Actions of dense subgroups of compact groups and II_1-factors with the Haagerup property

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Abstract

Let M be a finite von Neumann algebra with the Haagerup property, and let G be a compact group that acts continuously on M and that preserves some finite trace τ. We prove that if Γ is a countable subgroup of G which has the Haagerup property, then the crossed product algebra $M \rtimes \Gamma$ has also the Haagerup property. In particular, we study some ergodic, non-weakly mixing actions of groups with the Haagerup property on finite, injective von Neumann algebras, and we prove that the associated crossed products von Neumann algebras are II_1-factors with the Haagerup property. If moreover the actions have Property (τ), then the latter factors are full.

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1 Introduction

It is well known that the Haagerup property does not behave well under semidirect product groups [3] or under crossed products von Neumann algebras [14]. Consider then an action α of a countable group Γ on a finite von Neumann algebra M, such that α preserves some finite trace on M. Assume that both Γ and M have the Haagerup property. We look for sufficient conditions on α or Γ which ensure that the crossed product von Neumann algebra $M \rtimes_\alpha \Gamma$ has the same property.

For instance, Theorem 3.2 of [14] provides such conditions: if Γ is the middle term of a short exact sequence $1 \to H \to \Gamma \to Q \to 1$, if Q is amenable and if $M \rtimes_{\alpha|H} H$ has the Haagerup property, then so does $M \rtimes_\alpha \Gamma$.

We propose here another condition:

Theorem 1.1 *Suppose that Γ embeds into a compact group G, and that α is the restriction to Γ of a continuous action of G on M. Then $M \rtimes_\alpha \Gamma$ has the Haagerup property if Γ and M do.*

A typical situation where the above conditions are satisfied is when Γ is maximally almost periodic and G is a compact group containing Γ. Then the action of G on itself by left translations gives a continuous action on $M = L^\infty(G)$ and thus $L^\infty(G) \rtimes \Gamma$ has the Haagerup property if Γ does. Furthermore, taking the closure of Γ if necessary, we can assume that it is dense in G, thus the corresponding crossed product is a type II_1 factor. Observe that such actions are non-weakly mixing.

We give next three families of examples of such factors which are moreover full; the first one is inspired by Chapter 7 in [18], and the second one, by [1]:

Theorem 1.2 *Let $\mathbb{H}$ be the usual Hamiltonian quaternion algebra, let $G = \mathbb{H}^*/Z(\mathbb{H}^*)$ be the corresponding $\mathbb{Q}$-algebraic group and let p be any odd prime number. Embed $\Gamma_p := G(\mathbb{Z}[\frac{1}{p}])$ diagonally into $G(\mathbb{R}) \times G(\mathbb{Q}_p) = SO(3) \times PGL_2(\mathbb{Q}_p)$, and consider the corresponding action α of Γ_p on the 2-dimensional sphere S^2. Then the crossed product algebra $L^\infty(S^2) \rtimes_\alpha \Gamma_p$ is a full II_1-factor with the Haagerup property.*

Theorem 1.3 *Let $\mathbb{G}$ be the $\mathbb{Q}$-algebraic group $SO(n,1)$ or $SU(n,1)$, let p be any prime number and denote by G_p the closure of $\Gamma = \mathbb{G}(\mathbb{Z})$ in $\mathbb{G}(\mathbb{Z}_p)$. Then $L^\infty(G_p) \rtimes \Gamma$ is a full II_1-factor with the Haagerup property.*

Our last class of examples uses diagonal actions of $\Gamma = SL(2,\mathbb{Z})$ on products of quotient groups by some principal congruence subgroups $\Gamma(m)$:

Theorem 1.4 *Let $\mathfrak{m} = (m_i)_{i\geq 1}$ be a sequence of integers $2 \leq m_1 < m_2 < \ldots$ which are pairwise coprime. Let $G_i = \Gamma/\Gamma(m_i)$ and let $G(\mathfrak{m}) = \prod_{i\geq 1} G_i$ be the associated compact group on which Γ acts diagonally. Then $L^\infty(G(\mathfrak{m}))\rtimes\Gamma$ is a full II_1-factor with the Haagerup property.*

In all these theorems, fullness comes from the strong ergodicity of the actions, and, as we will see, it is implied by the fact Γ has Property (τ) with respect to suitable families of subgroups. See for instance [1] and [18].

Finally, we give examples of crossed product factors that are not full but for which central sequences are under control:

Theorem 1.5 *Set again $\Gamma = SL(2,\mathbb{Z})$, let Z_0 denote its center and let Λ be the restricted direct product group $\bigoplus_{j\geq 1} Z_0$. Set $\tilde{\Gamma} = \Gamma \times \Lambda$ and let $Z = Z_0 \times \Lambda$ denote the center of $\tilde{\Gamma}$. Choose a sequence $\mathfrak{m} = (m_j)_{j\geq 1}$ as in Theorem 1.4 and assume that $m_1 \geq 3$. Let $\tilde{\Gamma}$ act on $G(\mathfrak{m}) = \prod_{i\geq 1} G_i$ as follows:*

$$(g, (z_j)_{j\geq 1}) \cdot (x_j)_{j\geq 1} = (g z_j x_j)_{j\geq 1}.$$

Set finally $N = L^\infty(G(\mathfrak{m})) \rtimes \tilde{\Gamma}$. Then N is a type II_1 factor with the Haagerup property and with Property gamma. Moreover, every central sequence in N is equivalent to a central sequence $(c_n)_{n\geq 1}$ contained in the abelian von Neumann subalgebra $L(Z)$.

Our article is organized as follows: the next section contains preliminaries on the Haagerup property, and on Property (τ) and its relationship to strong ergodicity. Section 3 is devoted to the proof of Theorem 1.1 and the last section contains the proofs of Theorems 1.2, 1.3, 1.4 and 1.5.

2 Preliminaries

2.1 Von Neumann algebras and the Haagerup property

Throughout the present article, M, N, denote finite von Neumann algebras with separable preduals, A denotes preferably an abelian von Neumann algebra, and τ denotes some finite, faithful, normal, normalized trace on any of these. Such a state will be called simply a *trace*.

We denote by N_* the predual of N, by $L^2(N,\tau)$ the standard Hilbert space associated with τ and by $\xi_\tau \in L^2(N,\tau)$ the unit vector which implements τ, namely such that $\tau(x) = \langle x\xi_\tau, \xi_\tau\rangle$ for every $x \in N$. We also denote by $\|\cdot\|_{2,\tau}$ the associated Hilbert norm on both N and $L^2(N,\tau)$. If the choice of τ is fixed and that there is no danger of confusion, we simply write $L^2(N)$ and $\|\cdot\|_2$. We also set $L^2(N,\tau)_0 := \{\xi \in L^2(N,\tau)\ ;\ \xi \perp \xi_\tau\}$.

