FUCHSIAN POLYHEDRA IN LORENTZIAN SPACE-FORMS

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Abstract. A Fuchsian polyhedron in a Lorentzian space-form is a polyhedral surface invariant under the action of a group of isometries fixing a point x_0 and acting cocompactly on the time-like units vectors at x_0. The induced metric on a convex Fuchsian polyhedron is isometric to a constant curvature metric with conical singularities of negative singular curvature on a compact surface of genus greater than one. We prove that these metrics are actually realised by exactly one convex Fuchsian polyhedron (up to global isometries) — in the spherical case, we must add the condition that the lengths of the contractible geodesics are $> 2\pi$.

This extends theorems of A.D. Alexandrov and Rivin–Hodgson [Ale42, RH93] concerning the sphere to the higher genus cases, and it is also the polyhedral version of a theorem of Labourie–Schlenker [LS00].

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1. Definitions and statements

1.1. **Metrics with conical singularities and convex polyhedra.** Let M_K^- be the Lorentzian space-form of dimension 3 with constant curvature K, $K \in \{-1, 0, 1\}$: M_0^- is the Minkowski space $\mathbb{R}^3_1$, M_1^- is the de Sitter space dS^3 and M_{-1}^- is the anti-de Sitter space AdS^3. A *convex polyhedron* is an intersection of half-spaces of M_K^-. The number of half-spaces may be infinite, but the intersection is asked to be locally finite: each face must be a polygon with a finite number of vertices, and the number of edges at each vertex must be finite. A *polyhedron* is a connected union of convex polyhedra. In all this paper, we consider only polyhedra with space-like faces. A *polyhedral surface* is the boundary of a polyhedron and a *convex polyhedral surface* is the boundary of a convex polyhedron. A *convex (polyhedral) cone* in M_K^- is a convex polyhedral surface with only one vertex. In the de Sitter case, we will call *polyhedral surface of hyperbolic type* a polyhedral surface dual of a hyperbolic polyhedral surface (the definition of duality is recalled in Section 2). The sum of the angles between the edges on the faces of a convex cone in M_K^- (of hyperbolic type for M_1^-) is strictly greater than 2π.

A *metric of curvature K with conical singularities of negative singular curvature* on a compact surface S is a (Riemannian) metric of constant curvature K on S minus n points $(x_1, \dots, x_n)$ such that the neighbourhood of each x_i is isometric to the induced metric on the neighbourhood of the vertex of a convex cone in M_K^-. The x_i are called the *singular points*. By definition the set of singular points is discrete, hence finite since the surface is compact. The angle α_i around a singular

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point x_i is the *cone-angle* at this point and the value $(2\pi - \alpha_i)$ is the *singular curvature* at x_i.

Let P be a convex polyhedral surface in M_1^- of hyperbolic type homeomorphic to the sphere (note that the de Sitter space is not contractible). The induced metric on P is isometric to a spherical metric with conical singularities of negative singular curvature on the sphere. Moreover the lengths of the closed geodesics for this metric are $> 2\pi$, see [RH93]. The following theorem says that all the metrics of this kind can be obtained by such polyhedral surfaces:

Theorem 1.1 (Rivin–Hodgson, [RH93])**.** *Each spherical metric on the sphere with conical singularities of negative singular curvature such that the lengths of its closed geodesics are $> 2\pi$ can be isometrically embedded in the de Sitter space as a unique convex polyhedral surface of hyperbolic type homeomorphic to the sphere.*

This result was extended to the cases where P is not of hyperbolic type in [Sch98a, Sch01] (actually the results contained in these references cover larger classes of metrics on the sphere). It is an extension to negative singular curvature of a famous theorem of A.D. Alexandrov. We denote by M_K^+ the Riemannian space-form of curvature K.

Theorem 1.2 (A.D. Alexandrov)**.** *Each metric of curvature K on the sphere with conical singularities of positive singular curvature can be isometrically embedded in M_K^+ as (the boundary of) a unique convex compact polyhedron.*

A conical singularity with *positive singular curvature* is a point which has a neighbourhood isometric to the induced metric on the neighbourhood of the vertex of a convex cone in M_K^+. The sum of the angles between the edges on the faces of a convex cone in M_K^+ is strictly between 0 and 2π.

By the Gauss–Bonnet formula, we know that there doesn't exist other constant curvature metrics with conical singularities of constant sign singular curvature on the sphere than the ones described in Theorem 1.1 and Theorem 1.2. In the present paper we extend these results to the cases of surfaces of higher genus (actually > 1):

Theorem A. *Let S be a compact surface of genus > 1.*

1) *A spherical metric with conical singularities of negative singular curvature on S such that the lengths of its closed contractible geodesics are $> 2\pi$ is realised by a unique convex Fuchsian polyhedron in the de Sitter space.*
2) *A flat metric with conical singularities of negative singular curvature on S is realised by a unique convex Fuchsian polyhedron in the Minkowski space.*
3) *A hyperbolic metric with conical singularities of negative singular curvature on S is realised by a unique convex Fuchsian polyhedron in the anti-de Sitter space.*

An *invariant polyhedral surface* is a pair (P, F), where P is a polyhedral surface in M_K^- and F is a discrete group of isometries of M_K^- such that $F(P) = P$ and F acts freely on P. The group F is called the *acting group*. If there exists an invariant polyhedral surface (P, F) in M_K^- such that the induced metric on P/F is isometric to a metric h on a surface S, we say that P *realises* the metric h (obviously the singular points of h correspond to the vertices of P, and F is isomorphic to the fundamental group of S). In this case we say that h is *realised by a unique invariant polyhedral surface* (P, F) if P is unique up to isometries of M_K^-. For example, if S is the sphere the acting group is the trivial one.

We denote by $\mathrm{Isom}_+^+(M_K^-)$ the group of orientation-preserving and time orientation-preserving isometries of M_K^-. We call a *Fuchsian group of* M_K^- a discrete subgroup of $\mathrm{Isom}_+^+(M_K^-)$ fixing a point c_K and acting cocompactly on the time-like units vectors at c_K. Such a group also leaves invariant and acts cocompactly on all the surfaces in the future-cone of c_K which are at constant distance from c_K. These surfaces have the properties to be strictly convex, umbilical and complete. The induced metric on exactly one of these surfaces is hyperbolic, denoted by O_K. A *Fuchsian polyhedron* of M_K^- is a (space-like) polyhedral surface invariant under the action of a Fuchsian group of M_K^- and contained in the future-cone of c_K. The necessity of the condition on the lengths of the geodesics in the spherical case will be explained in Section 2.

The analog of Theorem A for smooth metrics was proved in [LS00]. The part *1)* of Theorem A was already done in [Scha, thm 4.22], using a different proof, in particular for the main point which is the infinitesimal rigidity, see Subsection 3.3 for an explanation. The other parts are new from what I know.

1.2. **Examples of convex Fuchsian polyhedra.** In M_K^- take n points $(x_1,\dots,x_n)$ on O_K, and let a Fuchsian group F act on these points. We denote by E the boundary of the convex hull of the set of points fx_i, for all $f\in F$ and $i=1\dots n$. By construction, the convex polyhedral surface E is globally invariant under the action F: it is a convex Fuchsian polyhedron, see Figure 1.

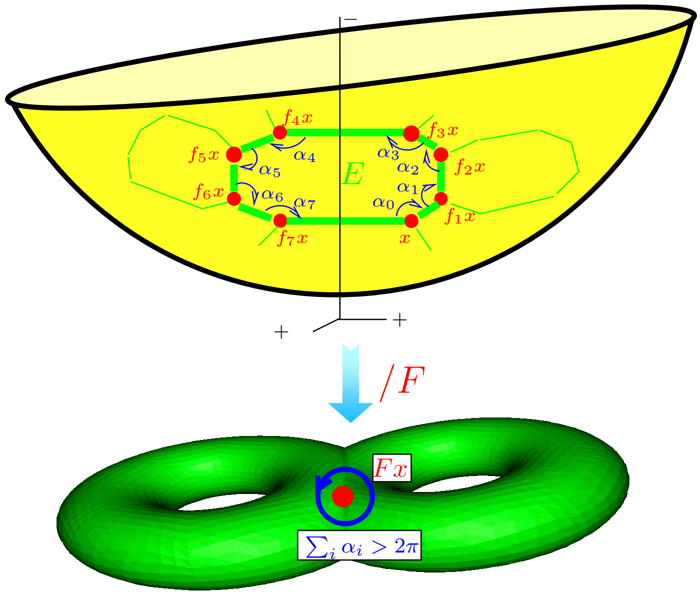

FIGURE 1. Example of polyhedral surface giving a flat metric on a compact surface of genus 2 with one conical singularity of negative singular curvature. Consider the hyperbolic plane $\mathbb{H}^2$ ($=O_0$) as a quadric in the Minkowski space $\mathbb{R}^3_1$ and the images of a point of $\mathbb{H}^2$ for the action of a suitable Fuchsian group F. The convex hull E in $\mathbb{R}^3_1$ of these points gives a convex Fuchsian polyhedron. The quotient of E under F gives the wanted metric.

1.3. **Fuchsian polyhedra in hyperbolic space — Towards a general result.** In the hyperbolic space a Fuchsian polyhedron is a polyhedral surface invariant under the action of a Fuchsian group of hyperbolic space $\mathbb{H}^3$. A *Fuchsian group of hyperbolic space* is a discrete group of orientation-preserving isometries leaving globally invariant a totally geodesic surface, denoted by $P_{\mathbb{H}^2}$, on which it acts cocompactly and without fixed points. In [Fil] it is proved that:

Theorem 1.3. *A hyperbolic metric with conical singularities of positive singular curvature on a compact surface S of genus > 1 is realised by a unique convex Fuchsian polyhedron in hyperbolic space.*

By the Gauss–Bonnet formula, we know that there doesn't exist other constant curvature metrics with conical singularities of constant sign singular curvature on surfaces of genus > 1 than the ones described in Theorem A and Theorem 1.3.

A *parabolic polyhedron* is a polyhedral surface of hyperbolic or de Sitter space invariant under the action of a cocompact group of isometries fixing a point on the boundary at infinity. Convex parabolic polyhedra provide constant curvature metrics with conical singularities with constant sign singular curvature on the torus. We think that every such metric is realised by a unique convex parabolic polyhedron [FI06], that would lead, together with theorems 1.1, 1.2, 1.3 and A, to the following general statement. Let S be a compact surface with fundamental group Γ.

1. *Each constant curvature K metric with conical singularities with constant sign singular curvature $\epsilon \in \{-,+\}$ on S can be realised in M_K^ϵ by a unique convex (space-like) polyhedral surface invariant under the action of a representation of Γ in a group of isometries of dimension 3 — with a condition on the lengths of contractible geodesics in the cases $K = 1, \epsilon = -$.*

1.4. **Outline of the proof — organisation of the paper.** Actually the general outline of the proof is very classical, starting from Alexandrov's work, and very close to the one used in [Fil]. Roughly speaking, the idea is to endow with suitable topology both the space of convex Fuchsian polyhedra of M_K^- and the space of corresponding metrics, and to show that the map from one to the other given by the induced metric is a homeomorphism.

The difficult steps are (always) to show local injectivity and properness of the maps "induced metric". The local injectivity is equivalent to statements about infinitesimal rigidity of convex Fuchsian polyhedra. Actually, due to the so-called infinitesimal Pogorelov maps, it suffices to prove the infinitesimal rigidity in the de Sitter space (and this proof uses an infinitesimal Pogorelov map itself).

In the remainder of this section we present some consequences and possible extensions of Theorem A — note that the theorems proved in this paper are labeled with letters instead of numbers.

In the following section we present "projective models" of the space-forms and the so-called infinitesimal Pogorelov maps, and we recall some facts about Teichmüller space which will be used in the sequel.

Section 3 is devoted to the proof of the infinitesimal rigidity of convex Fuchsian polyhedra among convex Fuchsian polyhedra (that corresponds to the local injectivity of the maps "induced metric").

Subsection 4.1 studies the topologies of the spaces of polyhedra, Subsection 4.2 studies the ones of the spaces of metrics, and Subsection 4.3 gives a sketch of the proof of Theorem A.

Finally, we prove in Section 5 the properness of the maps "induced metric", that was the last thing to check according to the sketch of the preceding section.

1.5. **Global rigidity of Fuchsian polyhedron.** A direct consequence of the uniqueness of the convex Fuchsian polyhedron realising the induced metric is

Theorem B. *Convex Fuchsian polyhedra in Lorentzian space-forms are globally rigid among convex Fuchsian polyhedra.*

In all the paper "globally rigid" must be understood in the sense of "uniquely determined by its metric".

1.6. **Andreev's Theorem.** Theorem 1.1 can be seen as a generalisation of the famous Andreev's Theorem about compact hyperbolic polyhedra with acute dihedral angles [And70, RHD06]. It is proved in [Hod92] and it seems that the genus does not intervene in this proof. It follows that the part *1)* of Theorem A would be seen as a generalisation of the Andreev's Theorem for surface of genus > 1.

1.7. **Dual statements.** Using the duality between hyperbolic space and de Sitter space (see the following section), part *1)* of Theorem A can be reformulated as a purely hyperbolic statement:

Theorem C. *Let S be a compact surface of genus > 1 with a spherical metric h with conical singularities of negative singular curvature such that its closed contractible geodesics have lengths $> 2\pi$.*

There exists a unique convex Fuchsian polyhedron in the hyperbolic space such that its dual metric is isometric to h (up to a quotient).

We can do the same statement in the anti-de Sitter case, which is its own dual.

1.8. **Hyperbolic manifolds with polyhedral boundary.** Take a convex Fuchsian polyhedron (P, F) of the hyperbolic space and consider the Fuchsian polyhedron (P', F) obtained by a symmetry relative to the plane $P_{\mathbb{H}^2}$ (the one fixed by the Fuchsian group action). Next cut the hyperbolic space along P and P', and keep the component bounded by P and P'. The quotient of this manifold by the acting group F is a kind of hyperbolic manifold called *Fuchsian manifold* (with convex polyhedral boundary): they are compact hyperbolic manifolds with convex boundary with an isometric involution fixing a hyperbolic surface (the symmetry relative to $P_{\mathbb{H}^2}/F$). All the Fuchsian manifolds can be obtained in this way: the lifting to the universal covers of the canonical embedding of a component of the boundary in the Fuchsian manifold gives a Fuchsian polyhedron of the hyperbolic space. Theorem C says exactly that for a choice of a (certain kind of) spherical metric g on the boundary, there exists a unique hyperbolic metric on the Fuchsian manifold such that the dual metric of the induced metric of the boundary is isometric to g:

Theorem D. *The metric on a Fuchsian manifold with convex polyhedral boundary is determined by the dual metric of its boundary.*

This is a part of the following statement. Let M be a compact connected manifold with boundary ∂M of dimension 3, which admits a hyperbolic metric such that ∂M is polyhedral and convex. We know that the dual metric of the induced metric on ∂M is a spherical metric with conical singularities of negative singular curvature such that the lengths of its closed contractible geodesics are $> 2\pi$.

