Free groups and reduced 1-cohomology of unitary representations

Florian Martin*, Alain Valette

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To Alain Connes, with admiration

Abstract

Guichardet [Gui72] showed that every unitary representation of the free group $F_n$ ($2 \leq n < \infty$) has non-zero 1-cohomology. We construct a continuum of pairwise inequivalent, irreducible, unitary representations of $F_n$, with vanishing reduced 1-cohomology and such that the $C^*$-algebra generated by each representation is the unitized algebra of the compact operators.

1 Introduction

If $G$ is a countable discrete group and $\pi$ a unitary representation of $G$, we denote by $H^1(G, \pi)$ the first cohomology of $G$ with coefficients in $\pi$, i.e. the quotient of the space $Z^1(G, \pi)$ of 1-cocycles by the space $B^1(G, \pi)$ of 1-coboundaries. Endowed with the topology of pointwise convergence, $Z^1(G, \pi)$ becomes a Fréchet space, and the reduced 1-cohomology $\overline{H^1}(G, \pi)$ is defined as the quotient of $Z^1(G, \pi)$ by the closure of the space of 1-coboundaries.

Reduced 1-cohomology was first considered by Guichardet [Gui72] and its relevance to questions of rigidity and geometric group theory was emphasized more recently in papers of Shalom (see [Sha00], [Sha04]).

This paper focuses on the free group on $n$ generators $G = F_n$ ($2 \leq n \leq \infty$): its 1-cohomology has the following interesting property established

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by Guichardet (Example 1 in [Gui72]): $H^1(\mathbb{F}_2, \pi) \neq 0$ for every unitary representation $\pi$ of $\mathbb{F}_2$. Using the dictionary between 1-cohomology and affine isometric actions on Hilbert spaces (see e.g. [BHV08], p.73), the geometric equivalent of this observation is: every unitary representation of $\mathbb{F}_2$ is the linear part of some affine isometric action without globally fixed point.

We illustrate the difference between reduced and ordinary 1-cohomology by establishing:

**Theorem 1.1** Fix $n \in \mathbb{N} \cup \{\infty\}$ ($n \geq 2$). There exists a continuum of pairwise inequivalent, unitary, irreducible representations $\sigma$ of $\mathbb{F}_n$ such that

1) $\overline{H}^1(\mathbb{F}_n, \sigma) = 0$.

2) The $C^*$-algebra generated by $\sigma(\mathbb{F}_n)$ is $\tilde{K}$, the unitized $C^*$-algebra of the algebra $K$ of compact operators on an infinite-dimensional separable Hilbert space.

There are other instances of the fact that, for a given group, the vanishing of the 1-cohomology is not equivalent to the vanishing of its reduced counterpart: for example, let $\lambda_G$ be the left regular representation of a countably infinite amenable group $G$: then $H^1(G, \lambda_G) \neq 0$, by Théorème 1 in [Gui72], while $\overline{H}^1(G, \lambda_G) = 0$ by [MV07]. Let us mention however a remarkable result by Shalom [Sha00]: for a compactly generated locally compact group, the vanishing of reduced 1-cohomology for all unitary representations, is equivalent to the vanishing of 1-cohomology for all unitary representations (the latter being equivalent to Kazhdan’s property (T), by the Delorme-Guichardet theorem, see Chapter 2 in [BHV08]).

## 2 Proof of Theorem 1.1

Let us denote by $\text{Im } T$ the range of the linear operator $T$.

**Lemma 2.1** Fix an integer $n \geq 2$. Let $U_1, \ldots, U_n$ be unitary operators on a Hilbert space such that:

1) $1$ is not an eigenvalue of $U_i$, for $1 \leq i \leq n$;

2) for $2 \leq j \leq n$:

$$\text{Im}(U_j - 1) \cap \left(\sum_{i=1}^{j-1} \text{Im}(U_i - 1)\right) = \{0\}.$$
Then the assignment \( \pi(x_i) = U_i^* \) defines a unitary representation \( \pi \) of the free group \( \mathbb{F}_n \) on \( n \) generators \( x_1, \ldots, x_n \), such that \( \overline{H^1(\mathbb{F}_n, \pi)} = 0 \).