Let $\mathrm{Aut}(M,\tau)$ be the group of all τ-preserving automorphisms of M. It is a Polish group with respect to the topology of pointwise $\|\cdot\|_{2,\tau}$-convergence: a sequence (θ_n) converges to θ if and only if, for all $x \in M$, one has $\|\theta_n(x) - \theta(x)\|_{2,\tau} \to 0$ as $n \to \infty$.

Let Γ be a countable group. The *group von Neumann algebra of* Γ is denoted by $L(\Gamma)$ and it is the commutant of the right regular representation of Γ on $\ell^2(\Gamma)$. Assume Γ acts on N and that the action α preserves some trace τ. We briefly recall the definition and a realization of the corresponding crossed product $M \rtimes_\alpha \Gamma$. We denote by $g \mapsto \lambda_g$ (respectively $g \mapsto \rho_g$) the left (respectively right) regular representation on $\ell^2(\Gamma)$. We also set $\lambda(g) = 1 \otimes \lambda_g$, which is a unitary operator acting on $L^2(M,\tau) \otimes \ell^2(\Gamma)$. Then $N := M \rtimes_\alpha \Gamma$ is the von Neumann algebra generated by $M \cup \{\lambda(g)\ ;\ g \in \Gamma\}$, where $x \in M$ acts on $L^2(M,\tau) \otimes \ell^2(\Gamma)$ as follows:

$$(x \cdot \xi)(g) = \alpha_{g^{-1}}(x)\xi(g)$$

for all $\xi \in L^2(M,\tau) \otimes \ell^2(\Gamma)$, so that $\lambda(g)x\lambda(g^{-1}) = \alpha_g(x)$ for all g and $x \in M$. In this realization, N is a von Neumann subalgebra of $M\bar{\otimes}B$ (where B denotes the algebra of all linear, bounded operators on $\ell^2(\Gamma)$), and more precisely, it is the fixed point algebra under the action θ of Γ defined by $\theta_g = \alpha_g \otimes \mathrm{Ad}(\rho_g)$. We still denote by τ the extended trace on N, and we let E_M denote the τ-preserving conditional expectation of N onto M. Every operator $x \in N$ admits a "Fourier expansion" $\sum_{g\in\Gamma} x(g)\lambda(g)$ such that $x(g) = E_M(x\lambda(g^{-1})) \in M$ for all g and $\sum \|x(g)\|_2^2 = \|x\|_2^2$. If $\alpha : G \to \mathrm{Aut}(M,\tau)$ is a continuous action of some group G on M, we also denote by $g \mapsto \alpha_g$ the corresponding unitary representation of G on $L^2(M,\tau)$ given by

$$\alpha_g(x\xi_\tau) = \alpha_g(x)\xi_\tau \quad \forall x \in M.$$

The restriction to the invariant subspace $L^2(M,\tau)_0 := \{\xi \in L^2(M,\tau)\ ;\ \xi \perp \xi_\tau\}$ is denoted by α^0.

If $\Phi : N \to N$ is a completely positive map such that $\tau \circ \Phi \leq \tau$, then Φ is automatically normal and it extends to a contraction operator T_Φ on $L^2(N,\tau)$ via the equality:

$$T_\Phi(x\xi_\tau) = \Phi(x)\xi_\tau \quad \forall x \in N.$$

We say that Φ is L^2-*compact* if T_Φ is a compact operator. Following [14], we say that N has the *Haagerup property* if there exists a trace τ on N and a sequence $(\Phi_n)_{n\geq 1}$ of completely positive, normal maps on N which satisfy:

(i) $\tau \circ \Phi_n \leq \tau$ and Φ_n is L^2-compact for every n;

(ii) for every $x \in N$, $\|\Phi_n(x) - x\|_{2,\tau} \to 0$ as $n \to \infty$.

In fact, it follows from Proposition 2.2 of [14] and from Corollary 2, p. 39 of [15] that if N satisfies conditions (i) and (ii) above with respect to τ, then each Φ_n can be chosen so that $\Phi_n(1) = 1$, $\tau \circ \Phi_n = \tau$ and T_{Φ_n} is a selfadjoint operator. Moreover, Proposition 2.4 of [14] implies that N satisfies the same conditions with respect to any other trace τ' on M.

The above property was introduced by M. Choda in [6] where it was proved that when N is the group von Neumann algebra $L(\Gamma)$ of some countable group Γ, then $L(\Gamma)$ has the Haagerup property if and only if there exists a sequence $(\varphi_n)_{n\geq 1}$ of positive type, normalized functions on Γ with the following two properties:

(i') for every n, φ_n tends to 0 at infinity of Γ;

(ii') for every $g \in \Gamma$, the sequence $(\varphi_n(g))_{n\geq 1}$ tends to 1 as $n \to \infty$.

See [3] for much more on the Haagerup property for locally compact groups.

Finally, consider a type II_1 factor N. A *central sequence* is a bounded sequence $(x_n)_{n\geq 1} \subset N$ such that, for every $x \in N$,

$$\lim_{n\to\infty} \|x_n x - x x_n\|_2 = 0.$$

Two bounded sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ in N are *equivalent* if

$$\lim_{n\to\infty} \|x_n - y_n\|_2 = 0.$$

The factor N is *full* if every central sequence is *trivial*, i.e. if it is equivalent to the scalar sequence $(\tau(x_n))_{n\geq 1}$, and it has *Property gamma* (of Murray and von Neumann) if it is not full.

2.2 Strong ergodicity and Property (τ)

Recall from [20] that a measure-preserving action of Γ on a probability space (X, μ) is *strongly ergodic* if every sequence (B_n) of Borel subsets of X that satisfies

$$\lim_{n\to\infty} \mu(B_n \triangle gB_n) = 0 \quad \forall g \in \Gamma$$

is trivial, i.e.

$$\lim_{n\to\infty} \mu(B_n)(1 - \mu(B_n)) = 0.$$

It generalizes easily to actions on finite von Neumann algebras [7]: a τ-preserving action α of Γ on the finite von Neumann algebra M is *strongly ergodic* if every operator-norm bounded sequence $(x_n) \subset M$ such that $\|\alpha_g(x_n) - x_n\|_2 \to 0$ as $n \to \infty$ for every $g \in \Gamma$ is equivalent to the scalar sequence $(\tau(x_n))$, in the sense of Subsection 2.1.