Conjecture 1.4. *Each such dual metric on ∂M is induced on ∂M by a unique hyperbolic metric on M.*

Note that the theorem of Rivin–Hodgson is another part of this conjecture, in the case where M is the ball. The hyperbolic statement of Conjecture 1.4 is stated in [Fil], and the Fuchsian particular case corresponds to Theorem 1.3. Both conjectures in the case where the boundary is smooth and strictly convex have been proved in [Sch06].

It is also possible to do analogous statements for "anti-de Sitter manifolds with convex boundary", related to so-called *maximal globally hyperbolic anti-de Sitter manifolds*, see *e.g.* [Mes, Sch03]. The part *iii)* of Theorem A would describe a particular case of this statement.

1.9. **Global Pogorelov map and complete hyperbolic metrics.** We can enlarge the definition of Fuchsian polyhedron of the hyperbolic space (Klein projective model, see further) by considering polyhedra with vertices lying on the boundary at infinity (*ideal vertices*) or vertices lying outside the closed ball (*hyperideal vertices*). A vertex lying inside the open ball is called a *finite vertex.* We require that the edges always meet the hyperbolic space. The induced metric on (the quotient of) such a convex Fuchsian polyhedron — called *generalised Fuchsian polyhedron* — is isometric to a complete hyperbolic metric on a surface of topological finite type (the metric may have conical singularities of positive singular curvature). One can naturally asks:

Question 1.5. *Is a complete hyperbolic metric on a surface of finite topological type (genus > 1) realised by a unique convex generalised Fuchsian polyhedron?*

Some particular cases are known: the one with only finite vertex is Theorem 1.3, the one with only ideal vertices is done in [Scha] (and its dual statement in [Rou04]), the one with finite and ideal vertices is done in [Fil06], the one with hyperideals vertices (maybe ideals) is done in [Schb] — this last reference uses different tools than here. The case of the sphere is contained in [Sch98a]. It may be possible to enlarge Conjecture 1.4 using complete hyperbolic metrics on the boundary.

There exists remarkable maps invented by Pogorelov, called *(global) Pogorelov maps*, which sends pair of convex isometric surfaces (maybe polyhedral surfaces) of a constant curvature space (Riemannian or Lorentzian) to a pair of convex isometric surfaces of a flat space [Pog56, Pog73]. Those maps have the property that surfaces of a pair are congruent if and only if their images by the Pogorelov maps have the same property (the infinitesimal Pogorelov maps that we will introduce below can be seen as derivatives of the global Pogorelov maps along the diagonal). Moreover, one of these maps sends convex generalised Fuchsian polyhedra to convex Fuchsian polyhedra of the Minkowski space [Sch98a, LS00, Fil06]. But as Theorem B says that two isometric convex Fuchsian polyhedra of the Minkowski space are congruent, the property of the Pogorelov map says immediatly that:

Theorem E. *Convex generalised Fuchsian polyhedra are globally rigid among convex generalised Fuchsian polyhedra.*

And this statement is the uniqueness part of Question 1.5.

1.10. **Convention.** In all the paper, we call *length* of a time-like geodesic the imaginary part of the "distance" between the endpoints of the geodesic. We will also call *distance* between two points joined by a time-like geodesic the length (in the sense we have just defined) of the geodesic. It follows that distances and lengths will be real numbers. In this mind, for a time-like vector X, we denote the modulus of its "norm" by $\|X\|$ instead of $|\,\|X\|\,|$.

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2. Backgrounds

2.1. **Klein projective models.** For this subsection and nexts, details can be found in [RH93, Sch98a, Sch01, Fil06].

The notation $\mathbb{R}^n_p$ means $\mathbb{R}^n$ endowed with a non-degenerate bilinear form of signature $(n-p,p)$ — read $(+,-)$. Its associated "norm" is denoted by $\|.\|_p$. Recall that Riemannian and Lorentzian space-forms can be seen as pseudo-spheres in flat spaces:

$$\mathbb{H}^3 = \{x \in \mathbb{R}^4_1 |\, \|x\|^2_1 = -1, x_4 > 0\},$$
$$\mathrm{dS}^3 = \{x \in \mathbb{R}^4_1 |\, \|x\|^2_1 = 1\},$$
$$\mathrm{AdS}^3 = \{x \in \mathbb{R}^4_2 |\, \|x\|^2_2 = -1\}$$

(note that the negative directions are always the last ones).

In each M^-_K we choose the following particular point c_K:

$$c_1 = (1,0,0,0) \in \mathbb{R}^4_1,$$
$$c_0 = (0,0,0) \in \mathbb{R}^3_1,$$
$$c_{-1} = (0,0,0,1) \in \mathbb{R}^4_2.$$

With this definition of c_1, for example, the hyperbolic surface O_1 is given by the intersection in $\mathbb{R}^4_1$ of the pseudo-sphere dS^3 and the hyperplane $\{x_1 = \sqrt{2}\}$.

The *Klein projective models* of the hyperbolic and de Sitter spaces are the images of both spaces under the *projective map* given by the projection onto the hyperplane $\{x_4 = 1\}$: $x \mapsto x/x_4$ (naturally endowed with a Euclidean structure in $\mathbb{R}^4_1$).

The hyperbolic space is sent to the open unit ball and the de Sitter space is sent to the exterior of the closed unit ball (actually, to have a diffeomorphism, we involve in this projection only the upper part of the de Sitter space given by $\{x_4 \geq 0\}$, but this is not restrictive anyway because all the surfaces we will consider will be contained in the future-cone of c_1, itself contained in the upper part of the de Sitter space).

In this model, the geodesics correspond to the lines, and for the de Sitter space they are space-like if they don't intersect the closed ball, light-like if they are tangent to the sphere and time-like if they intersect the open ball. It follows that convex polyhedral surfaces of dS^3 (resp. $\mathbb{H}^3$) in this model are exactly convex polyhedral surfaces of the Euclidean space less the closed unit ball (resp. of the unit open ball).

The projective map sends the point c_1 to infinity, and its future-cone is sent to an infinite cylinder with basis a unit disc centered at the origin, and this disc corresponds to the hyperbolic plane dual to c_1 (see below), denoted by $P_{\mathbb{H}^2}$. For both spaces, the unit sphere is the *boundary at infinity* ∂_∞ in this model. The intersection of the closure of a surface with the boundary at infinity is called the *boundary at infinity of the surface.* In this model, O_1 is a half-ellipsoid with as boundary at infinity the horizontal unit circle (the boundary at infinity is the same than the one of $P_{\mathbb{H}^2}$)), see Figure 2.

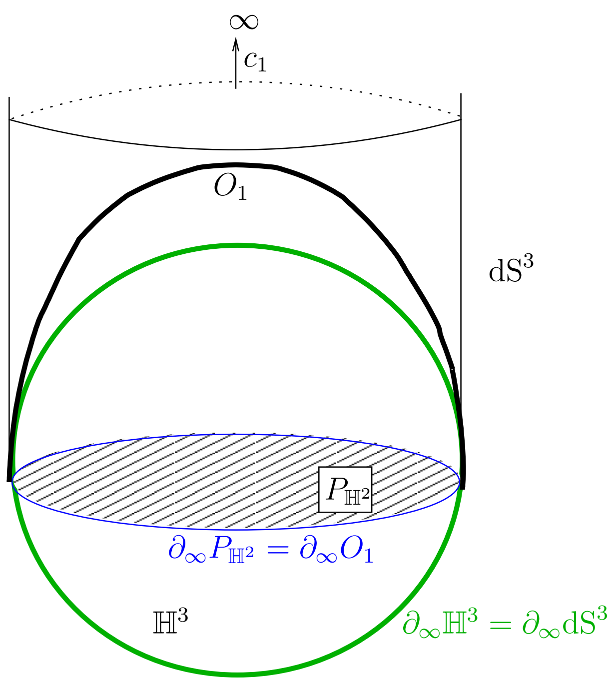

FIGURE 2. Klein projective model of hyperbolic and de Sitter spaces.

2.2. **De Sitter–Euclidean infinitesimal Pogorelov map.** A *Killing field* of a constant curvature space M_K^ϵ, $\epsilon \in \{-,+\}$, is a vector field of M_K^ϵ such that the elements of its local 1-parameter group are isometries (see *e.g.* [O'N83, GHL90]). An *infinitesimal isometric deformation* of a polyhedral surface consists of

- a triangulation of the polyhedral surface given by a triangulation of each face, such that no new vertex arises,
- a Killing field on each face of the triangulation such that two Killing fields on two adjacent triangles are equal on the common edge.

An infinitesimal isometric deformation is called *trivial* if it is the restriction to the polyhedral surface of a global Killing field. If all the infinitesimal isometric deformations of a polyhedral surface are trivial, then the polyhedral surface is said to be *infinitesimally rigid.*

The following construction is an adaptation of a map invented by Pogorelov [Pog73], which allows to transport infinitesimal deformation problems in a constant curvature space to infinitesimal deformation problems in a flat space, see for example [LS00, Rou04, Sch06].

Let $Z(x)$ be a vector of $\mathrm{T}_x\mathrm{dS}^3$, where x lies inside the future-cone of c_1. We denote by $\mathcal{C}$ the sphere obtained by the intersection of the de Sitter space and the hyperplane $\{x_4 = 0\}$ in $\mathbb{R}^4_1$. The *radial component* of $Z(x)$ is the projection of $Z(x)$ on the radial direction, which is given by the derivative at x of the (time-like) geodesic l_x in dS^3 passing through x and orthogonal to $\mathcal{C}$. The *lateral component* of $Z(x)$ is the component orthogonal to the radial one. We denote by ν the length of the geodesic l_x, and by Z_r and Z_l the radial and lateral components of Z. The definitions are the same in Euclidean space, taking the origin instead of $\mathcal{C}$. We denote by φ the projective map sending dS^3 to the Klein projective model. Its derivative sends the radial direction onto the radial direction.

The *de Sitter–Euclidean infinitesimal Pogorelov map* Φ is a map sending a vector field Z of the de Sitter space to a vector field $\Phi(Z)$ of Euclidean space, defined as follow: the radial component of $\Phi(Z)(\varphi(x))$ has same direction and same norm as $Z_r(x)$, and the lateral component of $\Phi(Z)(\varphi(x))$ is $d_x\varphi(Z_l)$, that is, if R is the radial direction of the Euclidean space:

$$\Phi(Z)(\varphi(x)) := d_x\varphi(Z_l) + \|Z_r\|_{\mathrm{dS}^3}\, R(\varphi(x)).$$

If we see a polyhedral surface P in the Klein projective model, then the infinitesimal Pogorelov map is a map sending a vector field on P to another vector field on P. We have

$$\|Z_r\|_{\mathrm{dS}^3} = \|\Phi(Z)_r\|_{\mathbb{R}^3}\,,$$
$$\|Z_l\|_{\mathrm{dS}^3} = \sinh(\nu)\,\|\Phi(Z)_l\|_{\mathbb{R}^3}\,.$$

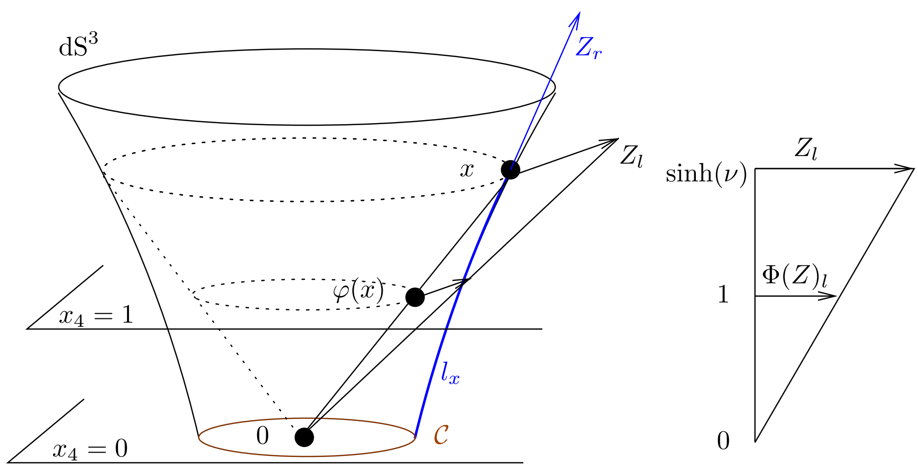

FIGURE 3. $\|Z_l\|_{\mathrm{dS}^3} = \sinh(\nu)\,\|\Phi(Z)_l\|_{\mathbb{R}^3}$.

The first one is the definition, the second one comes from a direct computation or an elementary property of the geometry of the plane (sometimes called the Thales Theorem, see Figure 3 — the pictures have often one less dimension than the objects they describe). We will often make the confusion consisting in missing out the point at which we evaluate a vector field. The infinitesimal Pogorelov map has the following remarkable property:

Lemma 2.1 (Fundamental property of the infinitesimal Pogorelov map [Sch05])**.** *Let* V *be a vector field on* dS^3*, then* V *is a Killing field if and only if* $\Phi(V)$ *is a Killing field of the Euclidean space.*

As an infinitesimal isometric deformation of a polyhedral surface is the data of a Killing field on each triangle of a triangulation, this lemma says that the image of an infinitesimal isometric deformation of a polyhedral surface P by the infinitesimal Pogorelov map is an infinitesimal isometric deformation of the image of P by the projective map. And one is trivial when the other is.

2.3. **Minkowski projective models.** Using the projection onto the hyperplane $\{x_1 = 1\}$ (naturally isometric to $\mathbb{R}^3_1$), we can define another projective model for both hyperbolic and de Sitter spaces, called *Minkowski projective model.* In this model, geodesics are always straight lines, the hyperbolic space is sent to the interior of the upper branch of the hyperboloid, the light cone to this upper branch and the image of the de Sitter space lies between the light-cone and this upper branch (once again, only a half of these spaces are involved in this projection, but up to a global isometry they contain the surfaces that we will consider). The point c_1 is sent to the origin and its future-cone is sent to the future-cone of the origin, see Figure 4 left.

There exists also such a map for the anti-de Sitter space, using the projection onto the hyperplane $\{x_4 = 1\}$. The anti-de Sitter space is sent to the interior of the hyperboloid with one branch, c_{-1} is sent to the origin, its future-cone to the future-cone of the origin and O_{-1} is sent to infinity, see Figure 4 right.

