

# Quaternionic-perfect forms in dimension 12

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## 1 General facts and credits

The group  $Q$  of unit quaternions (Hurwitz) in dimension 4 is attached to the form  $D_4$ , which is the unique  $Q$ -perfect form in dimension 4. In dimension 8, it is rather easy to check that  $E_8$  is the unique  $Q^2$ -perfect form.

The situation is more complicated in dimension 12, where one finds 12  $Q^3$ -perfect forms. This list was known to me before appearance of the paper [BMS], and was not published at the time. A few years later, the complete computation (with orbits under the various automorphism groups) was done by my then PhD-student David-Olivier Jaquet-Chiffelle, again without publication. He deserves therefore also credit for this classification.

I am grateful to Achille Schuermann, who cross-checked the whole computation (2007 – 2008). He should also be part of this story. I shall give below my matrices, resp. his (together with his notation, which keeps track of Voronoi's algorithm), to allow comparison with his database.

## 2 The $Q^3$ -perfect forms

Forms are listed with decreasing density. Apart from JMAX and JMIN, which are laminated lattices [CS], the given names are my own invention. All are perfect in the ordinary sense.

No=1;

Name=JMAX;

Matrices (S vs Sch)

JMAX=

[4, 2, 2, 2, 0, -2, 2, 0, 2, 1, 1, 1; 2, 4, 0, 0, 2, 0, 2, 2, 1, 2, 0, 0; 2, 0, 4, 0, -2, -2, 0, 0, 1, 0, 2, 0;  
2, 0, 0, 4, 0, -2, 0, 0, 1, 0, 0, 2; 0, 2, -2, 0, 4, 2, 2, 0, 0, 0, 0; -2, 0, -2, -2, 2, 4, 0, 0, 0, 0, 0, 0;  
2, 2, 0, 0, 2, 0, 4, 0, 0, 0, 0, 0; 0, 2, 0, 0, 2, 0, 0, 4, 0, 0, 0, 0; 2, 1, 1, 1, 0, 0, 0, 0, 4, 2, 2, 2;  
1, 2, 0, 0, 0, 0, 0, 0, 2, 4, 0, 0; 1, 0, 2, 0, 0, 0, 0, 0, 2, 0, 4, 0; 1, 0, 0, 2, 0, 0, 0, 0, 2, 0, 0, 4];

f5=

[4, 0, 1, 2, 2, 2, 2, 0, 1, 2, 0, 1; 0, 4, 0, -2, 2, 0, 0, 2, 0, 0, 2, 0; 1, 0, 4, -1, 0, 2, 0, 0, 2, 0, 0, 2;  
2, -2, -1, 4, 0, 1, 2, -2, 0, 2, 0, 0; 2, 2, 0, 0, 4, 0, 2, 2, 0, 0, 2, 0; 2, 0, 2, 1, 0, 4, 1, 0, 2, 1, 0, 2;  
2, 0, 0, 2, 2, 1, 4, 0, 1, 0, 2, 1; 0, 2, 0, -2, 2, 0, 0, 4, 0, -2, 0, 0; 1, 0, 2, 0, 0, 2, 1, 0, 4, -1, 0, 0;  
2, 0, 0, 2, 0, 1, 0, -2, -1, 4, 0, 1; 0, 2, 0, 0, 2, 0, 2, 0, 0, 0, 4, 0; 1, 0, 2, 0, 0, 2, 1, 0, 0, 1, 0, 4];

MINIMUM=4; DETERMINANT=1024; GAMMA=2.2449; KISSING NO=324;  
NEIGHBOURS=["JMIN",[384];"QMAX",[384]];

No=2;

Name="JMIN";

Matrices (S vs Sch)

JMIN=

[4, 2, 2, 2, 2, 0, 2, 1, 0, 0, 0, 1; 2, 4, 0, 0, 2, 2, 1, 1, 0, 0, -1, 1; 2, 0, 4, 0, 0, -1, 2, 0, 0, 1, 0, 0;  
2, 0, 0, 4, 1, -1, 0, 2, -1, -1, 0, 0; 2, 2, 0, 1, 4, 2, 2, 2, 2, 1, 1, 1; 0, 2, -1, -1, 2, 4, 0, 0, 1, 2, 0, 0;  
2, 1, 2, 0, 2, 0, 4, 0, 1, 0, 2, 0; 1, 1, 0, 2, 2, 0, 0, 4, 1, 0, 0, 2; 0, 0, 0, -1, 2, 1, 1, 1, 4, 2, 2, 2;  
0, 0, 1, -1, 1, 2, 0, 0, 2, 4, 0, 0; 0, -1, 0, 0, 1, 0, 2, 0, 2, 0, 4, 0; 1, 1, 0, 0, 1, 0, 0, 2, 2, 0, 0, 4];