**Proof of the lemma:** We start the same way as Guichardet in Example 1 in [Gui72], in his proof of \( H^1(\mathbb{F}_2, \sigma) \neq 0 \) for every unitary representation \( \sigma \) of \( \mathbb{F}_2 \). For a unitary representation \( \pi \) of \( \mathbb{F}_n \) on a Hilbert space \( V \), the map

\[
Z^1(\mathbb{F}_n, \pi) \rightarrow V^n : b \mapsto (b(x_1), \ldots, b(x_n))
\]

is a topological isomorphism (surjectivity follows from the freeness of the group: a 1-cocycle can be defined arbitrarily on generators). In that isomorphism, \( B^1(\mathbb{F}_n, \pi) \) corresponds to the image of the map

\[
\psi : V \rightarrow V^n : v \mapsto ((\pi(x_1) - 1)v, \ldots, (\pi(x_n) - 1)v)
\]

So \( \overline{H^1(\mathbb{F}_n, \pi)} = 0 \) if and only if \( \psi \) has dense image, if and only if \( \psi^* : V^n \rightarrow V \) is injective. But

\[
\psi^*(v_1, \ldots, v_n) = \sum_{i=1}^n (\pi(x_i)^* - 1)v_i.
\]

With \( U_i = \pi(x_i)^* \), we see that our assumptions on \( U_1, \ldots, U_n \) exactly mean that \( \psi^* \) is injective. \( \square \)

We now come to a problem in operator theory, namely construct families of unitary operators satisfying the conditions in Lemma 2.1. We will elaborate on Dixmier’s elegant construction [Dix49] (see also Theorem 3.6 in [FW71]) to answer that question.

**Proof of Theorem 1.1:** On \( V = L^2[0, 2\pi] \) with the trigonometric orthonormal system \((e_k)_{k \in \mathbb{Z}}\), let us construct unitary operators \( U_n (n \geq 1) \) such that \( U_n - 1 \) is trace-class, 1 is not an eigenvalue of \( U_n \) and \( \text{Im}(U_n - 1) \cap (\sum_{m=1}^{n-1} \text{Im}(U_m - 1)) = \{0\} \) for \( n \geq 2 \). Moreover \( U_1, U_2 \) will be shown to act together irreducibly on \( V \). Taking into account the fact that every irreducible \( C^* \)-algebra intersecting \( \mathcal{K} \) non-trivially, must contain it (see [Dix77], Corollary 4.1.10), we get the second statement in the Theorem. For \( n < \infty \), the first statement (vanishing of \( \overline{H^1} \)) will follow straight from Lemma 2.1.

For \( n = \infty \), we observe that if a group \( \Gamma \) is the increasing union of subgroups \( \Gamma_n \) with \( \overline{H^1(\Gamma_n, \sigma|_{\Gamma_n})} = 0 \), then clearly \( \overline{H^1(\Gamma, \sigma)} = 0 \).

To construct a continuum of such families of unitary operators, fix a real-valued rapidly decreasing sequence \( a = (a_k)_{k \in \mathbb{Z}} \), such that \( 0 \neq a_k \neq a_m \) for \( k \neq m \in \mathbb{Z} \), and define a diagonal, trace-class operator \( A^{(a)} \) on \( V \) by

\[
A^{(a)}e_k = a_ke_k \quad (k \in \mathbb{Z}).
\]
Note that $A^{(a)}$ is injective with all eigenvalues of multiplicity 1, by our choice of $a$. Now let $(x_n)_{n>0}$ be a strictly increasing sequence in $[0,2\pi]$, with $x_1 = 0$ and $\frac{x_n}{\pi}$ irrational. Define a function $\xi_n$ on $[0,2\pi]$ by

$$\xi_n(x) = \begin{cases} -1 & \text{if } 0 \leq x < x_n \\ 1 & \text{if } x_n \leq x < 2\pi \end{cases}$$

Let $M_n$ be the operator of multiplication by $\xi_n$: this is a self-adjoint unitary operator on $V$; note that $M_1 = 1$. Set $A^{(a)}_n = M_n A^{(a)} M_n^*$; the following holds:

Claim: For $n \geq 2$:

$$\text{Im} \ A^{(a)}_n \cap \left( \sum_{i=1}^{n-1} \text{Im} \ A^{(a)}_i \right) = \{0\}.$$  

To prove the claim, observe that, because $a$ is rapidly decreasing, $\text{Im} \ A^{(a)}_n$ is contained in the space of restrictions of real-analytic functions to $[0,2\pi]$. Then $\sum_{i=1}^{n-1} \text{Im} \ A^{(a)}_i$ is contained in the space of functions on $[0,2\pi]$ whose restrictions to all intervals $[x_1,x_2]$, $[x_2,x_3]$, ..., $[x_{n-1},2\pi]$ are real analytic. On the other hand non-zero functions in $\text{Im} \ A^{(a)}_n$ are not analytic in the neighborhood of $x_n \in ]x_{n-1},2\pi[$, proving the claim.