In [7], a slightly, but strictly stronger property is considered: if α, Γ and (M, τ) are as above, we say that α is *s-strongly ergodic* if, for every sequence of unit vectors $(\xi_n)_{n\geq 1}$ in $L^2(M)$ that satisfy $\|\alpha_g(\xi_n) - \xi_n\|_2 \to 0$ as $n \to \infty$ for all $g \in \Gamma$, one has $\|\xi_n - \langle \xi_n, \xi_\tau\rangle \xi_\tau\|_2 \to 0$ as $n \to \infty$. Here are characterizations of s-strong ergodicity:

Lemma 2.1 *Let α, Γ and (M, τ) be as above. Then the following conditions on α are equivalent:*

(1) α is s-strongly ergodic;

(2) τ is the unique α-invariant state on M;

(3) there exists $\delta > 0$ and a finite subset F of Γ such that

$$(\star) \quad \delta^2 \|x - \tau(x)\|_2^2 \leq \sum_{g \in F} \|\alpha_g(x) - x\|_2^2 \quad \forall x \in M.$$

Proof. The equivalence between (1) and (2) is Theorem 2 of [5] and it is obvious that (3) implies (1). It remains to prove that (1) implies (3).

Thus suppose that α is s-strongly ergodic and let $1 \in F_1 \subset F_2 \subset \ \ldots \ \Gamma$ be an exhaustive sequence of finite subsets of Γ. If one could not find $\delta > 0$ and F satisfying $(\star)$, there would exist a sequence $(x_n)_{n \geq 1} \in M$ such that $\|x_n\|_2 = 1$, $\tau(x_n) = 0$ and

$$\sum_{g \in F_n} \|\alpha_g(x_n) - x_n\|_2^2 \leq \frac{1}{n^2}$$

for all n. As $\bigcup_n F_n = \Gamma$, $\xi_n = x_n \xi_\tau$ satisfies the condition of s-strong ergodicity, but $\|\xi_n - \langle \xi_n, \xi_\tau \rangle \xi_\tau\|$ does not converge to 0, which is a contradiction. Q.E.D.

Remark. K. Schmidt gives in 2.7 of [20] an example of a strongly ergodic action of the free group $\mathbb{F}_3$ that has more than one invariant state, thus which is not s-strongly ergodic.

The use of s-strongly ergodic actions in the context of crossed products is explained in the next lemma which is adapted from [4]:

Lemma 2.2 *Let M be a finite von Neumann algebra equipped with some trace τ and let α be a τ-preserving, s-strongly ergodic and free action of a countable group Γ on M. Denote by Z the center of Γ and assume that Γ/Z is not inner amenable. Then every central sequence in the crossed product type II_1 factor $N = M \rtimes_\alpha \Gamma$ is equivalent to a central sequence $(c_n)_{n \geq 1}$ contained in $L(Z)$ which satisfies: for every finite subset K of $Z \setminus \{1\}$, one has*

$$\lim_{n \to \infty} \sum_{z \in K} |c_n(z)|^2 = 0.$$

In particular, if Z is finite, then N is a full II_1 factor.

Proof. Let $(x_n)_{n \geq 1} \subset N$ be a central sequence. One can assume that $\|x_n\|_2 = 1$ for every n. Let $\sum_g x_n(g)\lambda(g)$ be the Fourier expansion of x_n. Then we have for every fixed $g \in \Gamma$:

$$\begin{aligned}
\sum_{h \in \Gamma} |\|x_n(ghg^{-1})\|_2 - \|x_n(h)\|_2|^2 &\leq \sum_{h \in \Gamma} \|x_n(ghg^{-1}) - \lambda(g)x_n(h)\lambda(g^{-1})\|_2^2 \\
&= \sum_{h \in \Gamma} \|\alpha_{g^{-1}}(x_n(ghg^{-1})) - x_n(h)\|_2^2 \\
&= \|x_n\lambda(g) - \lambda(g)x_n\|_2^2 \to_{n \to \infty} 0.
\end{aligned}$$

Since Γ/Z is not inner amenable, one has

$$\sum_{g \notin Z} \|x_n(g)\|_2^2 \to_{n \to \infty} 0.$$

This implies that (x_n) is equivalent to its projection $(E_{M\rtimes Z}(x_n))$ onto the von Neumann subalgebra $M \rtimes Z$. One assumes then that $x_n(g) = 0$ for every $g \notin Z$. Set $Z^* = Z \setminus \{1\}$ and let $\delta > 0$ and the finite set $F \subset \Gamma$ be as in Lemma 2.1. Then

$$\begin{aligned}\sum_{g\in F} \|\lambda(g)x_n\lambda(g^{-1}) - x_n\|_2^2 &= \sum_{z\in Z}\sum_{g\in F} \|\alpha_g(x_n(z)) - x_n(z)\|_2^2 \\ &\geq \delta^2 \sum_{z\in Z} \|x_n(z) - \tau(x_n(z))\|_2^2.\end{aligned}$$

Thus, (x_n) is equivalent to $(c_n) \subset L(Z)$ where $c_n = E_{L(Z)}(x_n)$ and hence $c_n(z) = \tau(x_n(z))$ for all n and z.

Fix next a non empty finite subset K of Z^* and set $k = |K|$. For each $z \in K$, α_z is a properly outer automorphism, hence there exists a non zero projection $e_z \in M$ such that

$$(\star) \quad \tau(e_z\alpha_z(e_z)) \leq \frac{1}{2}\tau(e_z).$$

Indeed, by Theorem 1.2.1 of [8], one takes a non zero projection e_z such that $\|e_z\alpha_z(e_z)\| \leq \frac{1}{2}$, and, as $e_z\alpha_z(e_z)e_z \leq \frac{1}{2}e_z$, we get $(\star)$. This implies that $\|e_z - \alpha_z(e_z)\|_2^2 = 2\tau(e_z) - 2\tau(e_z\alpha_z(e_z)) \geq \tau(e_z) > 0$ for every $z \in K$. Set $c = \min\{\tau(e_z)\ ;\ z \in K\} > 0$ and choose a finite subset T of $\Gamma \setminus Z$ of cardinality k such that $tZ \cap t'Z = \emptyset$ for all $t, t' \in T$, $t \neq t'$. This is possible since Γ/Z is infinite (being non inner amenable). Finally, choose some bijection $z \mapsto t_z$ from K onto T, set $x(t_z) = e_z$ for every $z \in K$ and put $x = \sum_{t\in T} x(t)\lambda(t) \in N$. Observe that, for every $z \in K$, one has

$$\sum_{t\in T} \|x(t) - \alpha_z(x(t))\|_2^2 \geq \|e_z - \alpha_z(e_z)\|_2^2 \geq c.$$

We get then for every n:

$$\begin{aligned}\|xc_n - c_nx\|_2^2 &= \sum_{z\in Z^*} |c_n(z)|^2 \sum_{t\in T} \|x(t) - \alpha_z(x(t))\|_2^2 \\ &\geq c\sum_{z\in K} |c_n(z)|^2.\end{aligned}$$

This proves that $\sum_{z\in K} |c_n(z)|^2 \to 0$ as $n \to \infty$. Q.E.D.