f0=

[4, 0, 0, 2, 2, 2, 2, 0, 1, 2, 0, 1; 0, 4, 0, -2, 2, 0, 0, 2, -1, 0, 2, 1; 0, 0, 4, -2, 0, 2, -1, 1, 2, -1, -1, 2;  
2, -2, -2, 4, 0, 0, 2, -2, 0, 2, 0, -1; 2, 2, 0, 0, 4, 0, 2, 2, 0, 0, 2, 1; 2, 0, 2, 0, 0, 4, 0, 0, 2, 1, -1, 2;  
2, 0, -1, 2, 2, 0, 4, 0, 0, 0, 2, 1; 0, 2, 1, -2, 2, 0, 0, 4, 0, -2, 0, 1; 1, -1, 2, 0, 0, 2, 0, 0, 4, -1, -1, 0;  
2, 0, -1, 2, 0, 1, 0, -2, -1, 4, 0, 0; 0, 2, -1, 0, 2, -1, 2, 0, -1, 0, 4, 0; 1, 1, 2, -1, 1, 2, 1, 1, 0, 0, 0, 4];

MINIMUM=4; DETERMINANT=1024; GAMMA=2.2449; KISSING NO=312;

NEIGHBOURS=["JMAX",[16];"JMIN",[384,128,128,128,128,96,16,2];"QMAX",[128,128,32];

"QMIN",[384,128,128];"S",[384];"T",[384,96]];

No=3;

Name="QMAX";

Matrices (S vs Sch)

QMAX=

[6, 3, 3, 3, -2, -3, 1, -1, 0, 0, 0, 2; 3, 6, 0, 0, 1, -2, 2, 2, 0, 0, -2, 2; 3, 0, 6, 0, -3, -2, -2, 0, 0, 2, 0, 0;  
3, 0, 0, 6, -1, -2, 0, -2, -2, -2, 0, 0; -2, 1, -3, -1, 6, 3, 3, 3, 0, -3, 0, 0; -3, -2, -2, -2, 3, 6, 0, 0, 3, 0, 3, 0;  
1, 2, -2, 0, 3, 0, 6, 0, 0, -3, 0, 3; -1, 2, 0, -2, 3, 0, 0, 6, 0, 0, -3, 0; 0, 0, 0, -2, 0, 3, 0, 0, 6, 3, 3, 3;  
0, 0, 2, -2, -3, 0, -3, 0, 3, 6, 0, 0; 0, -2, 0, 0, 0, 3, 0, -3, 3, 0, 6, 0; 2, 2, 0, 0, 0, 0, 3, 0, 3, 0, 0, 6];

f3=

[10, 0, 0, 5, 5, 5, 5, 1, 2, 5, -1, 3; 0, 8, 0, -5, 4, 0, -1, 4, -3, 1, 4, 2; 0, 0, 8, -5, 0, 4, -2, 3, 4, -3, -2, 4;  
5, -5, -5, 10, 0, 0, 5, -5, 0, 5, -1, -2; 5, 4, 0, 0, 8, 0, 5, 4, -1, 1, 4, 3; 5, 0, 4, 0, 0, 8, 0, 1, 4, 2, -3, 4;  
5, -1, -2, 5, 5, 0, 10, 0, 0, 0, 4, 3; 1, 4, 3, -5, 4, 1, 0, 8, 0, -4, 0, 3; 2, -3, 4, 0, -1, 4, 0, 0, 8, -3, -3, 0;  
5, 1, -3, 5, 1, 2, 0, -4, -3, 10, 0, 0; -1, 4, -2, -1, 4, -3, 4, 0, -3, 0, 8, 0; 3, 2, 4, -2, 3, 4, 3, 3, 0, 0, 0, 8];

MINIMUM=6; DETERMINANT=202500; GAMMA=2.1675; KISSING NO=228;