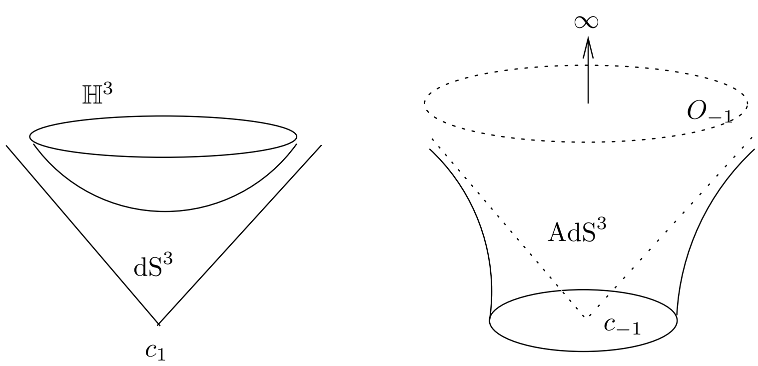

FIGURE 4. Minkowski projective models for hyperbolic and de Sitter space (left) and for anti-de Sitter space (right).

For each of these projections there exists an associated infinitesimal Pogorelov map, defined in the same way than the de Sitter–Euclidean one above, and having the same fundamental property (see [LS00, Rou04, Fil06] for more details).

The properties of these infinitesimal Pogorelov maps are related to the Darboux–Sauer Theorem, which says that the infinitesimal rigidity (for surfaces in the Euclidean space) is a property which remains true under projective transformations.

Actually all these results are contained inside a general statement of J. A. Volkov, which says that each times that there is a map from a (Riemannian in Volkov's text, and certainly pseudo-Riemannian) manifold to a flat manifold sending geodesics to geodesics then there exists a (unique) map of the associated tangent bundles sending Killing fields to Killing fields [Vol74]. Concerning the polyhedral surfaces, there exists a more geometric way to define the infinitesimal Pogorelov maps [SW].

2.4. **Description of Fuchsian polyhedra.**

Lemma 2.2. *A convex Fuchsian polyhedron P lies between two space-like surfaces $S_{\min}$ and $S_{\max}$ realising the minimum and the maximum of the distance from c_K.*

Proof. As the action of the Fuchsian group is cocompact and as P is asked to stay inside the future-cone of c_K, there exists a minimum of the distance of P from c_K, greater or equal to 0. This minimum can't be 0: otherwise, c_K is a point of P, which is space-like and convex, that means that it must stay out of the time-cone of each of its points, in particular out of the time-cone of c_K, that is impossible. It follows that $S_{\min}$ is a space-like surface.

Now it remains to check that the distance from c_K is bounded from above. It is clear in the anti-de Sitter space as all the points inside the future-cone of c_{-1} are at distance less than 2π from c_{-1}. For the de Sitter space, we see easily in the Klein projective model that if a point is lying on the boundary at infinity, by convexity it must be inside a light-like face. For the Minkowski space, if there exists a vertex y at infinite distance from c_0, consider a vertex x concurrent to y and at finite distance from c_0 (it must exist such a vertex, as all the vertices can't be at infinite distance from c_0). But as the light-cones of c_0 and x are parallel, the geodesic joining x and y must be inside the future-cone of x, that means that it is time-like, that is impossible as P is space-like. These arguments prove the lemma, as the group action is cocompact and as there is a finite number of vertices inside each fundamental domain. □

For example, if c_0 is the origin of the Minkowski space, a convex Fuchsian polyhedron is contained between the upper-branches of two hyperboloids. It is easy to check that in the Klein projective model of the de Sitter space a surface (in the future-cone of c_1) at constant distance from c_1 is sent to a half-ellipsoid contained in the upper half-space, which boundary at infinity is the unit circle in the horizontal plane. In this model, a convex Fuchsian polyhedron lies between two such hyperboloids, and in particular its boundary at infinity is the same as the closure of the half-ellipsoids.

As O_K is at constant distance from c_K, the time-like geodesics from c_K are orthogonal to O_K, then they define an orthogonal projection of the future-cone of c_K onto O_K that will be denoted by p_K. Moreover

Lemma 2.3. *The maps p_K are homeomorphisms between each convex Fuchsian polyhedron P and O_K.*

Proof. We will prove it in the de Sitter case, the others cases follow immediately using the projective maps. First, the orthogonal projection of P onto the horizontal plane is one-to-one, as the convex hull of P is the union of P and of the closed unit disc of the horizontal plane. It follows that this horizontal projection is a homeomorphism. Moreover, as O_1 is a convex cap, its orthogonal projection onto the horizontal plane is a homeomorphisms. And it suffices to compose these projections to get p_1. □

2.5. **Duality.** There exists a well-known duality between hyperbolic space and de Sitter space [Cox43, Thu80, Riv86, RH93, Thu97, Sch98a]. It corresponds in the Klein projective model to the classical projective duality with respect to the light-cone (*i.e.* the unit sphere in the Euclidean space). In $\mathbb{R}^4_1$, the dual of a unit vector (a point on a pseudosphere) is its orthogonal hyperplane and the dual of a plane is its orthogonal plane. It follows that the dual of a convex polyhedral surface is a convex polyhedral surface, and the duality is an involution.

The dual of a convex polyhedral surface of the hyperbolic space is a (space-like convex) polyhedral surface of the de Sitter space. Polyhedral surface of the de Sitter space obtained by this way are called *polyhedral surfaces of hyperbolic type.* There exists (space-like convex) polyhedral surfaces in the de Sitter space which are not of hyperbolic type (their dual is not contained in the hyperbolic space), see [Sch01].

Lemma 2.4. *A convex Fuchsian polyhedron of the de Sitter space is of hyperbolic type.*

Proof. Using a projective description of the duality in term of cross-ratio, it is easy to check that the dual of a surface at constant distance from c_1 is a hyperbolic surface at constant distance from the hyperbolic plane dual to c_1 [Sch98b, Fil06]. As a convex Fuchsian polyhedron lies between two surfaces at constant distance from c_1, it follows that its dual lies between two hyperbolic surfaces, then it lies entirely in the hyperbolic space. □

And the lengths of the closed geodesics for the induced metric on a convex poyhedral surface of hyperbolic type are $> 2\pi$ [RH93], that explains the additional condition in the part *1)* of Theorem A.

Note that the proof of this Lemma also says that the dual of a convex Fuchsian polyhedron of the hyperbolic space is contained inside the future-cone of c_1.

Lemma 2.5. *The dual of a convex Fuchsian polyhedron is a convex Fuchsian polyhedron.*

Proof. A Fuchsian polyhedron in the hyperbolic space is invariant under the action of a group F of isometries which leaves invariant $P_{\mathbb{H}^2}$, and this one is given by the intersection of the hyperbolic space with a time-like hyperplane V in $\mathbb{R}^4_1$. The group F is given by orientation-preserving and time orientation-preserving isometries of Minkowski space leaving invariant V. These isometries also fix the unit space-like vector v normal to V (which corresponds to the point c_1 of the de Sitter space), and they also fix the de Sitter space. Moreover these isometries act cocompactly on all the hyperplanes orthogonal to V, in particular the one which defines O_1. It follows that the restrictions of these isometries to the de Sitter space are Fuchsian isometries. The converse holds in the same manner. □

Moreover, the dual metric of a polyhedral surface in hyperbolic space is isometric to the metric induced on its dual [RH93, CD95, Fil06], that explains Theorem C.

The same definition of duality holds without any problem in the anti-de Sitter space (which is its own dual).

2.6. **Teichmüller space.**

Z-V-C coordinates for Teichmüller space. For more details about Z-V-C coordinates (Z-V-C stands for Zieschang–Vogt–Coldewey, [ZVC80]) we refer to [Bus92, 6.7].

Definition 2.6. *Let $g \geq 2$. A (geodesically convex) polygon of the hyperbolic plane with edges (in the direct order) $b_1, b_2, \overline{b}_1, \overline{b}_2, b_3, b_4, \ldots, \overline{b}_{2g}$ and with interior angles $\theta_1, \overline{\theta}_1, \ldots, \theta_{2g}, \overline{\theta}_{2g}$ is called* (normal) canonical *if, with $l(c)$ the length of the geodesic c,*

i) $l(b_k) = l(\overline{b}_k)$, $\forall k$;
ii) $\theta_1 + \ldots + \overline{\theta}_{2g} = 2\pi$;
iii) $\theta_1 + \theta_2 = \overline{\theta}_1 + \theta_2 = \pi$.

Two canonical polygons P and P' with edges $b_1, \ldots, \overline{b}_{2g}$ and $b'_1, \ldots, \overline{b}'_{2g}$ are said equivalent *if there exists an isometry from P to P' such that the edge b_1 is sent to the edge b'_1 and b_2 is sent to b'_2.*

If we identify the edges b_i with the edges $\overline{b}_i$, we get a compact hyperbolic surface of genus g. This surface could also be written $\mathbb{H}^2/F$, where F is the sub-group of $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$ generated by the translations along the edges b_i (the translation length is the length of b_i). The interior of the polygon is a fundamental domain for the action of F. This leads to a description of the Teichmüller space T_g:

Proposition 2.7 ([Bus92, 6.7.7])**.** *Let P_g be the set of equivalence classes of canonical polygons. An element of P_g is described by the $(6g-6)$ real numbers (the Z-V-C coordinates):*

$$(b_3, \ldots, b_{2g}, \theta_3, \overline{\theta}_3, \ldots, \theta_{2g}, \overline{\theta}_{2g}).$$

Endowed with this topology, P_g is in analytic bijection with T_g.

Fenchel–Nielsen coordinates for Teichmüller space. A compact hyperbolic surface S of genus g can be described as a gluing of pants. Such a gluing leads to the choice of $(6g-6)$ real numbers: the lengths of the geodesics along which we glue the pants and the angles of the twists. These numbers (the Fenchel–Nielsen coordinates) describe the Teichmüller space of S [Bus92].

We denote by $l(x)$ the length of the curve x. It is possible to compute the twist parameters knowing the lengths of certain geodesics:

Proposition 2.8 ([Bus92, 3.3.12])**.** *Let γ be the geodesic along which are glued two pants. We denote by δ the geodesic along which we can cut the X-piece to get the other pant decomposition than the one given by γ.*

We do a twist with parameter α around γ, and we denote by δ^α the geodesic which is in the homotopy class of the image of δ by the twist. Then there exists real analytic functions of $l(\gamma)$ u and v ($v > 0$) such that

$$\cosh\left(\frac{1}{2} l(\delta^\alpha)\right) = u + v \cosh(\alpha l(\gamma)). \tag{1}$$

Let $(S_k, h_k)_k$ be a sequence of (equivalence classes) of hyperbolic surfaces in the Teichmüller space of S. The metric on S is h_0. We denote by f_k the homeomorphism between S and S_k, and by $\mathrm{L}_{h_k}(\gamma)$ the length of the geodesic corresponding to the element $(f_k)_*(\gamma)$ of the fundamental group of S_k ($\gamma \in \pi_1(S)$) for the metric h_k.

Lemma 2.9. *If there exists a constant $c > 0$ such that, for all $\gamma \in \pi_1(S)$, $\mathrm{L}_{h_k}(\gamma) \geq \frac{1}{c}\mathrm{L}_{h_0}(\gamma)$, then $(h_k)_k$ converges (up to extract a subsequence).*

Proof. We will prove that in this case $(h_k)_k$ is contained inside a compact of the Teichmüller space using the Fenchel–Nielsen coordinates. First, the lengths of the geodesics along which we cut up to obtain the pants are bounded from above: if not, because of the Gauss–Bonnet Formula, it would exist another geodesic with decreasing length (otherwise the area of the surface may become arbitrary large), that is impossible as the lengths are bounded from below.

It remains to check that the twist parameters are bounded. Take a geodesic γ along which a twist is done. We suppose now that the twist parameter α_k associated to γ becomes arbitrary large (up to consider a k sufficiently large and up to extract a sub-sequence, in regard of the considerations above we can suppose that the length of γ doesn't change). We look at the geodesic δ defined as in Proposition 2.8 for the metric h_k. If now we do the twists in the counter order to go from h_k to h_0 (*i.e.* we do twists with parameters $-\alpha_k$), then by (1), $\mathrm{L}_{h_0}(\delta)$ is arbitrariness larger than $\mathrm{L}_{h_k}(\delta)$. □

Geodesic lengths coordinates for Teichmüller space. With the same notations than above, here is a result that can be found *e.g.* as [tra91, Exposé 7, Proposition 5]:

Lemma 2.10. *There exists a finite number of elements $(\gamma_1, \dots, \gamma_n)$ of the fundamental group of S such that, if for all i $\mathrm{L}_{h_k}(\gamma_i)$ is uniformly bounded from above, then $(h_k)_k$ converges (up to extract a subsequence).*

3. Fuchsian infinitesimal rigidity

3.1. **Fuchsian polyhedral embeddings.** To define the "Fuchsian infinitesimal rigidity" we need to describe Fuchsian polyhedra as polyhedral embeddings.

Definition 3.1. *A (space-like)* polyhedral embedding *of a surface S into M_K^- is a cellulation of S together with a homeomorphism from S to a (space-like) polyhedral surface of M_K^-, sending polygons of the cellulation to (space-like) geodesic polygons of M_K^-.*

A Fuchsian polyhedral embedding *in M_K^- is a triple (S, ϕ, ρ), where*

- *S is a compact surface of genus > 1,*
- *ϕ is a polyhedral embedding of the universal cover $\widetilde{S}$ of S into M_K^-,*
- *ρ is a representation of the fundamental group Γ of S into $\mathrm{Isom}_+^+(M_K^-)$,*

such that ϕ is equivariant *under the action of Γ:*

$$\forall \gamma \in \Gamma, \forall x \in \widetilde{S},\ \phi(\gamma x) = \rho(\gamma)\phi(x),$$

and $\rho(\Gamma)$ is Fuchsian.

The number of vertices of the Fuchsian polyhedral embedding *is the number of vertices of the cellulation of S.*

The Fuchsian polyhedral embedding is convex *if its image is a convex polyhedral surface of M_K^-.*

We consider the Fuchsian polyhedral embeddings up to homeomorphisms and up to global isometries: (S_1, ϕ_1, ρ_1) and (S_2, ϕ_2, ρ_2) are equivalent if there exists a homeomorphism h between S_1 and S_2 and an isometry I of M_K^- such that, for a lift $\widetilde{h}$ of h to $\widetilde{S}_1$ we have

$$\phi_2 \circ \widetilde{h} = I \circ \phi_1. \tag{2}$$

As two lifts of h only differ by conjugation by elements of Γ, using the equivariance property of the embedding, it is easy to check that the definition of the equivalence relation doesn't depend on the choice of the lift.