We describe next how to get s-strongly ergodic actions. Let Γ be a countable group and let $\mathcal{L} = (\Gamma_\iota)_{\iota\in I}$ be a family of normal subgroups of Γ, each Γ_ι having finite index in Γ. Denote by $R(\mathcal{L})$ the family of all irreducible unitary representations $(\rho, \mathcal{H}_\rho)$ for which there exists $\iota \in I$ such that $\Gamma_\iota \subset \ker(\rho)$. In other words, $R(\mathcal{L})$ is the subset of the unitary dual $\hat{\Gamma}$ formed by representations that factor through some finite quotient group Γ/Γ_ι. Let us recall Definition 4.3.1 of [18]:

Definition 2.3 *Let Γ and $\mathcal{L}$ be as above. We say that Γ has Property (τ) with respect to the family $\mathcal{L}$ if the trivial representation 1_Γ is isolated in $R(\mathcal{L})$.*

This means that one can find a positive number ε and a finite subset F of Γ such that

$$W(\varepsilon, F) \cap R(\mathcal{L}) = \{1_\Gamma\},$$

where $W(\varepsilon, F)$ is the set of $(\rho, \mathcal{H}_\rho) \in \hat{\Gamma}$ for which there exists a unit vector $\xi \in \mathcal{H}_\rho$ such that

$$\max_{g\in F} \|\rho(g)\xi - \xi\| \leq \varepsilon.$$

We will use Property (τ) in two distinct situations.

In the first one, we consider a countable subset $\mathcal{L}' = (\Gamma_i)_{i\geq 1}$ of $\mathcal{L}$. Set $X_{\mathcal{L}'} = \prod_{i\geq 1} \Gamma/\Gamma_i$ gifted with its natural probability measure μ, and with the diagonal action of Γ:

$$g \cdot (g_i\Gamma_i)_{i\geq 1} = ((gg_i)\Gamma_i)_{i\geq 1}.$$

Set also $A = A(\mathcal{L}') = L^\infty(X_{\mathcal{L}'}, \mu)$ and denote by α the corresponding action on $A(\mathcal{L}')$. Integration with respect to μ defines a trace τ on $A(\mathcal{L}')$.

Then Property (τ) interprets in terms of the action α as follows:

Lemma 2.4 *Let Γ, $\mathcal{L}$, $\mathcal{L}'$, $A(\mathcal{L}')$, μ and τ be as above. Assume moreover that:*

(a) Γ has Property (τ) with respect to $\mathcal{L}$;

(b) for every finite subset $\{\Gamma_{m_1}, \ldots, \Gamma_{m_n}\}$ of $\mathcal{L}'$ there exists $\iota \in I$ such that

$$\Gamma_\iota \subset \bigcap_{j=1}^{n} \Gamma_{m_j};$$

(c) the action of Γ on $\prod_{i\geq 1} \Gamma/\Gamma_i$ is ergodic.

Then α is s-strongly ergodic.

Proof. Since the families $\mathcal{L}$ and $\mathcal{L}'$ are fixed, we drop the corresponding subscripts everywhere. We prove that there exist $\delta > 0$ and $F \subset \Gamma$ that satisfy ($\star$) in Lemma 2.3.

Condition (a) implies existence of $0 < \varepsilon < 1/2$ and F such that $W(\varepsilon, F) \cap R = \{1_\Gamma\}$. Set $\delta = \varepsilon/2$. It suffices to see that $\sum_{g\in F} \|\alpha_g(a) - a\|_2^2 \geq \delta^2$ for every $a \in A$ such that $\|a\|_2 = 1$ and $\tau(a) = 0$. Suppose the contrary. There exists then $a \in A$, $\|a\|_2 = 1$ and $\tau(a) = 0$ such that

$$\sum_{g\in F} \|\alpha_g(a) - a\|_2^2 < \delta^2.$$

For $n \geq 1$, set $A_n = L^2(X_1 \times \ldots \times X_n) = L^\infty(X_1 \times \ldots \times X_n) \subset A$, where $X_i = \Gamma/\Gamma_i$, and let E_n be the trace preserving conditional expectation of A onto A_n. If n is large enough so that $\|a - E_n(a)\|_2 < \delta$, then we still have $\tau(E_n(a)) = 0$ and $\|E_n(a)\|_2 \geq 1 - \delta$. Set

$$b = \frac{E_n(a)}{\|E_n(a)\|_2} \in A_n,$$

which satisfies $\|b\|_2 = 1$ and $\tau(b) = 0$. Then, denoting by $\alpha^{(n)}$ the restriction of α to the Γ-invariant subalgebra A_n, one has:

$$\begin{aligned}\sum_{g\in F} \|\alpha_g^{(n)}(b) - b\|_2^2 &= \frac{1}{\|E_n(a)\|_2^2} \sum_{g\in F} \|E_n(\alpha_g(a) - a)\|_2^2 \\ &\leq \frac{\delta^2}{(1-\delta)^2} < \varepsilon^2.\end{aligned}$$

There exists $\iota \in I$ such that $\Gamma_\iota \subset \Gamma_i$ for every $i \leq n$. As all the Γ_i's are normal subgroups of Γ, it follows that

$$\alpha_g^{(n)}(b)(x_1, \ldots, x_n) = b(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$ and all $g \in \Gamma_\iota$. Hence $\Gamma_\iota \subset \ker(\alpha^{(n)0})$, where the latter representation is the restriction of $\alpha^{(n)}$ to $L^2(X_1 \times \ldots \times X_n)_0$. Thus, there exists an irreducible subrepresentation ρ of $\alpha^{(n)0}$ which belongs to $W(\varepsilon, F) \cap R$. However, ρ cannot be the trivial representation since $\alpha^{(n)0}$ does not contain 1_Γ because the action of Γ is ergodic. This is a contradiction. Q.E.D.