NEIGHBOURS=["JMAX",[1];"JMIN",[8,8,2];"QMAX",[2];"QMIN",[8];"S",[12]];

No=4;

Name="QMIN";

Matrices (S vs Sch)

QMIN=

[6, 3, 3, 3, -2, -3, 1, -1, 1, 0, 1, 3; 3, 6, 0, 0, 1, -2, 2, 2, 1, 1, -2, 3; 3, 0, 6, 0, -3, -2, -2, 0, 0, 2, 1, 0;  
3, 0, 0, 6, -1, -2, 0, -2, -2, -3, 0, 1; -2, 1, -3, -1, 6, 3, 3, 3, 0, -2, -1, 0; -3, -2, -2, -2, 3, 6, 0, 0, 2, 0, 2, -1;  
1, 2, -2, 0, 3, 0, 6, 0, 1, -2, 0, 3; -1, 2, 0, -2, 3, 0, 0, 6, 0, 1, -3, 0; 1, 1, 0, -2, 0, 2, 1, 0, 6, 3, 3, 3;  
0, 1, 2, -3, -2, 0, -2, 1, 3, 6, 0, 0; 1, -2, 1, 0, -1, 2, 0, -3, 3, 0, 6, 0; 3, 3, 0, 1, 0, -1, 3, 0, 3, 0, 0, 6];

f1=  
 [8, -1, -1, 4, 4, 4, 4, -1, 2, 4, 0, 1; -1, 8, 1, -5, 4, 1, 0, 4, -1, -1, 4, 3; -1, 1, 8, -5, 0, 4, -3, 2, 4, -2, -2, 4;  
 4, -5, -5, 8, -1, -1, 4, -5, -1, 4, 0, -3; 4, 4, 0, -1, 8, 1, 4, 4, 1, -1, 4, 2; 4, 1, 4, -1, 1, 8, 0, 0, 4, 2, -1, 4;  
 4, 0, -3, 4, 4, 0, 8, -1, -1, 0, 5, 2; -1, 4, 2, -5, 4, 0, -1, 8, 1, -5, 0, 2; 2, -1, 4, -1, 1, 4, -1, 1, 8, -2, -2, 0;  
 4, -1, -2, 4, -1, 2, 0, -5, -2, 8, -1, -1; 0, 4, -2, 0, 4, -1, 5, 0, -2, -1, 8, 1; 1, 3, 4, -3, 2, 4, 2, 2, 0, -1, 1, 8];

MINIMUM=6; DETERMINANT=202500; GAMMA=2.1675; KISSING NO=216;  
 NEIGHBOURS=["JMIN",[18,6,6];"QMAX",[6];"QMIN",[9,6,6];"R",[3];"S",[18,9]];

No=5;  
 Name="R";  
 Matrices (S vs Sch)

R=  
 [10, 5, 5, 5, -2, -5, 3, -1, 0, -2, 2, 4; 5, 10, 0, 0, 3, -2, 4, 4, 2, 0, -2, 6; 5, 0, 10, 0, -5, -4, -2, 0, -2, 2, 0, 0;  
 5, 0, 0, 10, -1, -4, 0, -2, -4, -6, 0, 0; -2, 3, -5, -1, 10, 5, 5, 5, 2, -4, 0, 2; -5, -2, -4, -4, 5, 10, 0, 0, 6, 2, 4, 0;  
 3, 4, -2, 0, 5, 0, 10, 0, 2, -4, 2, 6; -1, 4, 0, -2, 5, 0, 0, 10, 0, 0, -6, 2; 0, 2, -2, -4, 2, 6, 2, 0, 12, 6, 6, 6;  
 -2, 0, 2, -6, -4, 2, -4, 0, 6, 12, 0, 0; 2, -2, 0, 0, 0, 4, 2, -6, 6, 0, 12, 0; 4, 6, 0, 0, 2, 0, 6, 2, 6, 0, 0, 12];

f6=  
 [12, -2, -2, 6, 6, 6, 6, -2, 2, 6, 0, 2; -2, 14, 1, -8, 7, 1, 0, 7, -2, -2, 7, 4; -2, 1, 14, -8, 0, 7, -4, 3, 7, -4, -3, 7;  
 6, -8, -8, 12, -2, -2, 6, -8, -2, 6, 0, -4; 6, 7, 0, -2, 14, 1, 6, 7, 1, -2, 7, 3; 6, 1, 7, -2, 1, 14, 0, 0, 7, 2, -2, 7;  
 6, 0, -4, 6, 6, 0, 12, -2, -2, 0, 8, 4; -2, 7, 3, -8, 7, 0, -2, 14, 1, -8, 0, 3; 2, -2, 7, -2, 1, 7, -2, 1, 14, -4, -3, 0;  
 6, -2, -4, 6, -2, 2, 0, -8, -4, 12, -2, -2; 0, 7, -3, 0, 7, -2, 8, 0, -3, -2, 14, 1; 2, 4, 7, -4, 3, 7, 4, 3, 0, -2, 1, 14];