Definition 3.2. *The* genus of a Fuchsian group F of M_K^- *is the genus of the quotient of* O_K *by the restriction of* F *to it.*

The genus of a Fuchsian polyhedron (P, F) *is the genus of* F*.*

The number of vertices of a Fuchsian polyhedron (P, F) *is the number of vertices of* P *in a fundamental domain for the action of* F*.*

As S is a compact surface of genus $g > 1$, it can be endowed with hyperbolic metrics, and each of them provides a cocompact representation of Γ in the group of orientation-preserving isometries of O_K. The images of such representations are usually called *Fuchsian groups* (as O_K is isometric to $\mathbb{H}^2$), that explains the terminology used. Using the orthogonal projection p_K to prolong the action of a Fuchsian group of the hyperplane O_K to the entire future-cone of c_K, it is easy to check that (as done in [Fil]):

Lemma 3.3. *There is a bijection between the cocompact representations of the fundamental group of* S *in* $\mathrm{Isom}^+(\mathbb{H}^2)$ *and the Fuchsian groups of* M_K^- *of genus* g*.*

It follows that there is a bijection between the convex Fuchsian polyhedra of genus g *with* n *vertices and the convex Fuchsian polyhedral embeddings with* n *vertices of a compact surface of genus* g*.*

3.2. **Fuchsian deformations.** Let (S, ϕ, ρ) be a convex polyhedral Fuchsian embedding in M_K^- and let $(\phi_t)_t$ be a path of convex polyhedral embeddings of $\widetilde{S}$ in M_K^-, such that:

- $\phi_0 = \phi$,
- the induced metric is preserved at the first order at $t = 0$,
- there are representations ρ_t of $\Gamma = \pi_1(S)$ into $\mathrm{Isom}^+_+(M_K^-)$

such that

$$\phi_t(\gamma x) = \rho_t(\gamma)\phi_t(x)$$

and each $\rho_t(\Gamma)$ is Fuchsian.

We denote by

$$Z(\phi(x)) := \frac{d}{dt}\phi_t(x)_{|t=0} \in \mathrm{T}_{\phi(x)} M_K^-$$

and

$$\dot\rho(\gamma)(\phi(x)) = \frac{d}{dt}\rho_t(\gamma)(\phi(x))_{|t=0} \in \mathrm{T}_{\rho(\gamma)\phi(x)} M_K^-.$$

The vector field Z has a property of *equivariance* under $\rho(\Gamma)$:

$$Z(\rho(\gamma)\phi(x)) = \dot\rho(\gamma)(\phi(x)) + d\rho(\gamma).Z(\phi(x)).$$

This can be written

$$Z(\rho(\gamma)\phi(x)) = d\rho(\gamma).(d\rho(\gamma)^{-1}\dot\rho(\gamma)(\phi(x)) + Z(\phi(x))) \tag{3}$$

and $d\rho(\gamma)^{-1}\dot\rho(\gamma)$ is a Killing field of M_K^-, because it is the derivative of a path in the group of isometries of M_K^- (we must multiply by $d\rho(\gamma)^{-1}$, because $\dot\rho(\gamma)$ is not a vector field). We denote this Killing field by $\vec\rho(\gamma)$. Equation (3) can be written, if $y = \phi(x)$,

$$Z(\rho(\gamma)y) = d\rho(\gamma).(\vec\rho(\gamma) + Z)(y). \tag{4}$$

A *Fuchsian deformation* is an infinitesimal isometric deformation Z on a Fuchsian polyhedron which satisfies Equation (4), where $\vec{\rho}(\gamma)$ is a *Fuchsian Killing field*, that is a Killing field of the hyperbolic plane O_K extended to the future-cone of c_K along the geodesics orthogonal to O_K. More precisely, for a point $x \in M_K^-$ inside the future-cone of c_K, let d be the distance between x and $p_K(x)$. We denote by $p(d)$ the orthogonal projection onto O_K of the surface S_d which is at constant distance d from O_K (passing through x). Then the Killing field K at $p_K(x)$ is extended as $dp(d)^{-1}(K)$ at the point x. In other words, a Fuchsian Killing field of M_K^- is a Killing field of M_K^- which restriction to each surface S_d is a Killing field of S_d.

A Fuchsian polyhedron is *Fuchsian infinitesimally rigid* if all its Fuchsian deformations are trivial (*i.e.* are restriction to the Fuchsian polyhedron of Killing fields of M_K^-). We want to prove

Theorem F. i) *Convex Fuchsian polyhedra in de Sitter space are Fuchsian infinitesimally rigid;*
ii) *Convex Fuchsian polyhedra in Minkowski space are Fuchsian infinitesimally rigid;*
iii) *Convex Fuchsian polyhedra in anti-de Sitter space are Fuchsian infinitesimally rigid.*

We will first prove the part *i*) of Theorem F in a way close to the one used in the hyperbolic case [Fil]. As already noted, in the Klein projective model of the de Sitter space a convex Fuchsian polyhedron looks like a polyhedral convex cap (with infinite number of vertices accumulating on the boundary). It remains to check that the vertical component of the image of a Fuchsian deformation Z by the infinitesimal Pogorelov map (from de Sitter to Euclidean space) vanishes on the boundary, and the conclusion will be given by

Proposition 3.4 ([Fil, Proposition 1])**.** *If the vertical component of an infinitesimal isometric deformation of a polyhedral convex cap of the Euclidean space vanishes on the boundary, then the deformation is trivial.*

Note that this proposition (which statement is classical) allows polyhedral convex caps with an infinite number of vertices which accumulate at the boundary, and also infinitesimal deformations which diverge at the boundary.

After that, we use infinitesimal Pogorelov maps from de Sitter space to Minkowski space to have the part *ii*) of Theorem F, and from anti-de Sitter space to Minkowski space to have the part *iii*) (the key point will be that these infinitesimal Pogorelov maps send Fuchsian deformations to Fuchsian deformations).

3.3. **Remarks about the method employed.** Theorem F is not new, because the case of the Minkowski space was done in [Sch07, Thm 6.2] (see [Isk00, Thm B] for a partial result), using a completely different method than the one used here, and the other cases can be easily deduced using the infinitesimal Pogorelov maps (as it has been used for the de Sitter case in [Scha, Rou04]). Theorem F can also be seen as a direct consequence of the Fuchsian infinitesimal rigidity in the hyperbolic space proved in [Fil], using an infinitesimal Pogorelov map from the hyperbolic space to the Minkowski space. In a counter point of view, the Fuchsian infinitesimal rigidity in the hyperbolic space itself can be seen as a consequence of the Fuchsian infinitesimal rigidity in the Minkowski space.

Here we prove the infinitesimal rigidity in the case of the de Sitter space in a more direct way, but it is not excluded that it may be possible to have another proof without using any infinitesimal Pogorelov map, using a method close to the one used to prove Proposition 3.4. It may be possible to use the method presented here and in the hyperbolic case to give a more direct proof of the infinitesimal rigidity in the anti-de Sitter case, this is explained at the end of this section. I don't know if there is a possible adaption of the method used here for the Minkowski case to get a more direct proof than the one in [Sch07].

Also note that the present proof of Theorem F is also true without any change for strictly convex smooth Fuchsian surfaces, using a smooth analog of Proposition 3.4.

3.4. **Proof of part** *i*) **of Theorem F.** Let $P = (\phi, \rho)$ be a convex Fuchsian polyhedron of the de Sitter space. The derivative at a point $x \in P$ of the unique time-like geodesic from c_1 to x is called the *vertical direction* at x, and the directions orthogonal to this one are *horizontal directions*. So a vector field Z can be decomposed into a vertical component Z_v and into a horizontal component Z_h. We denote by $(Z_r)_h$ the horizontal component of the radial component of Z, etc. We have

$$Z_r = (Z_r)_h + (Z_r)_v, Z_r = (Z_h)_r + (Z_v)_r.$$

The first one is obvious and the second one comes from the linearity of the projectors. By definition, a Fuchsian Killing field has no vertical component, it follows that

Lemma 3.5. *The vertical component Z_v of a Fuchsian deformation Z is* invariant *under the action of* Γ*,* i.e. *$\forall x \in P$:*

$$Z_v(\rho(\gamma)x) = d\rho(\gamma)Z_v(x).$$

By cocompacity it follows that there exists a constant c_v such that, for all $x \in P$,

$$\|Z_v(x)\| \leqslant c_v, \qquad (5)$$

and the vector field Z_h is equivariant under the action of Γ*.*

Recall that $p(d)$ is the restriction of the orthogonal projection onto O_1 to the surface which is at constant distance d from O_1. In the Klein projective model, we denote by x_o the intersection of O_1 with the geodesic passing through the origin of the Klein projective model, denoted by c_h, and c_1 (see Figure 5). The *radial direction* of O_1 at a point $y \in O_1$, denoted by $\mathrm{rad}(y)$, is the tangent vector at y of the geodesic of O_1 joining x_o and y.

For a point x on a Fuchsian polyhedron, at distance d from O_1, we call *radial-horizontal* the component of Z (at x) in the direction $dp(d)^{-1}(\mathrm{rad}(p_1(x)))$. This component is denoted by Z_{rh}, and it is a horizontal vector. We denote by W the projection onto O_1 of the horizontal component of Z (it is equivariant under the action of $\rho(\Gamma)$). We denote by W_r its radial component. Then $dp(d)^{-1}(W_r) =: (Z_h)_{rh}$ is the radial-horizontal component of the horizontal component of Z.

Proposition 3.6 ([Fil, Proposition 4])**.** *Let H be a vector field of $\mathbb{H}^2$ equivariant under the action of a Fuchsian group. There exists a constant $c_{\dot g}$ such that, for all $x \neq p$,*

$$\|H_r(x)\|_{\mathbb{H}^2} \leq c_{\dot g} \mathrm{d}_{\mathbb{H}^2}(p, x),$$

where p is any fixed point of the hyperbolic plane.

We apply this result to the vector field W on O_1 with $p = x_o$. As the hyperbolic metric is the metric induced by the de Sitter one on O_1, we get $\|W_r\|_{\mathbb{H}^2} = \|W_r\|_{\mathrm{dS}^3}$, and it is easy to check that:

$$\|(Z_h)_{rh}(x)\|_{\mathrm{dS}^3} = \sinh(\tau)\,\|W_r(p_1(x))\|_{\mathrm{dS}^3}\,,$$

where τ is the length of the geodesic joining x and c_1 in dS^3, and τ is uniformly bounded. It follows that there exists a constant c_{rh} such that:

$$\|(Z_h)_{rh}(x)\|_{\mathrm{dS}^3} \leq c_{rh}\mu, \tag{6}$$

where $\mu = \mathrm{d}_{\mathbb{H}^2}(x_o, p_1(x))$.

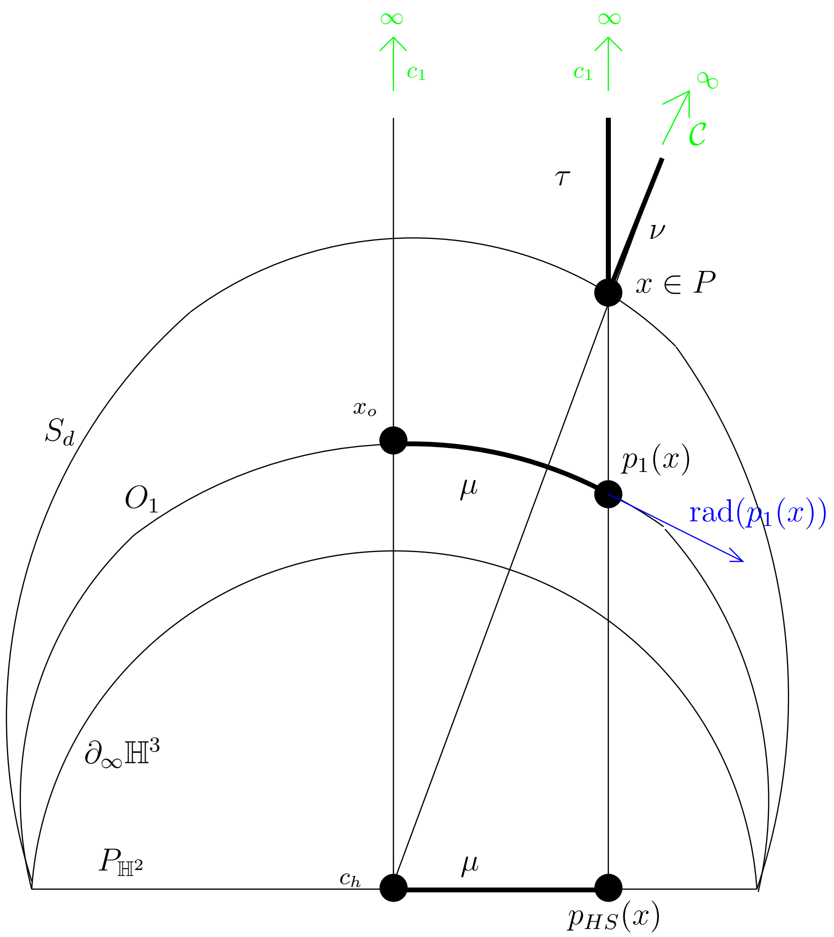

FIGURE 5. Notations in the Klein projective model of the de Sitter space.

The *vertical plane* is the vector space of $\mathrm{T}_x\mathrm{dS}^3$ spanned by the orthogonal vectors Z_v and Z_{rh} (Z_{rh} is in the horizontal plane by definition, then orthogonal to the vertical direction). We can see the vertical plane as the tangent plane (at x) to the totally geodesic plane passing through c_1 and x (and orthogonal to O_1). We denote by Z_{lv} the projection onto the vertical plane of the lateral component of Z.

The vector Z_r belongs to the vertical plane, because it can be decomposed in a horizontal component, which is in the radial-horizontal direction, and a vertical component. The lateral component is orthogonal to the radial component, thus the vector Z_{lv} is orthogonal to Z_r in the vertical plane.

Lemma 3.7. *Let V be the projection of a component of Z onto the vertical plane. There exists a constant c such that*

$$\|V\|_{\mathrm{dS}^3} \leq c(1+\mu).$$

Proof. We denote by Π_V the projection onto the vertical plane, considered as spanned by the orthogonal vectors Z_{rh} and Z_v. We can write $\Pi_V(Z) = Z_{rh} + Z_v$. As V is already in the vertical plane, and as we project a component of Z, we can write:

$$\begin{aligned} \|V\|_{\mathrm{dS}^3} = \|\Pi_V(V)\|_{\mathrm{dS}^3} &\leqslant \|\Pi_V(Z)\|_{\mathrm{dS}^3} \\ &\leqslant \|Z_{rh}\|_{\mathrm{dS}^3} + \|Z_v\|_{\mathrm{dS}^3} \\ &\leqslant \|(Z_h)_{rh}\|_{\mathrm{dS}^3} + \|(Z_v)_{rh}\|_{\mathrm{dS}^3} + \|Z_v\|_{\mathrm{dS}^3} \\ &\leqslant \|(Z_h)_{rh}\|_{\mathrm{dS}^3} + 2\,\|Z_v\|_{\mathrm{dS}^3}\,, \end{aligned}$$

and as the overestimation of these two last norms are known (by Formulas (5) and (6)) we get

$$\|V\|_{\mathrm{dS}^3} \leqslant c_{rh}\mu + 2c_v,$$

that is, if c is greater than c_{rh} and $2c_v$,

$$\|V\|_{\mathrm{dS}^3} \leqslant c(1+\mu).$$

□

We denote $u := \Phi(Z)$, where Φ is the de Sitter-Euclidean infinitesimal Pogorelov map, and we denote by α the angle between u_v and u_{lv} (the definitions of the decompositions of u are the same as for Z, in particular, u_r and u_{lv} form an orthogonal basis of the vertical plane).

