(\xi_{sg}|\xi_g) \geq \frac{1}{2}(|o - x_{sg}| + |o - x_g| - |x_{sg} - x_g|),
\]

so that \((\xi_{sg}|\xi_g)\) is bounded below by \(|g^{-1} o - o|\) plus an absolute additive constant. This proves the claim.
Since \( \Lambda \) has at least 3 points, the group \( \Gamma \) is non-amenable (namely it is well-known that \( \Lambda \) is a minimal set, and that an amenable group stabilises one or two points in \( \partial X \)). So by proposition 1 and by Lemma 1, we must prove that \( f \) does not belong to \( i(\ell^p(\Gamma)) + \mathbb{C} \). Assume it does, then \( f(g) \) tends to a constant number when the length of \( g \) in \( \Gamma \) tends to \( +\infty \). This contradicts the fact that \( u \) is non-constant on \( \Lambda \).

\[ \square \]

References


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