MINIMUM=10; DETERMINANT=107495424; GAMMA=2.1415; KISSING NO=180;  
 NEIGHBOURS=["QMIN",[6];"S",[9]];

No=6; Name="S";  
 Matrices (S vs Sch)

S=  
 [8, 4, 4, 4, -2, -4, 2, -1, 0, -1, 1, 3; 4, 8, 0, 0, 2, -2, 3, 3, 1, 0, -2, 4; 4, 0, 8, 0, -4, -3, -2, 0, -1, 2, 0, 0;  
 4, 0, 0, 8, -1, -3, 0, -2, -3, -4, 0, 0; -2, 2, -4, -1, 8, 4, 4, 4, 1, -3, 0, 1; -4, -2, -3, -3, 4, 8, 0, 0, 4, 1, 3, 0;  
 2, 3, -2, 0, 4, 0, 8, 0, 1, -3, 1, 4; -1, 3, 0, -2, 4, 0, 0, 8, 0, 0, -4, 1; 0, 1, -1, -3, 1, 4, 1, 0, 8, 4, 4, 4;  
 -1, 0, 2, -4, -3, 1, -3, 0, 4, 8, 0, 0; 1, -2, 0, 0, 0, 3, 1, -4, 4, 0, 8, 0; 3, 4, 0, 0, 1, 0, 4, 1, 4, 0, 0, 8];

f2=  
 [10, -1, -1, 5, 5, 5, 5, -1, 2, 5, 0, 2; -1, 10, 1, -6, 5, 1, 0, 5, -2, -1, 5, 3; -1, 1, 10, -6, 0, 5, -3, 3, 5, -3, -2, 5;  
 5, -6, -6, 10, -1, -1, 5, -6, -1, 5, 0, -3; 5, 5, 0, -1, 10, 1, 5, 5, 0, -1, 5, 3; 5, 1, 5, -1, 1, 10, 0, 1, 5, 2, -2, 5;  
 5, 0, -3, 5, 5, 0, 10, -1, -1, 0, 6, 3; -1, 5, 3, -6, 5, 1, -1, 10, 1, -6, 0, 3; 2, -2, 5, -1, 0, 5, -1, 1, 10, -3, -3, 0;  
 5, -1, -3, 5, -1, 2, 0, -6, -3, 10, -1, -1; 0, 5, -2, 0, 5, -2, 6, 0, -3, -1, 10, 1; 2, 3, 5, -3, 3, 5, 3, 3, 0, -1, 1, 10];

MINIMUM=8; DETERMINANT=7496644; GAMMA=2.1389; KISSING NO=180;  
 NEIGHBOURS=["JMIN",[4];"QMAX",[2];"QMIN",[4,2];"R",[1];"T",[2]];

No=7;  
 Name="T";  
 Matrices (S vs Sch)

T=

[6, 3, 3, 3, -1, -3, 2, -1, 0, -1, 1, 2; 3, 6, 0, 0, 2, -1, 3, 2, 1, 0, -1, 3; 3, 0, 6, 0, -3, -3, -1, 0, -1, 1, 0, 0;  
 3, 0, 0, 6, 0, -2, 0, -1, -2, -3, 0, 0; -1, 2, -3, 0, 6, 3, 3, 3, 1, -2, 0, 1; -3, -1, -3, -2, 3, 6, 0, 0, 3, 1, 2, 0;  
 2, 3, -1, 0, 3, 0, 6, 0, 1, -2, 1, 3; -1, 2, 0, -1, 3, 0, 0, 6, 0, 0, -3, 1; 0, 1, -1, -2, 1, 3, 1, 0, 6, 3, 3, 3;  
 -1, 0, 1, -3, -2, 1, -2, 0, 3, 6, 0, 0; 1, -1, 0, 0, 0, 2, 1, -3, 3, 0, 6, 0; 2, 3, 0, 0, 1, 0, 3, 1, 3, 0, 0, 6];