Recall that ν is the distance between x and $\mathcal{C}$ in the de Sitter space ($\mathcal{C}$ is the intersection of dS^3 with the hyperplane $\{x_1 = 0\}$ in the Minkowski space of dimension 4). From the properties of the infinitesimal Pogorelov map and Lemma 3.7 (Z_{lv} and Z_r lie on the vertical plane):

$$\begin{aligned} u_v &= \cos(\alpha)u_{lv} + \sin(\alpha)u_r, \\ \|u_v\|_{\mathbb{R}^3} &\leqslant \|u_{lv}\|_{\mathbb{R}^3} + \sin(\alpha)\,\|u_r\|_{\mathbb{R}^3} \\ &\leqslant \sinh^{-1}(\nu)\,\|Z_{lv}\|_{\mathrm{dS}^3} + \sin(\alpha)\,\|Z_r\|_{\mathrm{dS}^3} \\ &\leqslant \sinh^{-1}(\nu)c(1+\mu) + \sin(\alpha)c(1+\mu), \end{aligned}$$

and at the end we get

$$\|u_v\|_{\mathbb{R}^3} \leqslant c(1+\mu)(\sinh^{-1}(\nu) + \sin(\alpha)). \tag{7}$$

The orthogonal projection in the Klein projective model gives an isometry between O_1 and $P_{\mathbb{H}^2}$. We denote by p_{HS} this projection. Note that $p_{HS}(x_o) = c_h$. Let δ' be a positive real number such that the half-ellipsoid of radii $(1, 1, \delta')$ is contained in the upper-part of the unit ball of the Euclidean space. We denote by $S_{\delta'}$ this surface seen in the hyperbolic space (Klein projective model), and obviously:

$$\mu := \mathrm{d}_{O_1}(x_o, p_1(x)) = \mathrm{d}_{P_{\mathbb{H}^2}}(c_h, p_{HS}(x)) \leq \mathrm{d}_{\mathbb{H}^3}(c_h, y)$$

where y is a point of $S_{\delta'}$, preimage of $p_{HS}(x)$ by the orthogonal projection of $S_{\delta'}$ onto $P_{\mathbb{H}^2}$. Going on from (7):

$$\begin{aligned} \|u_v\|_{\mathbb{R}^3} &\leqslant c(1+\mu)(\sinh^{-1}(\nu) + \sin(\alpha)) \\ &\leqslant c(1+\mathrm{d}_{\mathbb{H}^3}(c_h, y))(\sinh^{-1}(\nu) + \sin(\alpha)). \end{aligned}$$

A direct computation (that is a straightforward adaptation of [Fil, Lemma 8] or of the next lemma) gives:

$$\mathrm{d}_{\mathbb{H}^3}(c_h, y) \underset{\delta' \to 0}{\approx} -c_{\delta'} \ln(\delta'), \tag{8}$$

where $c_{\delta'}$ is a positive constant.

We denote by δ the Euclidean distance between the point of the Fuchsian polyhedron P and $P_{\mathbb{H}^2}$. Obviously, $\delta \mapsto 0$ if and only if $\delta' \mapsto 0$. Near the boundary at infinity, using an easy equivalence between the sine and δ, together with Formula (8):

$$c(1+\mathrm{d}_{\mathbb{H}^3}(c_h, y))(\sinh^{-1}(\nu) + \sin(\alpha)) \underset{\delta \to 0}{\approx} \overline{c}(1 - \ln(\delta'))(\sinh^{-1}(\nu) + \delta).$$

Remember that $S_{\min}$ is the surface realising the minimum of the distance between P and c_1. We denote by $x_{\min}$ the intersection of $S_{\min}$ with the geodesic joining x and $\mathcal{C}$, and we denote by $\nu_{\min}$ the distance in dS^3 between $x_{\min}$ and $\mathcal{C}$ and $\delta_{\min}$ the distance in $\mathbb{R}^3$ between $x_{\min}$ and $P_{\mathbb{H}^2}$ (guess the definition of $\delta_{\max}$). We have

$$\nu_{\min} \leq \nu,$$

$$\delta' \leq \delta_{\max} \leq \delta \leq \delta_{\min}.$$

Lemma 3.8. *When $x_{\min}$ goes near the boundary, we have the approximation*

$$\nu_{\min} \underset{\delta_{\min} \to 0}{\approx} -c_{\min} \ln(\delta_{\min}),$$

where $c_{\min}$ is a positive constant.

Proof. It is easy to check that a sphere at constant distance d of $\mathcal{C}$ in dS^3_+ is sent by the projective map to a sphere of radius $\coth(d)$ in $\mathbb{R}^3$. It follows that $\nu_{\min} = \coth^{-1}(\|x_{\min}\|_{\mathbb{R}^3})$, that means

$$\nu_{\min} = \ln\left(\frac{\|x_{\min}\|_{\mathbb{R}^3} + 1}{\|x_{\min}\|_{\mathbb{R}^3} - 1}\right) \underset{\|x_{\min}\|_{\mathbb{R}^3} \to 1}{\approx} -\ln(\|x_{\min}\|_{\mathbb{R}^3} - 1).$$

As the image of $S_{\min}$ in $\mathbb{R}^3$ is a half-ellipsoid, $\delta_{\min}$ verifies the equation

$$x_1^2 + x_2^2 + \frac{\delta_{\min}^2}{r^2} = 1,$$

where r is a positive constant, strictly greater than 1. Adding and removing a $\delta_{\min}^2$, and reordering the terms, we get

$$\|x_{\min}\|_{\mathbb{R}^3} - 1 = \delta_{\min}^2 \frac{r^2 - 1}{r^2},$$

because $\delta_{\min}^2 = x_3^2$, and this gives the announced result. □

We have

$$\overline{c}(1-\ln(\delta'))(\sinh^{-1}(\nu)+\delta)\leq\overline{c}(1-\ln(\delta'))(\sinh^{-1}(\nu_{\min})+\delta)$$

and, using the equivalence between the exponential and the hyperbolic sine,

$$\overline{c}(1-\ln(\delta'))(\sinh^{-1}(\nu_{\min})+\delta)\underset{\delta_{\min}\to 0}{\approx}\overline{\overline{c}}(1-\ln(\delta'))(\delta_{\min}+\delta),$$

and as

$$\overline{\overline{c}}(1-\ln(\delta'))(\delta_{\min}+\delta)\leq\overline{\overline{c}}(1-\ln(\delta_{\max}))(\delta_{\min}+\delta_{\min}),$$

and as this last term goes to 0 when δ (and thus $\delta_{\min}$ and $\delta_{\max}$) goes to 0, it follows that $\|u_v\|_{\mathbb{R}^3}$ goes to zero when the point goes near the boundary at infinity. It was that we had to check to prove part i) of Theorem F.

3.5. **Proof of part ii) and iii) of Theorem F.**

Lemma 3.9. *A convex polyhedral surface contained in the future-cone of c_0 in the Minkowski space and invariant under the action of a Fuchsian group is a convex Fuchsian polyhedron,* i.e. *it is space-like.*

Proof. Suppose that the polyhedral surface is not space-like. Without loss of generality, we can suppose that one of the vertex x belonging to such a non space-like face H lies on the hyperbolic space O_0. The images of x by the action of the Fuchsian group F are all lying on O_0. Moreover by cocompacity their limit set corresponds to the entire boundary at infinity of O_0 (seen in the Klein projective model).

If H belongs to a time-like affine hyperplane, this one separates O_0 in two components, each one containing an infinite number of points on the boundary at infinity. By convexity, all the images of x under the action of F must stay in the same side of H, that is impossible as all the points of the boundary at infinity must be reached.

If H belongs to a light-like affine hyperplane, its intersection with O_0 is a horocycle, it follows that all the images of x by the action of F must be "outside" the horocycle (*i.e.* in its non-convex part in the Klein projective model). To get a contradiction with the convexity, it is enough to show that if $(x_k)_k$ is a sequence such that $x_k := f_k x$, $f_k \in F$, converging to the "center" of the horocycle on the boundary at infinity, then the sequence must meet the "inner part" of the horocycle (*i.e.* its convex part in the Klein projective model). But this is a well-known fact of the geometry of the hyperbolic plane, as F is a cocompact group containing only hyperbolic isometries.

There is no such problem if H belongs to a space-like affine hyperplane, because in this case its intersection with O_0 is a circle (maybe reduced to a point if the hyperplane is tangent to the hyperboloid) and all the images of the vertex are sent outside of the disc bounded by the circle. □

Remember that we have defined projective maps from de Sitter and anti-de Sitter spaces to Minkowski space, which send respectively c_1 to c_0 and c_{-1} to c_0 (the origin of the Minkowski space).

Lemma 3.10. *The projective maps from de Sitter and anti-de Sitter spaces to Minkowski space send convex Fuchsian polyhedra to convex Fuchsian polyhedra.*

Proof. We already know that these projective maps send convex polyhedral surfaces contained in the future-cone of c_1 (or c_{-1}) to convex polyhedral surfaces contained in the future-cone of c_0. We will see that they also act on the representations of Γ. This, together with Lemma 3.9, will prove the statement.

We denote by $\mathrm{Isom}_{c_1}(\mathrm{dS}^3)$ the subgroup of $\mathrm{Isom}_+^+(\mathrm{dS}^3)$ which fix the point c_1. The fact is that: the projective map $\varphi : \mathrm{dS}^3 \longrightarrow \mathbb{R}^3_1$ induces an isomorphism $G : \mathrm{Isom}_{c_1}(\mathrm{dS}^3) \longrightarrow \mathrm{Isom}_{c_0}(\mathbb{R}^3_1)$ which commutes with the projective map, that is, if $f \in \mathrm{Isom}_{c_1}(\mathrm{dS}^3)$, then

$$\varphi(f(x)) = G(f)(\varphi(x)).$$

The projective map from AdS^3 to $\mathbb{R}^3_1$ has the same property. Now we prove this fact.

For the de Sitter–Minkowski case, by definition of the projective map, the Minkowski space $\mathbb{R}^3_1$ is seen as the intersection of $\mathbb{R}^4_1$ with the hyperplane $\{x_1 = 1\}$. It allows us to extend the isometries of the Minkowski space of dimension 3: an isometry of $\mathbb{R}^3_1$ sending x to y can be prolonged to an isometry of $\mathbb{R}^4_1$ which sends the point (t, tx) to the point (t, ty). Furthermore, these isometries of $\mathbb{R}^4_1$ preserve the de Sitter space: there restrictions to the hyperboloid are isometries of the de Sitter space. It follows that the isometries of $\mathbb{R}^3_1$ which fix the origin correspond to isometries of de Sitter space which fix the point c_1.

It is the same thing in the anti-de Sitter case, isometries which send x to y are prolonged to isometries sending (tx, t) to (ty, t). The properties of commutations are then obvious. And by construction, a cocompact group is sent to a cocompact group. □

To prove the remainder of Theorem F it suffices to show that:

Lemma 3.11. *The de Sitter–Minkowski and anti-de Sitter–Minkowski infinitesimal Pogorelov maps send Fuchsian Killing fields to Fuchsian Killing fields.*

Proof. We know that these maps send Killing fields to Killing fields. Moreover, they send radial component to radial component, and a Fuchsian Killing field is characterised by the fact that it has no vertical component. □

Lemma 3.12. *The de Sitter–Minkowski and anti-de Sitter–Minkowski infinitesimal Pogorelov maps send Fuchsian deformations to Fuchsian deformations.*

Proof. A Fuchsian deformation Z verifies

$$Z(\phi(\gamma x)) = d\rho(\gamma)(\vec{\rho}(\gamma) + Z)(\phi(x)),$$

and, on one hand the infinitesimal Pogorelov maps sends Fuchsian Killing fields to Fuchsian Killing fields, and on the other hand, considering the radial component or applying the projection onto the lateral component are linear operations. These arguments and Lemma 3.10 suffice to prove this lemma, but here are the details. We do them for the de Sitter case, it is word by word the same for the anti-de Sitter space.

We denote by r the radial direction of the de Sitter space, by R the radial direction of the Minkowski space of dimension 3 and by φ the projective map from dS^3 to $\mathbb{R}^3_1$. The proof of Lemma 3.10 gives the existence of a morphism G between $\mathrm{Isom}_{c_1}(\mathrm{dS}^3)$ and $\mathrm{Isom}_{c_0}(\mathbb{R}^3_1)$ such that

$$\varphi(\rho(\gamma)(\phi(x))) = G(\rho(\gamma))(\varphi(\phi(x))).$$

If Φ is the infinitesimal Pogorelov map from de Sitter to Minkowski, then:

$$\Phi(Z)(\varphi \circ \phi(\gamma x)) = d\varphi(Z_l)(\phi(\gamma x)) + \|Z_r(\phi(\gamma x))\|\, R(\varphi(\phi(\gamma x))).$$

We first examine the first term of the right member of the equation above:

$$\begin{aligned} d\varphi(Z_l)(\phi(\gamma x)) &= d\varphi d\rho(\gamma)(\vec{\rho_l}(\gamma)+Z_l)(\phi(x)) \\ &= dG(\rho(\gamma))d\varphi(\vec{\rho_l}(\gamma)+Z_l)(\varphi\circ\phi(x)). \end{aligned}$$

Afterwards we examine the second term of the right member:

$$\begin{aligned} \|Z_r(\phi(\gamma x))\|\, R(\varphi(\phi(\gamma x))) &= \|d\rho(\gamma)(Z_r)(\phi(x))\|\, R(\varphi(\rho(\gamma)\phi(x))) \\ &= \|Z_r(\phi(x))\|\, R(\varphi(\rho(\gamma)\phi(x))) \\ &= \|\|Z_r\|\, r(\phi(x))\|\, R(G(\rho(\gamma))(\varphi\circ\phi(x))) \\ &= \|Z_r\|\, R(G(\rho(\gamma))(\varphi\circ\phi(x))) \\ &= dG(\rho(\gamma))(\|Z_r\|\, R(\varphi\circ\phi(x))). \end{aligned}$$

And at the end, using both computations above, we have what we wanted:

$$\begin{aligned} \Phi(Z)(\varphi\circ\phi(\gamma x)) &= dG(\rho(\gamma))(d\varphi(\vec{\rho_l}(\gamma))+d\varphi(Z_l)+\|Z_r\|\, R)(\varphi\circ\phi(x)) \\ &= dG(\rho(\gamma))(\Phi(\vec{\rho}(\gamma))+\Phi(Z))(\varphi\circ\phi(x)). \end{aligned}$$

Note that we used the fact that the Fuchsian isometries fix the point from which the radial direction is defined to write:

$$R(G(\rho(\gamma))(\varphi\circ\phi(x))) = dG(\rho(\gamma))R(\varphi\circ\phi(x)).$$

This proves that a Fuchsian deformation in dS^3 or AdS^3 is sent to a Fuchsian deformation of the Minkowski space. Proving that a Fuchsian deformation of the Minkowski space is sent to a Fuchsian deformation of dS^3 or AdS^3 is exactly the same, as the inverse of the Pogorelov map sends the lateral component to its image by the inverse of the projective map and it sends the radial component to a radial vector having the same norm. □

We must take care that the projective map from de Sitter space doesn't reach all the Fuchsian surfaces in the Minkowski space, because the projective map sends surfaces inside the part of the Minkowski space bounded by the light-cone and the upper branch of the hyperboloid. But we can obtain all the Fuchsian surfaces inside the light-cone by simple homotheties.