The second situation is inspired by [1]. Suppose that $\mathcal{L} = (\Gamma_n)_{n\geq 1}$ is a decreasing sequence of finite index normal subgroups of Γ such that $\bigcap_n \Gamma_n = \{1\}$. Let $\Gamma_c = \operatorname{proj\,lim} \Gamma/\Gamma_n$ be the projective limit of the sequence of finite groups (Γ/Γ_n) with respect to the natural projections $\Gamma/\Gamma_{n+1} \to \Gamma/\Gamma_n$. Then Γ_c is a compact group containing Γ as a dense subgroup. In fact, Γ_c is the completion of Γ in the topology for which the Γ_n's form a base of neighbourhoods of 1. We denote again by α the action of Γ on $L^\infty(\Gamma_c)$ by left translation.

Lemma 2.5 *Let Γ, $\mathcal{L} = (\Gamma_n)$ and Γ_c be as above. If Γ has Property (τ) with respect to $\mathcal{L}$, then the action of Γ on $L^\infty(\Gamma_c)$ is s-strongly ergodic.*

Proof. For every n, set $A_n = L^\infty(\Gamma/\Gamma_n)$, so that A_n is a von Neumann subalgebra of A and that $\bigcup_n A_n$ is $\|\cdot\|_2$-dense in A. Let also $\alpha^{(n)}$ (respectively $\alpha^{(n)0}$) be the restriction of the action α to A_n (respectively to $\{a \in A_n \ ; \ \tau(a) = 0\}$).

Let $0 < \varepsilon < \frac{1}{2}$ and $F \subset \Gamma$ finite be such that $W(\varepsilon, F) \cap R(\mathcal{L}) = \{1_\Gamma\}$. Put $\delta = \varepsilon/2$. As in the proof of Lemma 2.5, assume by contradiction that there exists $a \in A := L^\infty(\Gamma_c)$ such that $\|a\|_2 = 1$, $\tau(a) = 0$ and

$$\sum_{g\in F} \|\alpha_g(a) - a\|_2^2 < \delta^2.$$

By the same arguments, there exists n and $b \in A_n$ with $\|b\|_2 = 1$, $\tau(b) = 0$ and such that

$$\sum_{g\in F} \|\alpha_g^{(n)}(b) - b\|_2^2 < \varepsilon^2.$$

Hence one can find an irreducible subrepresentation σ of $\alpha^{(n)0}$ such that $\sigma \in W(\varepsilon, F)$. Since $\Gamma_n = \ker(\alpha^{(n)}) \subset \ker(\sigma)$, we have $\sigma = 1_\Gamma$, but this contradicts the ergodicity of $\alpha^{(n)}$. Q.E.D.

3 Actions of compact groups

Let G be a compact group and let α be a (continuous) action of G on a von Neumann algebra M. If the action is ergodic, it follows from Corollary 4.2 of [13] that M is necessarily finite and injective, and that there is a unique G-invariant state on M that is a trace. Even if all our examples deal with ergodic actions, we state our main theorem for actions that are not necessarily ergodic.

Theorem 3.1 *Let α be a continuous action of a compact group G on a finite von Neumann algebra M that preserves some trace τ. Assume that G contains a countable subgroup Γ which has the Haagerup property and that M has the same property. Then the corresponding crossed product von Neumann algebra $N = M \rtimes_\alpha \Gamma$ has also the Haagerup property.*

Proof of Theorem 2.1 follows readily from the following two lemmas:

Lemma 3.2 *Retain hypotheses and notations above. There exists a sequence $(\Psi_m)_{m\geq 1}$ of completely positive, unital, normal, τ-preserving maps on M with the following properties:*

(i) Ψ_m *is L^2-compact for every m;*

(ii) $\alpha_g \circ \Psi_m = \Psi_m \circ \alpha_g$ *for all $g \in G$ and all m;*

(iii) for every $x \in M$, one has $\|\Psi_m(x) - x\|_2 \to 0$ as $m \to \infty$.

Proof. Choose a sequence $(\Phi_m)_{m\geq 1}$ of completely positive, unital, τ-preserving maps on M such that the corresponding operators T_{Φ_m} are all compact and selfadjoint, and such that, for every $x \in M$, $\|\Phi_m(x) - x\|_2 \to 0$ as $m \to \infty$. Define Ψ_m by:

$$\Psi_m(x) = \int_G \alpha_g \circ \Phi_m \circ \alpha_{g^{-1}}(x)dg \quad \forall x \in M.$$

Notice that the integral is defined in the weak sense: this means that, for $x \in M$, $\Psi_m(x)$ is the element of M characterized by

$$\varphi(\Psi_m(x)) = \int_G \varphi(\alpha_g \circ \Phi_m \circ \alpha_{g^{-1}}(x))dg \quad \forall \varphi \in M_*.$$

Each Ψ_m is a completely positive, unital, τ-preserving map on M, such that $\alpha_g \circ \Psi_m = \Psi_m \circ \alpha_g$ for every $g \in G$. This proves (ii) and the first properties of the sequence $(\Psi_m)_m$.

Let us prove that each Ψ_m is L^2-compact. As m is fixed for the moment, put $T = T_{\Psi_m}$ and $S = T_{\Phi_m}$. Since S is a selfadjoint, compact operator, there exists a sequence $(S_k)_{k\geq 1}$ of finite-rank, selfadjoint operators on $L^2(M)$ such that $\|S - S_k\| \to 0$, and $\|S_k\| \leq 1\ \forall k$ because S itself is a contraction. Moreover, one has $\|T - \int_G \alpha_g S_k \alpha_{g^{-1}} dg\| \leq \|S - S_k\|$ for every k. Thus, it remains to check that $S_{k,G} := \int_G \alpha_g S_k \alpha_{g^{-1}} dg$ is a compact operator. Let $\mathcal{B}$ denote the unit ball of $L^2(M)$. Since G is compact and S_k is a finite-rank operator, the set $\Omega_k := \{\alpha_g S_k \alpha_{g^{-1}}(\mathcal{B})\ ;\ g \in G\}$ is relatively compact. Finally, the image of $\mathcal{B}$ under $S_{k,G}$ is contained in the closed convex circled hull of Ω_k which is compact.
This proves claim (i).