f4=

[10, 0, 0, 5, 5, 6, 5, 1, 2, 5, -1, 3; 0, 8, 2, -5, 4, 1, -1, 4, -2, 1, 4, 3; 0, 2, 8, -6, 1, 4, -2, 4, 4, -3, -1, 4;  
 5, -5, -6, 10, 0, 0, 5, -5, -1, 5, -1, -2; 5, 4, 1, 0, 8, 2, 5, 4, 0, 1, 4, 4; 6, 1, 4, 0, 2, 8, 1, 2, 4, 2, -2, 4;  
 5, -1, -2, 5, 5, 1, 10, 0, 0, 0, 4, 4; 1, 4, 4, -5, 4, 2, 0, 8, 2, -4, 0, 3; 2, -2, 4, -1, 0, 4, 0, 2, 8, -4, -3, 0;  
 5, 1, -3, 5, 1, 2, 0, -4, -4, 10, 0, 0; -1, 4, -1, -1, 4, -2, 4, 0, -3, 0, 8, 2; 3, 3, 4, -2, 4, 4, 4, 3, 0, 0, 2, 8];

MINIMUM=6; DETERMINANT=252144; GAMMA=2.1282; KISSING NO=180;

NEIGHBOURS=["JMIN",[8,2];"S",[4];"U",[1]];

No=8;

Name="U";

Matrices (S vs Sch)

U=

[4, 2, 2, 2, -1, -2, 1, -1, 0, -1, 1, 1; 2, 4, 0, 0, 1, -1, 2, 1, 1, 0, 0, 2; 2, 0, 4, 0, -2, -2, -1, 0, -1, 0, 0, 0;  
 2, 0, 0, 4, 0, -1, 0, -1, -1, -2, 0, 0; -1, 1, -2, 0, 4, 2, 2, 2, 1, -1, 0, 1; -2, -1, -2, -1, 2, 4, 0, 0, 2, 1, 1, 0;  
 1, 2, -1, 0, 2, 0, 4, 0, 1, -1, 1, 2; -1, 1, 0, -1, 2, 0, 0, 4, 0, 0, -2, 1; 0, 1, -1, -1, 1, 2, 1, 0, 4, 2, 2, 2;  
 -1, 0, 0, -2, -1, 1, -1, 0, 2, 4, 0, 0; 1, 0, 0, 0, 0, 1, 1, -2, 2, 0, 4, 0; 1, 2, 0, 0, 1, 0, 2, 1, 2, 0, 0, 4];

f7=

[8, 0, 0, 4, 4, 5, 4, 0, 2, 4, 0, 2; 0, 6, 2, -4, 3, 1, 0, 3, -1, 0, 3, 3; 0, 2, 6, -5, 1, 3, -2, 3, 3, -2, -1, 3;  
 4, -4, -5, 8, 0, 0, 4, -4, -1, 4, 0, -2; 4, 3, 1, 0, 6, 2, 4, 3, 1, 0, 3, 3; 5, 1, 3, 0, 2, 6, 1, 1, 3, 2, -1, 3;  
 4, 0, -2, 4, 4, 1, 8, 0, 0, 0, 4, 3; 0, 3, 3, -4, 3, 1, 0, 6, 2, -4, 0, 2; 2, -1, 3, -1, 1, 3, 0, 2, 6, -3, -2, 0;  
 4, 0, -2, 4, 0, 2, 0, -4, -3, 8, 0, 0; 0, 3, -1, 0, 3, -1, 4, 0, -2, 0, 6, 2; 2, 3, 3, -2, 3, 3, 3, 2, 0, 0, 2, 6];

MINIMUM=4; DETERMINANT=2500; GAMMA=2.0840; KISSING NO=180;

NEIGHBOURS=["T",[15]];

## References

- [BMS] A.-M. Bergé, J. Martinet, and F. Sigrist. *Une généralisation de l'algorithme de Voronoï pour les formes quadratiques*. *Astérisque* **209** (1992), 137-158.
- [CS] J.H. Conway and N.J.A. Sloane. *Sphere Packings, Lattices, and Groups*. Springer-Verlag, 1992.