3.6. **Another possible proof of part** *ii*) **of Theorem F.** For the anti-de Sitter space, it is easy to see (using the projection onto $\{x_3 = 1\}$ instead of $\{x_4 = 1\}$) that there exists a projective model for which c_{-1} is sent to infinity, O_{-1} is sent to the horizontal disc, and surfaces at constant distance from c_{-1} are sent to half-ellipsoids with boundary the unit circle in the horizontal plane and the future-cone of c_{-1} is a half-cylinder above O_{-1}. It follows that a convex Fuchsian polyhedron lies between two such hyperboloids.

It can occur that one of the bounding half-ellipsoids is contained in the lower-space and the other in the upper-space. But if the Fuchsian polyhedron is convex, it must stay inside one of the two half-spaces delimited by the horizontal plane. Up to an isometry, we consider that it is the upper one (*i.e.* c_{-1} is in the concave side of the Fuchsian polyhedron). It follows that in this model, a convex Fuchsian polyhedron looks like a polyhedral convex cap (with infinite number of vertices accumulating on the boundary), see Figure 6.

Lemma 3.13. *A convex Fuchsian polyhedron in* AdS^3 *lies between* c_{-1} *and* O_{-1}.

Proof. In the projective model described above, the Fuchsian polyhedron lies above the horizontal plane which contains O_{-1}, and below a half-ellipsoid, as c_{-1} is sent to infinity in this projective model and as the distance to c_{-1} can't be 0 by Lemma 2.2. □

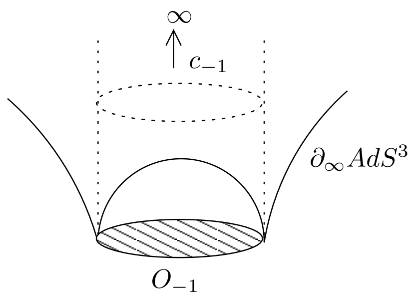

FIGURE 6. Another projective model of the anti-de Sitter space.

It is possible to define an infinitesimal Pogorelov map from the projective map described above — actually, it is exactly as the infinitesimal Pogorelov map that we have already defined, as the coordinates x_3 and x_4 play symmetric parts [Fil06]. Moreover, the polyhedral convex caps in the Minkowski space have the same property of infinitesimal rigidity than in the Euclidean space (Proposition 3.4), due to the following remarkable trick:

Lemma 3.14 ([GPS82],[Sch01, Lemma 3.3]). *A vector field* V *with coordinates* (X, Y, Z) *is a Killing field of the Euclidean space if and only if the vector field* $\overline{V}$ *with coordinates* $(X, Y, -Z)$ *is a Killing field of the Minkowski space.*

Proof. It is obvious:

$$0 = \langle X, dV(X)\rangle = \langle X, JdV(X)\rangle_m = \langle X, d\overline{V}(X)\rangle_m,$$

where $\langle , \rangle$ is the Euclidean scalar product, $\langle , \rangle_m$ the bilinear form of the Minkowski space and $J = \mathrm{diag}(1, 1, -1)$. □

It follows that it would be possible to prove the infinitesimal rigidity of convex Fuchsian polyhedra in the anti-de Sitter space by proving that the image of a Fuchsian deformation by this infinitesimal Pogorelov map has its vertical component going to 0 near the boundary.

4. Realisation of metrics

4.1. **Sets of Fuchsian polyhedra.** We denote by $\mathcal{P}_K(g, n)$ the set of convex Fuchsian polyhedral embeddings of a compact surface S of genus $g > 1$ with n vertices in M_K^-, modulo isotopies of S fixing the vertices of the cellulation and modulo the isometries of M_K^-. More precisely, the equivalence relation given by the isotopies is written as Equation (2), with the difference that $\tilde{h}$ is a lift of a homeomorphism h of S isotopic to the identity, such that if h_t is the isotopy (*i.e.* $t \in [0, 1]$, $h_0 = h$ and $h_1 = id$), then h_t fixes the vertices of the cellulation for all t.

Let (ϕ_1, ρ_1) and (ϕ_2, ρ_2) be two convex Fuchsian polyhedral embeddings describing the same element of $\mathcal{P}_K(g,n)$. As h is homotopic to the identity, $\forall x \in \widetilde{S}, \forall \gamma \in \Gamma$, we check from (2) that $\rho_2(\gamma)(I(\phi_1(x))) = I(\rho_1(\gamma)((\phi_1(x))))$. If two orientation-preserving and time orientation-preserving isometries of M_K^- are equal on an open set of a totally geodesic surface (a face of the Fuchsian polyhedron), then they are equal, it follows that for all $\gamma \in \Gamma$, $\rho_2(\gamma) = I \circ \rho_1(\gamma) \circ I^{-1}$. As ρ_1 and ρ_2 are also representations of Γ in $\mathrm{PSL}(2,\mathbb{R})$, it follows that they describe the same element of the Teichmüller space of S.

Moreover, it also clear from (2) that the projections onto O_K of the vertices of two polyhedral surfaces given by equivalent embeddings are the same (up to a global isometry): there is a natural map $\mathcal{S}_K$ from $\mathcal{P}_K(g,n)$ to $\mathrm{T}_g(n)$, the Teichmüller space of S with n marked points. And as we have seen that from any Fuchsian representation and any n points on the plane, we can build a convex Fuchsian polyhedron with n vertices (it is enough to take n points at the same distance from c_K): $\mathcal{S}_K$ is surjective.

To recall an element of $\mathcal{P}_K(g,n)$ from the data of its projection onto O_K, it remains to know the distance of the vertices from c_K. Such a distance is called the *height* of a vertex.

In the Minkowski and anti-de Sitter spaces, if the vertices of a convex polyhedral surface are all contained inside the future-cone of c_K, then the polyhedral surface is entirely contained inside the future-cone of c_K. It is not true in the de Sitter space: for example in the Klein projective model, some faces can intersect the closed ball: they are time-like or light-like (in case of tangency), but the vertices stay in the future-cone of c_1.

Lemma 4.1. *Take $[h] \in \mathrm{T}_g(n)$. For $K \in \{-1, 0\}$, $\mathcal{S}_K^{-1}([h])$ is diffeomorphic to the open unit ball of $\mathbb{R}^n$.*

Proof. We will prove that $\mathcal{S}_K^{-1}([h])$ is a contractible open subset of $(\mathbb{R}_+)^n$. We fix a fundamental domain for the action of $\rho(\Gamma)$ on O_K, with n marked points, which will be the projection of the vertices of the polyhedra along the geodesics from c_K (*i.e.* we fix $[h]$). We have to find the possible heights of the vertices for the resulting invariant polyhedral surface to be convex. Recall that we have built examples of convex Fuchsian polyhedra for any number of vertices: the spaces considered below are all non-empty.

In the anti-de Sitter space. We consider the projective model of the anti-de Sitter space, where c_{-1} is a point at infinity and O_{-1} is the unit disc in the horizontal plane, as in Figure 6. We have seen that in this model, the convex Fuchsian polyhedra are polyhedral convex caps. In this case we know that the set of possible Euclidean distances from O_{-1} (*i.e.* the horizontal plane) of the n vertices of a fundamental domain is a (non-empty) contractible open subset of $(\mathbb{R}_+)^n$ [Fil, Lemma 9].

If (a,b,z) are the Euclidean coordinates of a vertex, with z the Euclidean distance form the horizontal plane O_{-1}, it is not hard to check that the anti-de Sitter distance from c_{-1} of this vertex is

$$\cot^{-1}\left(\frac{z}{\sqrt{1-a^2-b^2}}\right),$$

where a and b are fixed by hypothesis. The set of heights is diffeomorphic to the set of Euclidean distance from O_{-1}: it is a contractible set.

In the Minkowski space. The proof for the anti-de Sitter space above says that the set of possible anti-de Sitter distances between c_{-1} and the vertices of the polyhedral surfaces for the polyhedral surface to be convex form a contractible set. And this is always true in the Minkowski projective model, for which c_{-1} corresponds to the origin. If t is such an anti-de Sitter distance a direct computation shows that the corresponding Minkowski distance is $\tan(t)$: the set of possible Minkowski heights is also a contractible set.

□

Recall from Subsection 2.6 that to each element of the Teichmüller space T_g is associated a canonical hyperbolic polygon (the Z-V-C coordinates). A (small) open set of $\mathrm{T}_g(n)$ is parametrised by a (small) deformation of a canonical polygon in O_K and a displacement of the marked points inside this polygon. With fixed heights for the vertices, a small displacement of a convex Fuchsian polyhedron (corresponding to a path in $\mathrm{T}_g(n)$), is always convex (the convexity is a property preserved by a little displacement of the vertices) and Fuchsian (by construction).

It follows that we can endow $\mathcal{P}_K(g,n)$ with the topology which makes it a fiber space based on $\mathrm{T}_g(n)$, with fibers homeomorphic to a connected open subset of $\mathbb{R}^n$, and as $\mathrm{T}_g(n)$ is a contractible manifold of dimension $(6g-6+2n)$:

Proposition 4.2. *For $K \in \{-1, 0\}$, the space $\mathcal{P}_K(g,n)$ is a contractible manifold of dimension $(6g-6+3n)$.*

For $K = 1$, $\mathcal{P}_K(g,n)$ is locally a manifold of dimension $(6g-6+3n)$.

Proof. It remains to prove it for the de Sitter case. Actually the description is the same than for the other topologies: $\mathcal{P}_K(g,n)$ can be parametrised by little deformations of the canonical polygon, a little displacement of the marked points inside it and a little variation of their heights. □

Definition 4.3. *A* (generalised) triangulation *of a compact surface S is a decomposition of S by images by homeomorphisms of triangles of the Euclidean space, with possible identification of the edges or the vertices, such that the interiors of the faces (resp. of the edges) are disjoint.*

This definition allows triangulations of the surface with only one or two vertices. For example, take a canonical polygon and take a vertex of this polygon. Join it with the other vertices of the polygon. By identifying the edges of the polygon, we have a triangulation of the resulting surface with only one vertex.

A simple Euler characteristic argument gives that, if e is the number of edges of a triangulation, g the genus of the surface and n the number of vertices:

$$e = 6g - 6 + 3n.$$

Take a subdivision of each faces of a convex Fuchsian polyhedron (P, F) in triangles (such that the resulting triangulation has no more vertices than the cellulation of the polyhedron, and is invariant under the action of F). For the data of such a triangulation on P, we get a map EPol_K which sends each Fuchsian polyhedron lying in a neighbourhood of P in $\mathcal{P}_K(g,n)$ to the square of the lengths of the edges of the triangulation in a fundamental domain for the Fuchsian group action. As this triangulation of P provides a triangulation of the surface S, the map EPol_K has its values in $\mathbb{R}^{6g-g+3n}$.

The map EPol_K associates to the n vertices $x_1, \ldots, x_n$ a set of $(6g - g + 3n)$ real numbers among all the $\mathrm{d}_{M_K^-}(fx_i, gx_j)^2$, $i, j = 1, \ldots, n$, $(f, g) \in \rho(\Gamma)^2$. It is in particular a C^1 map. By the local inverse Theorem, Theorem F says exactly that the map EPol_K is a local homeomorphism around P.

4.2. **Sets of metrics.** By standard methods involving Voronoi regions and Delaunay cellulations, it is known [Thu98, ILTC01, Riv] that for each constant curvature metric with conical singularities on S with constant sign singular curvature there exists a geodesic triangulation such that the vertices of the triangulation are exactly the singular points. This allows us to see such a metric as a gluing of (geodesic) triangles. Actually, we don't need this result, because in the following we could consider only the metrics given by the induced metric on convex Fuchsian polyhedra. In this case, the geodesic triangulation of the metric is given by a triangulation of the faces of the polyhedral surface.

We introduce the following spaces of metrics:

- $\mathcal{M}(g, n)$ the set of Riemannian metrics on a compact surface S of genus g minus n points. It is endowed with the following C^k topology: two metrics are close if their coefficients until those of their kth derivative in any local chart are close (we don't care which $k > 2$);
- $\widetilde{\mathrm{Cone}}_K^-(g, n) \subset \mathcal{M}(n)$ the space of metrics of curvature K on S with n conical singularities of negative singular curvature, seen as Riemannian metrics after removing the singular points;
- $\mathrm{Cone}_K^-(g, n)$ the quotient of $\widetilde{\mathrm{Cone}}_K^-(g, n)$ by the isotopies of S minus n marked points;
- $\widetilde{M}^T$ — where T is a geodesic triangulation of an element of $\widetilde{\mathrm{Cone}}_K^-(g, n)$ — the space of metrics belonging to $\widetilde{\mathrm{Cone}}_K^-(g, n)$ which admit a geodesic triangulation homotopic to T;
- $\mathrm{Conf}(g, n)$ the space of conformal structures on S with n marked points.

We denote by $\widetilde{\mathrm{EMet}}_K$ the map from $\widetilde{M}^T$ to $\mathbb{R}^{6g-6+3n}$ which associates to each element of $\widetilde{M}^T$ the square of the lengths of the edges of the triangulation. The (square of) the distance between two marked points of S is a continuous map from $\mathcal{M}(g, n)$ to $\mathbb{R}$. Around a point of $\widetilde{M}^T$, $\widetilde{\mathrm{EMet}}_K$ takes its values in an open set of $\mathbb{R}^{6g-6+3n}$: if we modify slightly the lengths of the $(6g - 6 + 3n)$ edges, the metric that we will obtain will always be in $\widetilde{M}^T$, because the conditions defining a totally geodesic triangle and the ones on the values of the cone-angles are open conditions.

Lemma 4.4. *The space* $\mathrm{Cone}_{-1}^-(g, n)$ *is a contractible manifold of dimension* $(6g - 6 + 3n)$.

Proof. Theorems of Picard–Mc Owen–Troyanov [McO88][Tro91, Theorem A] say that, if $g > 1$, there is a bijection between $\widetilde{\mathrm{Cone}}_{-1}^-(g, n)$ and $\mathrm{Conf}(g, n) \times A_n$, where A_n is a contractible sets of $\mathbb{R}^n$ given by Gauss–Bonnet conditions (A_n parametrises the values of the cone-angles).