It remains to prove statement (iii). As the linear span of the set of projections in M is norm-dense, it suffices to prove it for projections. Thus, fix a projection $f \in M$ and $\varepsilon > 0$. One has:

$$\begin{aligned}
\|\Psi_m(f) - f\|_2^2 &= \|\Psi_m(f)\|_2^2 + \|f\|_2^2 - 2\mathrm{Re}\tau(\Psi_m(f)f) \\
&\leq 2\|f\|_2^2 - 2\mathrm{Re}\tau(\Psi_m(f)f) \\
&= 2\left(\tau(f) - \tau(\Psi_m(f)f)\right).
\end{aligned}$$

As G is a compact group, one can find a finite set $F \subset G$ such that, for every $g \in G$, there exists $h = h(g) \in F$ that satisfies:

$$\|\alpha_g(f) - \alpha_h(f)\|_2 \leq \frac{\varepsilon^2}{6}.$$

Furthermore, since Φ_m tends to the identity map on M in the pointwise $\|\cdot\|_2$-topology, there exists an integer n such that

$$\|\Phi_m(\alpha_h(f)) - \alpha_h(f)\|_2 \leq \frac{\varepsilon^2}{6} \quad \forall h \in F \quad \text{and} \quad \forall m \geq n.$$

This implies that

$$\sup_{g\in G}\|\Phi_m(\alpha_g(f))\alpha_g(f)-\alpha_g(f)\|_2\leq\frac{\varepsilon^2}{2}\quad\forall m\geq n.$$

Indeed, if $m\geq n$ and if $g\in G$, let $h=h(g)\in F$ be as above. One has:

$$\begin{aligned}\|\Phi_m(\alpha_g(f))\alpha_g(f)-\alpha_g(f)\|_2 &\leq \|\Phi_m(\alpha_g(f)-\alpha_h(f))\alpha_g(f)\|_2+\\ &\quad\|\Phi_m(\alpha_h(f))\alpha_g(f)-\alpha_h(f)\alpha_g(f)\|_2+\\ &\quad\|\alpha_h(f)-\alpha_g(f)\|_2\\ &\leq \frac{\varepsilon^2}{2}.\end{aligned}$$

Since

$$\tau(\Psi_m(f)f)=\int_G\tau(\Phi_m(\alpha_g(f))\alpha_g(f))dg,$$

we get for $m\geq n$:

$$\begin{aligned}|\tau(f)-\tau(\Psi_m(f)f)| &= |\int_G\tau[\alpha_g(f)-\Phi_m(\alpha_g(f))\alpha_g(f)]dg|\\ &\leq \int_G|\tau[\alpha_g(f)-\Phi_m(\alpha_g(f))\alpha_g(f)]|dg\leq\frac{\varepsilon^2}{2}.\end{aligned}$$

This proves that $\|\Psi_m(f)-f\|_2\leq\varepsilon$ for all $m\geq n$. Q.E.D.

The next lemma is Theorem 3 in [4], but we sketch the proof for the reader's convenience.

Lemma 3.3 *Let M be a finite von Neumann algebra gifted with a normal, faithful, normalized trace τ and let α be a τ-preserving action of a countable group Γ. Assume that:*

(i) Γ has the Haagerup Property;

(ii) There exists a sequence $(\Psi_m)_{m\geq1}$ of τ-preserving, completely positive, unital, L^2-compact maps on M such that $\alpha_g\circ\Psi_m=\Psi_m\circ\alpha_g$ for all m and $g\in\Gamma$, and that $\|\Psi_m(x)-x\|_2\to0$ as $m\to\infty$, for every $x\in M$.

Then the crossed product von Neumann algebra $M\rtimes\Gamma$ has the Haagerup Property.

Outline of the proof. On the one hand, let $(\varphi_n)_{n\geq1}$ be a sequence of positive definite, normalized functions on Γ as in the definition of the Haagerup property for groups. Denote by Φ_n the completely positive multiplier on the von Neumann algebra $L(\Gamma)$ associated to φ_n. By [12], it extends to $M\rtimes\Gamma$ a completely positive map still denoted by Φ_n in such a way that

$$\Phi_n(x\lambda(g))=x\varphi_n(g)\lambda(g)\quad\forall x\in M,\ g\in\Gamma.$$

On the other hand, the restriction Ψ'_m to $M\rtimes\Gamma$ of the completely positive map $\Psi_m\otimes i_B$ on $M\bar{\otimes}B$ has range contained in $M\rtimes\Gamma$ because $\theta_g\circ\Psi_m\otimes i_B=\Psi_m\otimes i_B\circ\theta_g$ for every g, and $M\rtimes\Gamma=(M\bar{\otimes}B)^\theta$. It is straightforward to check that the sequence $(\Psi'_n\circ\Phi_n)$ satisfies then all required properties to ensure that $M\rtimes\Gamma$ has the Haagerup property. Q.E.D.

Remark. Let Γ be a countable group with the Haagerup property and let B be any finite von Neumann algebra with the Haagerup property of dimension at least 2 gifted with some trace τ_B. Consider the infinite tensor product algebra $M = \bigotimes_{g\in\Gamma}(B, \tau_B)$ on which Γ acts by Bernoulli shifts: $\beta_g(\otimes_h x_h) = \otimes_h x_{g^{-1}h}$. M has the Haagerup property, but we don't know whether the corresponding crossed product $M \rtimes_\beta \Gamma$ has the Haagerup property except if Γ is amenable. However, M. Choda claims on p. 88 of [7] that $M \rtimes_\beta \Gamma$ has that property, using Lemma 3.3, but there is a gap in her proof. Indeed, she constructs, from completely positive, trace-preserving, unital L^2-compact maps Φ on B, infinite tensor product maps $\tilde{\Phi} := \otimes_g \Phi_g$, where $\Phi_g = \Phi\ \forall g$. Such maps $\tilde{\Phi}$ make perfectly sense, are completely positive, unital, trace-preserving, but they are not L^2-compact in general so Lemma 3.3 cannot be applied.

Corollary 3.4 *Let M be a finite von Neumann algebra, let τ be a trace on M and let α be a τ-preserving action of a group Γ on M. Assume that both Γ and M have the Haagerup property and that the range of Γ through α is relatively compact in* $\mathrm{Aut}(M, \tau)$. *Then the crossed product $M \rtimes_\alpha \Gamma$ has the Haagerup property.*

Corollary 3.5 *Let Γ be a maximally almost periodic group with the Haagerup property and let G be a compact group such that Γ embeds into G. Then the crossed product algebra $L^\infty(G) \rtimes \Gamma$ has the Haagerup property.*

Corollary 3.6 *Let Γ and G be as in Corollary 3.5. Assume furthermore that Γ is dense in G and that the latter acts freely and ergodically on a standard probability space (X, μ), and that its action preserves μ. Then the crossed product $L^\infty(X) \rtimes \Gamma$ is a type II_1 factor with the Haagerup property.*

As is well known, the free group $\mathbb{F}_2$ embeds into $SU(2)$, into $SO(3)$ and also into $SO(n+1)$ for all odd $n \geq 3$ (see [10]). These instances provide examples of crossed products with the Haagerup property. In the final section we give examples of such factors that are full, and of factors whose central sequences are under control.