Let then $U^{(a)}_n$ be the Cayley transform of $A^{(a)}_n$:

$$U^{(a)}_n = (1 - iA^{(a)}_n)(1 + iA^{(a)}_n)^{-1}.$$

Then $U^{(a)}_n$ is unitary, 1 is not an eigenvalue of $U^{(a)}_n$, and $U^{(a)}_n$ is diagonal in the basis $(M_n e_k)_{k \in \mathbb{Z}}$, with all eigenvalues of multiplicity 1. Moreover

$$U^{(a)}_n - 1 = -2iA^{(a)}_n(1 + iA^{(a)}_n)^{-1}$$

so that $U^{(a)}_n - 1$ is trace-class, and $\text{Im}(U^{(a)}_n - 1) \cap \left( \sum_{m=1}^{n-1} \text{Im}(U^{(a)}_m - 1) \right) = \{0\}$ by the claim.

To prove that $U^{(a)}_1$, $U^{(a)}_2$ together act irreducibly on $V$, let $S$ be an operator on $V$ which commutes both with $U^{(a)}_1$ and $U^{(a)}_2$. Since $U^{(a)}_1$, $U^{(a)}_2$ have all eigenvalues of multiplicity 1, the operator $S$ must be diagonal both in the bases $(e_k)_{k \in \mathbb{Z}}$ and $(M_2 e_k)_{k \in \mathbb{Z}}$. From the Fourier series expansion of $M_2 e_k$:

$$M_2 e_k = \frac{1 - x_2^2}{\pi} e_k + \sum_{m \neq k} \frac{i}{\pi (m-k)} [1 - e^{i(k-m)x_2}] e_m$$
and the fact that all Fourier coefficients of $M_2e_k$ are non-zero (because $\frac{2\pi}{\pi}$ is irrational), it follows that $S$ must be scalar. Irreducibility then follows from Schur’s lemma.

Finally, to get a continuum of pairwise inequivalent representations, we notice that, since $U_1^{(a)} - 1$ is trace-class, the complex number

$$Tr(U_1^{(a)} - 1) = -2i Tr A^{(a)}(1 + iA^{(a)})^{-1} = -2i \sum_{k \in \mathbb{Z}} a_k(1 + ia_k)^{-1}$$

is an invariant of unitary equivalence of the associated representation. So varying $a$ in the space of real-valued rapidly decreasing sequences satisfying $0 \neq a_k \neq a_m$ for $k \neq m \in \mathbb{Z}$, we get the desired continuum. □

Theorem 1.1 motivates:

**Question 1** Is there a countable group $G$ such that $\overline{H}^1(G, \pi) \neq 0$ for every unitary representation $\pi$ of $G$?

Note that such a group, if it exists, must be non-amenable: indeed, by Corollary 5.2 in [MV07], a countable amenable group $G$ has $\overline{H}^1(G, \lambda_G) = 0$, where $\lambda_G$ is the left regular representation.

### 3 A remark

The irreducible representations $\sigma$ constructed in Theorem 2.1 satisfy $\sigma(g) - 1 \in \mathcal{K}$ for every $g \in \mathbb{F}_n$. We observe that this fact alone is responsible for the non-vanishing of $H^1$.

**Proposition 3.1** Let $G$ be a discrete group. Assume that there exists a unitary irreducible representation $\pi$ of $G$ with the property that $1 - \pi(g)$ is a compact operator for every $g \in G$. Then $B^1(G, \pi)$ is not closed in $Z^1(G, \pi)$; in particular $H^1(G, \pi) \neq 0$.

**Proof:** Observe that by irreducibility the $C^*$-algebra generated by $\pi(G)$ is $\tilde{\mathcal{K}}$, and consider the short exact sequence

$$0 \to \mathcal{K} \to \tilde{\mathcal{K}} \xrightarrow{\varphi} \mathbb{C} \to 0.$$

For $f \in \ell^1(G)$, we have $q(\pi(f)) = \sum_{g \in G} f(g)q(\pi(g)) = \sum_{g \in G} f(g)$ for every $f \in \ell^1(G)$, so that

$$|\sum_{g \in G} f(g)| \leq \|\pi(f)\|.$$
By Theorem 3.4.4 in [Dix77], this means exactly that \( \pi \) weakly contains the trivial representation of \( G \). We conclude by applying another result by Guichardet ([Gui72], Théorème 1): for a unitary representation without non-zero fixed vectors, the space of 1-coboundaries is not closed in the space of 1-cocycles if and only if the representation weakly contains the trivial representation.

□

References


Authors’ addresses:

Philip Morris International
Quai Jeanrenaud 56
CH-2000 Neuchâtel
SWITZERLAND
florian.martin@pmintl.com

Institut de Mathématiques
Rue Emile Argand 11-BP158
CH-2009 Neuchâtel
SWITZERLAND
alain.valette@unine.ch