As the Teichmüller space $\mathrm{T}_g(n)$ is the quotient of $\mathrm{Conf}(g, n)$ by the isotopies of S minus its marked points, $\mathrm{Cone}_{-1}^-(g, n)$ is in bijection with $\mathrm{T}_g(n) \times A_n$, and this last space is contractible. With the help of this bijection, we endow $\mathrm{Cone}_{-1}^-(g, n)$ with the topology of $\mathrm{T}_g(n) \times A_n$. □

Lemma 4.5. *The space* $\mathrm{Cone}_0^-(g,n)$ *is a contractible manifold of dimension* $(6g-6+3n)$.

Proof. A Theorem of M. Troyanov [Tro86, Tro91] says that, if $g>1$, there is a bijection between $\widetilde{\mathrm{Cone}_0^-}(g,n)$ up to the homotheties and $\mathrm{Conf}(g,n)\times B_n$, where B_n is a contractible sets of $\mathbb{R}^{n-1}$ given by Gauss–Bonnet conditions (B_n parametrises the values of the cone-angles). We endow $\mathrm{Cone}_0^-(g,n)$ with the topology which makes it a fiber space based on $\mathrm{T}_g(n)\times B_n$ with fiber $\mathbb{R}_+$. □

Lemma 4.6. *For* $K\in\{-1,0\}$, *the map* EMet_K *is a local homeomorphism between* $\mathrm{Cone}_K^-(g,n)$ *and* $\mathbb{R}^{6g-6+3n}$.

Proof. For the topology given by the one of the space of metrics, $\widetilde{\mathrm{EMet}}_K$ is a continuous map on $\widetilde{M}^T\subset\widetilde{\mathrm{Cone}_K^-}(g,n)$. For the cases $K\in\{-1,0\}$, we check that this property is always true for the topologies given by Lemma 4.4 and Lemma 4.5.

Let i_T be the canonical inclusion of $\widetilde{M}^T$ (endowed with the topology induced by the one of $\mathcal{M}(n)$) in $\widetilde{\mathrm{Cone}_{-1}^-}(g,n)$. For the hyperbolic case, the composition of i_T with the projection onto $\mathrm{Conf}(g,n)$ is the map which associates to each metric its conformal structure, this is a continuous map as by definition $\mathrm{Conf}(g,n)$ is the quotient of $\mathcal{M}(g,n)$ by the set of real-values functions on S minus its marked points. Moreover, the composition of i_T with the projection onto A_n is obviously continuous. It follows that i_T is continuous and injective: it is a local homeomorphism.

It is exactly the same for the flat case, because if we fix a point m in $\widetilde{M}^T$, around it $\widetilde{\mathrm{Cone}_0^-}(g,n)$ can be written as $\mathrm{Conf}(g,n)\times B_n\times\mathbb{R}_+$. To conclude it remains to note that, up to the isotopies of the surface, the map $\widetilde{\mathrm{EMet}}_K$ becomes an injective map EMet_K from $M^T\subset\mathrm{Cone}_K^-(g,n)$ (M^T is the quotient of $\widetilde{M}^T$ by the isotopies) to $\mathbb{R}^{6g-6+3n}$. This map takes its values in an open set of $\mathbb{R}^{6g-6+3n}$ and the dimension of $\mathrm{Cone}_K^-(g,n)$ is $(6g-6+3n)$, □

Things are not so simple for the spherical metrics, because if there exists a result of existence of the metrics (under a Gauss–Bonnet condition), the uniqueness is not known [Tro91] — actually, there exists uniqueness results for some particular cases, see [LT92, UY00, Ere04]. For this reason, we can endow $\mathrm{Cone}_1^-(g,n)$ only with the topology given by the one of $\mathcal{M}(n)$. We denote by $\mathrm{Cone}_1^{-,>2\pi}(g,n)$ the subset of $\mathrm{Cone}_1^-(g,n)$ containing the metrics with closed contractible geodesics of lengths $>2\pi$.

Lemma 4.7. *The space* $\mathrm{Cone}_1^{-,>2\pi}(g,n)$ *is locally a manifold of dimension* $(6g-6+3n)$.

Proof. With the help of triangulations, using $\widetilde{\mathrm{EMet}}_1$ as above, we get that $\mathrm{Cone}_1^-(g,n)$ is locally a manifold of dimension $(6g-6+3n)$. Moreover, the condition on the lengths of the closed contractible geodesics is an open condition [RH93, Theorem 6.3, Lemma 9.9]. □

4.3. **Final steps.** We denote by $\mathcal{I}_K(g,n)$ the map "induced metric" between $\mathcal{P}_K(g,n)$ and $\mathrm{Cone}_K^-(g,n)$. Let m be the induced metric on $P\in\mathcal{P}_K(g,n)$. We consider a triangulation of m given by a subdivision of the faces of P in triangles. Obviously, the (square of) the lengths of the edges of the triangulation of P are the same that the (square of) the lengths of the edges of the triangulation of $m := \mathcal{I}_K(g,n)(P)$. It means that locally:

$$\mathrm{EMet}_K \circ \mathcal{I}_K(g,n) \circ \mathrm{EPol}_K^{-1} = id.$$

From this we deduce immediately that $\mathcal{I}_K(g,n)$ is continuous and locally injective. In the next section we will show that it is proper: $\mathcal{I}_K(g,n)$ is a covering map onto its image. We know that $\mathcal{P}_K(g,n)$ and $\mathrm{Cone}_K^-(g,n)$ are connected and simply connected for $K \in \{-1,0\}$. It follows that $\mathcal{I}_K(g,n)$ is a homeomorphism for $K \in \{-1,0\}$.

Let $\mathrm{Mod}(n)$ be the quotient of the group of the homeomorphisms of S minus n points by its subgroup of isotopies. The homeomorphism $\mathcal{I}_K(g,n)$ gives a bijection between the quotient of $\mathcal{P}_K(g,n)$ by $\mathrm{Mod}(n)$ and the quotient of $\mathrm{Cone}_K^-(g,n)$ by $\mathrm{Mod}(n)$ for $K \in \{-1,0\}$, and this is exactly the statement of parts *2)* and *3)* of Theorem A.

Now it remains to prove that $\mathcal{I}_1(g,n)$ is a homeomorphism between $\mathcal{P}_1(g,n)$ and $\mathrm{Cone}_1^{-,>2\pi}(g,n)$. But we don't know anything about the connectedness of $\mathrm{Cone}_1^{-,>2\pi}(g,n)$: the conclusion is less straightforward than for Minkowski or anti-de Sitter spaces. In [Riv86, RH93] there is a result on a kind of "connectedness" for $\mathrm{Cone}_1^{-,>2\pi}(0;n)$, using the connectedness of a space of smooth metrics, and J.-M. Schlenker has noted [Scha] that the genus doesn't intervene in the proof. The only difference is that in our case we must consider the metrics up to isotopies, that changes nothing.

Proposition 4.8. *Each metric* $m_1 \in \mathrm{Cone}_1^{-,>2\pi}(g,n)$ *can be joined to a metric* $m_0 := \mathcal{I}_1(g,n)(P)$*, for a* $P \in \mathcal{P}_1(g,n)$*, by a continuous path* $(m_t)_t$*, with* $m_t \in \mathrm{Cone}_1^{-,>2\pi}(g,N)$*,* $N \geq n$*,* $t \in]0,1[$*, and such that* m_t *is realisable for* t *near* 0.

Sketch of the proof. For a suitable neighbourhood of the cone points of m_0 and m_1, it is possible to (continuously) smooth each cone point [RH93, 9.2] to obtain continuous paths $(\overline{m}_t)_t$, $t \in [0,t_1]$ and $(\overline{m}_{t'})_{t'}$, $t' \in [t_2,1]$, where $\overline{m}_{t_1}$ and $\overline{m}_{t_2}$ are smooth metrics with curvature $K \leq 1$ and lengths of contractible geodesics $L > 2\pi$ (obviously, $\overline{m}_0 = m_0$ and $\overline{m}_1 = m_1$). The space of such metrics is path-connected (that is proved using standard arguments [RH93, Sch96, LS00, Sch06]). It comes that m_0 and m_1 can be joined by a continuous path of (smooth or with conical singularities) metrics such that $K \leq 1$ and $L > 2\pi$.

Now take a geodesic cellulation of m_0 such that the cone points are the vertices, and subdivide each cell with as many (geodesic) triangles as necessary to each triangles to have a diameter strictly less than a given constant δ. We denote by N the number of vertices resulting of such a triangulation T_0. The deformation $(\overline{m}_t)_t$ gives a continuous family T_t of geodesic triangulations, and T_1 is a geodesic triangulation of m_1. Afterward we replace each triangle by a spherical triangle with the same edge length, and this gives us the announced path m_t between m_0 and m_1 (this new path can be taken very close to $(\overline{m}_t)_t$, such that its cone angles remain $> 2\pi$ and the lengths of its closed contractible geodesics remain $> 2\pi$).

It remains to prove that for t sufficiently small, m_t is realisable. The triangulation of m_0 gives a triangulation of P, and each m_t, $t \in [0,\epsilon]$ is obtained by pushing outward each vertex (of the triangulation) contained inside a face of P. The way to push each vertex is given by the change of the length of the edges of the triangulation. This technique is also used in [Ale05]. □

In the next section, we will show that $\mathcal{I}_1(g,n)$ is proper, and thus:

Corollary 4.9. *The map $\mathcal{I}_1(g,n)$ is surjective.*

Proof. With the same notations than above, we already know that m_t is realisable for $t \in [0,\epsilon[$. By properness of $\mathcal{I}_1(g,n)$, m_t is realisable for $t \in [0,\epsilon]$. By local injectivity and the fact that $\mathrm{Cone}_1^{-,>2\pi}(g,n)$ is locally an open manifold, the invariance of domain Theorem gives that the map $\mathcal{I}_1(g,N)$ is open: m_t is realisable for $t \in [0,\epsilon'[$, with $\epsilon' > \epsilon$, and so on. At the end, m_t is realisable for $t \in [0,1[$, and again by properness of $\mathcal{I}_1(g,N)$, m_1 is realisable. □

There is two ways to conclude that $\mathcal{I}_1(g,n)$ is a homeomorphism. The first is short but uses heavy tools:

Proposition 4.10. *The map $\mathcal{I}_1(g,n)$ is injective,* i.e. *convex Fuchsian polyhedra are rigid among convex Fuchsian polyhedra.*

Proof. As we have proved above the part *2)* of Theorem A, it implies that convex Fuchsian polyhedra in the Minkowski space are globally rigid. To conclude it remains to invoke the global Pogorelov map from de Sitter space to Minkowski space (see Subsection 1.9). □

The second way, proposed in [Scha], is more direct. We know that $\mathcal{I}_1(g,N)$ is a covering on the entire $\mathrm{Cone}_1^{-,>2\pi}(g,n)$. To conclude that $\mathcal{I}_1(g,N)$ is a homeomorphism, it remains to check that each fiber contains only one element. This is equivalent to prove that the covering of a loop is a loop, using a kind of "simple connectedness" of $\mathrm{Cone}_1^{-,>2\pi}(g,n)$, and this is given by a straightforward adaptation of Proposition 4.8:

Proposition 4.11. *For each $c : \mathbb{S}^1 \to \mathrm{Cone}_1^{-,>2\pi}(g,n)$ there exists a disc $D \subset \mathrm{Cone}_1^{-,>2\pi}(g,N)$, $N \geq n$, $t \in]0,1[$, such that $\partial D = c(\mathbb{S}^1)$.*

Note that we know now that all the metrics involved in this lemma are realisable.

Sketch of the proof. The proof is step by step the same as for Proposition 4.8, using the fact that the space of smooth metrics with curvature ≤ 1 and lengths of contractible geodesics $> 2\pi$ is simply connected — that is proved using standard arguments [RH93, Sch96, LS00, Sch06]. □

5. Properness

We will use the following characterisation of a proper map: $\mathcal{I}_K(g,n)$ is proper if, for each sequence $(P_k)_k$ in $\mathcal{P}_K(g,n)$ such that the sequence $(g_k)_k$ converges in $\mathrm{Cone}_K^-(g,n)$ (with $g_k := \mathcal{I}_K(g,n)(P_k)$) to $g_\infty \in \mathrm{Cone}_K^-(g,n)$, then $(P_k)_k$ converges in $\mathcal{P}_K(g,n)$ (may be up to the extraction of a sub-sequence). For $K = 1$, we consider $\mathrm{Cone}_1^{-,>2\pi}(g,n)$ instead of $\mathrm{Cone}_1^-(g,n)$.

We denote by d_k the restriction to P_k of the distance from c_K. We always denote by p_K the orthogonal projection onto O_K, and (ϕ_k,ρ_k) is the embedding of the surface S corresponding to P_k. Let denote by γ_k a geodesic on P_k given by an element of the fundamental group of S or a geodesic between two vertices, and by l_k the length of γ_k. By convergence of the sequence of induced metrics, l_k is bounded from above and below for all k. We denote these bounds by $l_{\min} \leq l_k \leq l_{\max}$. Note that this argument will avoid the collapsing of two singular points. We will suppose that the geodesics $\gamma_k(t)$ are parametrised by the arc-length, *i.e.* $g_k(\gamma_k'(t),\gamma_k'(t)) = 1$.

For each M_K^-, we call u_k the restriction of the coordinate function to P_k, that is

$$u_k := \begin{cases} \frac{1}{2}(\mathrm{d}_k)^2, \ K=0; \\ \cos(\mathrm{d}_k), \ K=-1; \\ \cosh(\mathrm{d}_k), \ K=1. \end{cases} \tag{9}$$

Lemma 5.1. *For $K \in \{0,1\}$, for each k, for each geodesic $\gamma(t)$ on P_k, $(u_k \circ \gamma)'$ has a positive jump at its singular points (which correspond to points where $\gamma(t)$ crosses an edge of P_k).*

Proof. Consider an edge e of P_k. We denote by $f_1$and f_2 its adjacent faces, and we look at a geodesic $\gamma(t)$ (for the induced metric) on $f_1 \cup f_2$. We denote by γ_i the part of γ which lies on f_i, and t_0 is such that $\gamma(t_0) \in e$. We denote by $\overline{\gamma}_1$ the prolongation of γ_1 on the plane containing the face f_1. Let d be the distance from c_K. The graph of $d \circ \overline{\gamma}_1$ is smooth, and, until t_0, the graph of $d \circ \gamma$ is also smooth.