4 Examples

Let Γ be a countable group with the Haagerup property which is embeddable into a compact group G. Thus, for our purposes, we assume that it is dense in G. Consider the action α of Γ by translation on G. It is ergodic and free, so that the associated crossed product $L^\infty(G) \rtimes_\alpha \Gamma$ is a type II_1 factor. It follows from Theorem 3.1 that it has the Haagerup property. We present here three families of examples of pairs (Γ, G) for which the corresponding factor $L^\infty(G) \rtimes_\alpha \Gamma$ is a full factor, and one family of factors that have Property gamma with some control on central sequences.

Here is the first family of examples which is inspired by Chapter 7 of [18]: Let $\mathbb{D} = \mathbb{D}(u, v)$ be a definite quaternion algebra defined over $\mathbb{Q}$: for a ring R, $\mathbb{D}(R) = \{x_0 + x_1 i + x_2 j + x_3 k\ ;\ x_\ell \in R\}$ where u and v are rational numbers and $i^2 = -u$, $j^2 = -v$, $k^2 = -uv$, $ij = -ji = k$. (When $u, v > 0$, for example, we get the standard Hamiltonian quaternion algebra.) Let $G = \mathbb{D}^*/Z(\mathbb{D}^*)$ be the $\mathbb{Q}$-algebraic group of invertible elements of $\mathbb{D}$ modulo the central ones, and let p be some prime number for which $\mathbb{D}$ splits in $\mathbb{Q}_p$ (e.g. p can be any odd prime in the case of Hamiltonian quaternions). Set $\Gamma_p = G(\mathbb{Z}[\frac{1}{p}])$ and embed it diagonally into $G(\mathbb{R}) \times G(\mathbb{Q}_p) = SO(3) \times PGL_2(\mathbb{Q}_p)$. Then the projection of Γ_p to $SO(3)$ gives an embedding to a dense subgroup of $SO(3)$. As the latter acts on the 2-sphere S^2, then so does Γ_p. One has then:

Theorem 4.1 *With the assumptions above, the crossed product algebras* $L^\infty(SO(3)) \rtimes \Gamma_p$ *and* $L^\infty(S^2) \rtimes \Gamma_p$ *are full type* II_1 *factors with the Haagerup property.*

Proof. The fact that the factors have both the Haagerup property follows from Corollaries 3.5 and 3.6. Furthermore, by Section 7.2 of [18], the natural Lebesgue measures on $L^\infty(SO(3))$ and on $L^\infty(S^2)$ are the unique Γ_p-invariant means on these algebras. In order to apply Lemmas 2.1 and 2.2, we check that Γ_p is not inner amenable. But the projection Γ_2 of Γ_p into $PGL(2, \mathbb{Q}_p)$ is faithful and it is a lattice. Corollary of Proposition 2 of [9] implies that Γ_2, hence Γ_p, is not inner amenable. In particular, its center is trivial and the associated factors are full. Q.E.D.

Our second family of examples is inspired by [1]:

Theorem 4.2 *Let* $\mathbb{G}$ *be the* $\mathbb{Q}$*-algebraic group* $SO(n, 1)$ *or* $SU(n, 1)$*, let* p *be any prime number and denote by* G_p *the closure of* $\Gamma = \mathbb{G}(\mathbb{Z})$ *in* $\mathbb{G}(\mathbb{Z}_p)$*. Then* $L^\infty(G_p) \rtimes \mathbb{G}(\mathbb{Z})$ *is a full* II_1*-factor with the Haagerup property.*

Proof. Set $\Gamma = \mathbb{G}(\mathbb{Z})$. It is a lattice in a Lie group with the Haagerup property, thus it has the same property by Theorem 4.0.1 and Proposition 6.1.5 of [3]. It follows from Theorem 3.1 that $L^\infty(G_p) \rtimes \Gamma$ is a II_1-factor with the Haagerup property. It remains to prove that it is a full factor. Notice first that the center of Γ is finite and it follows from [9] that $\Gamma/Z(\Gamma)$ is not inner amenable.

Next, for $n \geq 1$, let $\Gamma(n)$ denote the principal congruence subgroup

$$\Gamma(n) = \{g \in \Gamma \ ; \ g \equiv I \pmod{n}\}.$$

As is explained on page 509 of [1], Γ has Property (τ) with respect to the family $\mathcal{L} = (\Gamma(n))_{n\geq 1}$: this follows from [11], [17], [2] when $\mathbb{G}(\mathbb{R})$ is isomorphic to $SO(n, 1)$ and from [16] when it is isomorphic to $SU(n, 1)$. In particular, it also has that property with respect to the subfamilly $(\Gamma(p^n))_{n\geq 1}$, and G_p equals the projective limit $\Gamma_c = \operatorname{proj\,lim} \Gamma/\Gamma(p^n)$. Lemmas 2.2 and 2.5 imply that the associated factor is full. Q.E.D.

Remark. Generally, G_p is different from $\mathbb{G}(\mathbb{Z}_p)$, but it is always a finite index subgroup. This follows from Lemma 2 of [1]. Notice that, however, $G_p = SL(2, \mathbb{Z}_p)$ in the case $\Gamma = SL(2, \mathbb{Z})$.

Our last two classes of examples involve $\Gamma := SL(2, \mathbb{Z})$ and some of its subgroups. As in the proof of the above theorem, for $n \geq 1$, denote by $\Gamma(n)$ the principal congruence subgroup of Γ. Explicitely, $\Gamma(n)$ is the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $a \equiv d \equiv 1 \pmod{n}$ and $b \equiv c \equiv 0 \pmod{n}$. It is the kernel of the natural homomorphism from Γ onto $SL(2, \mathbb{Z}/n\mathbb{Z})$ (see [19], 4.2, for instance). In particular, $\Gamma(1) = \Gamma$.

Let $\mathfrak{m} = (m_i)_{i\geq 1}$ be a sequence of integers such that $2 \leq m_1 < m_2 < \ldots$ and that $(m_i, m_j) = 1$ for all $i \neq j$, and let $G_j = \Gamma/\Gamma(m_j)$ be gifted with the natural action of Γ. Γ embeds into the compact group $G(\mathfrak{m}) := \prod_{j\geq 1} G_j$ via the mapping $g \mapsto (g\Gamma(m_j))_{j\geq 1}$, because $\bigcap_j \Gamma(m_j) = \{I\}$.