As P_k is convex and c_K lies in the concave side of P_k, $d \circ \gamma_2$ is greater than $d \circ \overline{\gamma}$, and therefore the jump of $d \circ \gamma$ at t_0 is positive. As the geodesic lies on P_k, we can write that the jump of $\mathrm{d}_k \circ \gamma$ at t_0 is positive, and as the functions involved in (9) are increasing for $K \in \{0,1\}$, this is true for u_k. □

Lemma 5.2. *For all k, the distance d_k is uniformly bounded from below by a strictly positive constant.*

Proof. [Scha] We see a sequence of (closure of) fundamental domains on P_k for the action of $\rho_k(\Gamma)$ as a sequence $(D_k)_k$ of convex isometric space-like embeddings of the disc, with n singular points. Each D_k must stay out of the light-cone of its vertices, and inside the light-cone of c_K, it follows that if a vertex x_k goes to c_K, then the D_k will be in an arbitrarily neighborhood of a light-cone for k sufficiently large. But this is impossible: a light-cone (without its vertex) is a smooth surface, and it cannot be approximate by polyhedral surfaces with a fixed number of vertices. □

Lemma 5.3. *If the projection of the P_k onto a space-like surface N at constant distance from c_K is a dilating function, then the associated sequence of representations converges.*

Proof. The curvature of the induced metric on N is constant and strictly negative. For each k, $\rho_k(\Gamma)$ acts on N, and the quotient is isometric to a hyperbolic metric (up to a homothety) on the compact surface S. We denote by h_k this hyperbolic metric on S.

By hypothesis, the induced metrics g_k on P_k converge to g_∞. For n sufficiently large, there exists a constant c' such that $g_k \geq \frac{1}{c'} g_\infty$. As the surface is compact, there exists a constant c such that $\frac{1}{c'} g_\infty \geq \frac{1}{c} h_0$. And as the projection is dilating, we have, if $\mathrm{L}_g(\gamma)$ is the length of the geodesic corresponding to $\gamma \in \pi_1(S)$ for the metric g:

$$\mathrm{L}_{h_k}(\gamma) \geq \mathrm{L}_{g_k}(\gamma) \geq \frac{1}{c}\mathrm{L}_{h_0}(\gamma),$$

and Lemma 2.9 leads to the conclusion. □

Lemma 5.4. *Up to extract a subsequence, if the sequence of representation $(\rho_k)_k$ converges and the height of at least one vertex is bounded from above, then the sequence $(P_k)_k = (\phi_k, \rho_k)_k$ converges in $\mathcal{P}_K(g,n)$.*

Proof. It remains to prove that the vertices (*i.e.* $(\phi_k)_k$) converge. As the Fuchsian embeddings are defined up to global isometries and as we suppose that they lie inside the future-cone of c_K, up to compose on the left by a sequence of isometries of the future-cone of c_K, we can consider that there exists a vertex $x_k \in P_k$ which always remain on the same geodesic from c_K. As the representations converge, for k sufficiently large, all the vertices (in a fundamental domain) are contained inside a cone, built with the images of x_k under the action of generators of the fundamental group of the surface.

We can consider that the heights which are bounded from above are those of of x_k. For each k, consider the convex hull C_k of x_k together with the orbits of x_k under the action of the Fuchsian group. By hypothesis, these convex hulls converge to a convex polyhedral surface C. As P_k is convex and as c_K lies in the concave side of P_k, all the vertices of P_k must lie in the same side of C than c_K. It follows that the heights of all the vertices are bounded from above, and also from below by Lemma 5.2. It follows that the vertices (for a fundamental domain) are contained inside a truncated cone, that is a compact domain.

It follows that the sequence $(P_k)_k$ converges to a Fuchsian polyhedron, and this one must be convex with n vertices as the P_k are convex and as sequence of the induced metrics converges. □

5.1. **Properness in the anti-de Sitter space.** In the future-cone of c_{-1}, the anti-de Sitter metric can be written $\sin^2(t)\mathrm{can}_{\mathbb{H}^2} - dt^2$, where t is the distance to c_{-1} and $\mathrm{can}_{\mathbb{H}^2}$ the hyperbolic metric. In the projective model for which c_{-1} is sent to infinity, all the P_k lie above O_{-1} (Lemma 3.13), this means that the projection onto O_{-1} is dilating and by Lemma 5.3, the sequence of representations associated to the P_k converges. Moreover, in this model, the heights of the vertices are bounded, as all the P_k lie below the surface realising the minimum of the distance to c_{-1} and above the horizontal plane. Lemma 5.4 leads to the conclusion.

5.2. **Properness in the de Sitter space.** Almost of this part was done in [Scha].

Lemma 5.5. *The sequence of the representations associated to $(P_k)_k$ converges (up to extract a subsequence).*

Proof. Let γ be an element of the fundamental group of S as in Lemma 2.10. At this γ corresponds a minimising geodesic $\gamma_k(t)$ on P_k between a point $\phi_k(x) \in P_k$ and $\phi_k(\gamma x) \in P_k$. We denote by L_k the length of the projection of $\gamma_k(t)$ onto O_1. If we prove that L_k is bounded from above for all k, then Lemma 2.10 will lead to the conclusion.

We denote by g_k the induced metric on P_k, which can be written:

$$g_k = \sinh^2(\mathrm{d}_k)\mathrm{can}_{\mathbb{H}^2} - d\mathrm{d}_k^2$$

that leads to

$$g_k = (u_k^2 - 1)\mathrm{can}_{\mathbb{H}^2} - \frac{du_k^2}{u_k^2 - 1},$$

it follows that we can compute:

$$\begin{aligned}
L_k &= \int_0^{l_k} \sqrt{\mathrm{can}_{\mathbb{H}^2}(dp_1(\gamma_k'(t)), dp_1(\gamma_k'(t)))}dt \\
&= \int_0^{l_k} \sqrt{\frac{g_k(\gamma_k'(t),\gamma_k'(t))}{u_k^2(t)-1} + \frac{du_k^2(\gamma'(t))}{(u_k^2(t)-1)^2}}dt \\
&= \int_0^{l_k} \sqrt{\frac{1}{u_k^2(t)-1} + \frac{u_k'^2(t)}{(u_k^2(t)-1)^2}}dt \\
&= \int_0^{l_k} \sqrt{\frac{1}{u_k^2(t)-1} + (\mathrm{cotanh}^{-1}(u_k(t))')^2dt} \\
&\leq \int_0^{l_{\max}} \sqrt{\frac{1}{u_k^2(t)-1} + (\mathrm{cotanh}^{-1}(u_k(t))')^2dt} \\
&\leq \int_0^{l_{\max}} \sqrt{\frac{1}{u_k^2(t)-1}}dt + \int_0^{l_{\max}} |(\mathrm{cotanh}^{-1}(u_k(t)))'|dt \\
&\leq \frac{l_{\max}}{\sqrt{u_0^2-1}} + \int_0^{l_{\max}} |(\mathrm{cotanh}^{-1}(u_k(t)))'|dt
\end{aligned}$$

(we have used the fact that u_k is bounded from below by $u_0 > \cosh(0) = 1$, Lemma 5.2).

It remains to check that the variation of $\mathrm{cotanh}^{-1}(u_k)$ over $[0, l_{\max}]$ is bounded from above by a constant which does not depend on k. For this, we can decompose $[0, l_{\max}]$ into a finite number of subsets of the form $[x, y]$, where x is a local minimum (of u_k) and y a local maximum, which immediately follows x in the list of local extrema, and into a finite number of subsets of the form $[y, x]$, where y is a local maximum and x a local minimum, which immediately follows y in the list of local extrema.

First we consider a subset of the kind $[x, y]$, where x is a local minimum and y a local maximum, which immediately follows x in the list of local extrema. We want to study the variation

$$\int_x^y |(\mathrm{cotanh}^{-1}(u_k(s)))'|ds = |\,\mathrm{cotanh}^{-1}(u_k(x)) - \mathrm{cotanh}^{-1}(u_k(y))|. \tag{10}$$

There exists a brutal overestimation which is:

$$|\,\mathrm{cotanh}^{-1}(u_k(x)) - \mathrm{cotanh}^{-1}(u_k(y))| \leq \mathrm{cotanh}^{-1}(u_0),$$

but it is not satisfying: as the number of subsets in the decomposition of $[0, l_{\max}]$ actually depends on k, the bound may become very large together with k. We will use the above bound only in the case $|y - x| \geq \pi/4$, and we will compute another bound in the other case (the term $\pi/4$ has no particular role, the important thing is that the other case verifies $|y - x| < \pi/2$).

As y is a local maximum, together with Lemma 5.1, we have $u_k'(y) = 0$.

We also know that, if f is the restriction to a pseudo-sphere of a linear form (and u_k are such functions) then $\mathrm{Hess}(f) = -ug$, where g is the induced metric on the pseudo-sphere [GHL90, Ex. 2.65,b]. It gives that $u_k'' = -u_k$ on the regular points, and, again by Lemma 5.1, the derivative has a positive jump at the singular points.

With these facts, it is easy to check that, for $s \in [x, y]$,

$$u_k(y) \geq u_k(s) \geq u_k(y)\cos(y_j - s),$$

and with this we compute

$$\begin{aligned} |\operatorname{cotanh}^{-1}(u_k(x)) - \operatorname{cotanh}^{-1}(u_k(y))| &\leq \int_{u_k(x)}^{u_k(y)} \frac{dt}{t^2-1} \\ &\leq \frac{u_k(y) - u_k(x)}{u_k(x)^2 - 1} \\ &\leq \frac{u_k(x)}{u_k(x)^2-1}\left(\frac{1}{\cos(y-x)} - 1\right) \\ &\leq 4(y-x)^2 \frac{u_k(x)}{u_k(x)^2-1} \\ &\leq \frac{4(y-x)^2 u_0}{u_0^2 - 1}. \end{aligned}$$

The bound is the same in the case where a local minimum immediately follows a local maximum, by the symmetry in Equation (10). At the end we have the wanted bound:

$$\int_0^{l_{\max}} |(\operatorname{cotanh}^{-1}(u_k(t)))' dt| \leq \frac{4 l_{\max}(\operatorname{cotanh}^{-1}(u_0) - 1)}{\pi} + \frac{4u_0 l_{\max}^2}{u_0^2 - 1}.$$

□

Lemma 5.6. *The heights of the vertices converge.*

Proof. We want to prove that the height of no vertex of the P_k goes to infinity. If it is, by convexity the face containing such a vertex will become tangent to the sphere. But in this case the sequence of induced metrics will converge to a metric having a geodesic of length 2π [RH93, Fil06], that is impossible as the induced metrics are supposed to converge in $\mathrm{Cone}_1^{-,>2\pi}(g, n)$. □

The conclusion follows from Lemma 5.4.

5.3. **Properness in the Minkowski space.**

Lemma 5.7. *The sequence of the representations associated to $(P_k)_k$ converges (up to extract a subsequence).*

Proof. We will prove it as it had be done for Lemma 5.5. We take back the same notations as given in the first lines of the proof of this lemma.

It is easy to check that the induced metric g_k on P_k can be written:

$$g_k = u_k \mathrm{can}_{\mathbb{H}^2} - \frac{du_k^2}{u_k},$$

it follows that we can compute:

$$\begin{aligned} L_k &= \int_0^{l_k} \sqrt{\mathrm{can}_{\mathbb{H}^2}(dp_0(\gamma_k'(t)), dp_0(\gamma_k'(t)))}dt \\ &= \int_0^{l_k} \sqrt{\frac{g_k(\gamma_k'(t),\gamma_k'(t))}{u_k(t)} + \frac{du_k^2(\gamma_k'(t))}{u_k^2(t)}}dt \\ &= \int_0^{l_k} \sqrt{\frac{1}{u_k(t)} + \left(\frac{u_k'(t)}{u_k(t)}\right)^2} dt \\ &\leq \int_0^{l_{\max}} \sqrt{\frac{1}{u_k(t)} + \left(\frac{u_k'(t)}{u_k(t)}\right)^2} dt \\ &\leq \int_0^{l_{\max}} \frac{1}{\sqrt{u_k(t)}} + \left|\frac{u_k'(t)}{u_k(t)}\right| dt. \end{aligned} \tag{11}$$

As u_k is bounded from below by $u_0 > 0$ (Lemma 5.2), we get

$$L_k \leq \frac{l_{\max}}{\sqrt{u_0}} + \int_0^{l_{\max}} |(\ln(u_k(t))'|dt,$$

and it remains to check that the variation of $\ln(u_k(t))$ over $[0, l_{\max}]$ is bounded from above by a constant which does not depend on k.

We introduce the same decomposition of $[0, l_{\max}]$ than for the de Sitter case: we decompose $[0, l_{\max}]$ into a finite number of subsets of the form $[x, y]$, where x is a local minimum (of u_k) and y a local maximum, which immediately follows x in the list of local extrema, and into a finite number of subsets of the form $[y, x]$, where y is a local maximum and x a local minimum, which immediately follows y in the list of local extrema.

Without loss of generality, we suppose that $u_0 > 1$. First we consider a subset of the kind $[y, x]$, where x is a local minimum and y a local maximum, which immediately follows x in the list of local extrema. We want to study the variation

$$\int_y^x |(\ln(u_k(t)))'|dt = \ln(u_k(y)) - \ln(u_k(x)). \tag{12}$$

By Lemma 5.1, $u_k'(y) = 0$. Furthermore, $u_k'' = -1$ on the regular points (u_k is defined as (half) minus the squared norm, its Hessian is minus the bilinear form). Moreover, u_k' has a positive jump at certain points. From these facts, it is easy to check that, for $s \in [0, x - y]$:

$$u_k(y + s) \geq u_k(y) - \frac{s^2}{2},$$

in particular,

$$u_k(y) - u_k(x) \leq \frac{(x - y)^2}{2}.$$

From this we compute

$$\ln(u_k(y)) - \ln(u_k(x)) \leq \int_{u_k(x)}^{u_k(y)} \frac{dt}{t} \leq \frac{u_k(y) - u_k(x)}{u_k(x)} \leq \frac{(x - y)^2}{2u_0}. \tag{13}$$

The bound is the same in the case where a local maximum immediately follows a local minimum, by the symmetry in Equation (12). It implies that

$$\int_0^{l_{\max}} |(\ln(u_k(t)))'|dt \leq \frac{(l_{\max})^2}{2u_0}.$$

□

Lemma 5.8. *The height of at least one vertex converges.*

Proof. From (11) and (13) we can write:

$$L_k \leq \frac{l_{\max}}{\sqrt{\min_k}} + \frac{(l_{\max})^2}{2\min_k},$$

where $\min_k$ is, for each k, the minimum of the $u_k(x)$, where x is a local minimum for the restriction of u_k to γ_k (it may be not unique). If the height of no vertex is bounded, $\min_k$ will become big when k is large, and L_k will be close to 0. But all the L_k (built for each $\gamma \in \pi_1(S)$) can't be arbitrarily close to 0, because if it is, the area of a fundamental domain on $\mathbb{H}^2$ for the action of the Fuchsian representations will be close to 0, that is impossible by the Gauss–Bonnet Formula. □

The conclusion follows from Lemma 5.4.

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