Lemma 4.3 *Let* $\mathfrak{m}$ *be as above. Then, for every integer* $m_0 \geq 1$ *such that* $(m_0, m_j) = 1$ *for every* $j \geq 1$*, the diagonal action of* $\Gamma(m_0)$ *on* $G_1 \times \ldots \times G_n$ *is transitive for every* $n \geq 1$*. In particular,* $\Gamma(m_0)$ *embeds as a dense subgroup into* $G(\mathfrak{m})$ *and its action on* $L^\infty(G(\mathfrak{m}))$ *is ergodic and free.*

Proof. Fix $n \geq 1$; it suffices to prove that the orbit of $(\bar{1}, \ldots, \bar{1}) \in \prod_{j=1}^n G_j$ under the diagonal action of $\Gamma(m_0)$ equals $\prod_{j=1}^n G_j$. Let then $g_1, \ldots, g_n \in \Gamma$, and let us prove that there exists $g \in \Gamma(m_0)$ such that $g\Gamma(m_j) = g_j\Gamma(m_j)$ for every $j = 1, \ldots, n$. Write $g_j^{-1} = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$. Then we have to find $g = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in \Gamma(m_0)$ such that

$$g_j^{-1} g = \begin{pmatrix} a_j x + b_j z & a_j y + b_j t \\ c_j x + d_j z & c_j y + d_j t \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{m_j}$$

for every $j = 0, 1, \ldots, n$. Thus, we have to find integers x, y, z, t such that $xt - yz = 1$ and such that $x \equiv d_j$, $y \equiv -b_j$, $z \equiv -c_j$ and $t \equiv a_j \pmod{m_j}$ for every $j = 0, 1, \ldots, n$. Set $k = m_0 \cdot m_1 \cdot \cdots \cdot m_n$. As $(m_i, m_j) = 1$ for all $i \neq j$, it follows from the Chinese Remainder Theorem that one can find integers x', y', z', t' that are solutions mod k of the above systems, and such that $x't' - y'z' \equiv 1 \pmod{k}$. As the natural homomorphism $SL(2, \mathbb{Z}) \to SL(2, \mathbb{Z}/k\mathbb{Z})$ is onto, the existence of x, y, z, t is proved. Density of $\Gamma(m_0)$ in $G(\mathfrak{m})$ follows from the definition of the product topology on $G(\mathfrak{m})$. Q.E.D.

By Example 4.3.3 D in [18], $\Gamma = SL(2, \mathbb{Z})$ has Property (τ) with respect to the family $\mathcal{L}$ of all its principal congruence subgroups. Thus Lemmas 2.2, 2.3 and Theorem 3.1 give:

Theorem 4.4 *Let $\mathfrak{m}$ and m_0 be as above. Then $L^\infty(G(\mathfrak{m})) \rtimes \Gamma(m_0)$ is a full II_1-factor with the Haagerup property.*

Remark. In fact, the abelian von Neumann algebras $L^\infty(G_p)$ in Theorem 4.2 and $L^\infty(G(\mathfrak{m}))$ in Theorem 4.4 contain both increasing sequences of finite-dimensional, invariant von Neumann subalgebras that are $\| \cdot \|_2$-dense. Thus, Proposition 3.3 of [14] suffices to prove that the corresponding factors have the Haagerup property.

At last, we give a modified construction of the above framework in order to get examples of non full factors with controlled central sequences. To do that, set again $\Gamma = SL(2, \mathbb{Z})$, let $Z_0 = \{I, -I\}$ denote its center and let Λ be the restricted direct product group $\bigoplus_{j \geq 1} Z_0$. Set $\tilde{\Gamma} = \Gamma \times \Lambda$ and let $Z = Z_0 \times \Lambda$ denote the center of $\tilde{\Gamma}$. Choose a sequence $\mathfrak{m} = (m_j)_{j \geq 1}$ as in Theorem 1.4 and assume that $m_1 \geq 3$ so that $Z \cap \Gamma(m_j) = \{I\}$ for every j. Let $\tilde{\Gamma}$ act on $G(\mathfrak{m}) = \prod_{i \geq 1} G_i$ as follows:

$$(g, (z_j)_{j \geq 1}) \cdot (g_j \Gamma(m_j))_{j \geq 1} = (g z_j g_j \Gamma(m_j))_{j \geq 1}.$$

It is easy to check that the action is free, and, as the action of Γ is s-strongly ergodic, then so is the action of $\tilde{\Gamma}$. For future use, we define the following sequence of subsets of Z: for every positive integer k, let R_k be the set of all $(z_j)_{j \geq 0} \in Z$ such that $z_0 z_j = 1 \ \forall j \leq k$; in other words, $z = (z_j)_{j \geq 0}$ belongs to R_k if and only if either $z = (I, ..., I, z_{k+1}, z_{k+2}, ...)$ or $z = (-I, ..., -I, z_{k+1}, z_{k+2}, ...)$.

Theorem 4.5 *Retain notations above and let $N = L^\infty(G(\mathfrak{m})) \rtimes \tilde{\Gamma}$. Then N is a type II_1 factor with the Haagerup property and with Property gamma. Furthermore, every central sequence in N is equivalent to a central sequence $(c_n)_{n \geq 1}$ contained in the abelian von Neumann subalgebra $L(Z)$ which satisfies the following condition: $(\star)$ for every $k \geq 1$,*

$$\lim_{n \to \infty} \sum_{z \notin R_k} |c_n(z)|^2 = 0.$$

Conversely, every bounded sequence $(c_n)_{n \geq 1} \subset L(Z)$ that satisfies $(\star)$ is a central sequence in N.

Proof. By Lemma 2.2, we know that every central sequence in N is equivalent to a central sequence $(c_n)_{n\geq 1}$ contained in $L(Z)$. Fix a positive integer k and $x_j \in G_j$ for $j = 1, ..., k$. Set $a = \chi_{(x_1,...,x_k)} \in L^\infty(G_1 \times ... \times G_k) \subset L^\infty(G(\mathfrak{m}))$. If $z \in Z$, one has $a = \alpha_z(a)$ if and only if $z \in R_k$. Thus,

$$\|ac_n - c_n a\|_2^2 = \sum_{z \notin R_k} |c_n(z)|^2 \|a - \alpha_z(a)\|_2^2 = 2 \sum_{z \notin R_k} |c_n(z)|^2.$$

Hence, if (c_n) is a central sequence, then it satisfies $(\star)$, and conversely, if it satisfies $(\star)$, then it is a central sequence because the set of all $\chi_{(x_1,...,x_k)}$ as above is total in $L^\infty(G(\mathfrak{m}))$, and each c_n commutes to $\lambda(\tilde{\Gamma})$. Q.E.D.

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