Affine isometric actions on Hilbert spaces and amenability

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Introduction

The study of affine isometric actions of groups on Hilbert spaces (more generally on Banach spaces) opens up a new chapter of representation theory, with applications to rigidity, ergodic theory, geometric group theory, and even operator algebras.

During the semester "*Amenability beyond groups*" at the Erwin Schrödinger Institute in Vienna, I was invited to give a mini-course on affine isometric actions, from 5 to 9 March 2007. I express my deepest thanks to the organizers for giving me the opportunity to popularize the subject. The aim of the course was, after presenting several examples and giving the link with group cohomology, to give applications to amenability, ergodic theory, and geometric group theory.

During the course, I had the pleasant surprise of finding a very careful notetaker in the person of Piotr Sołtan; the present notes are just a mild editing of Piotr's notes, whom I thank heartily for giving me permission to publish them. The reader is asked to bear in mind the informal nature of the notes.

Notations: The letter G will be reserved to denote a group. This group will most of the time be a topological group and the topology will most often be assumed to be locally compact. The notation $K \Subset G$ then means that K is a compact subset of G.

By π we will always denote a representation of G. In case this representation acts on a Hilbert space (usually denoted $\mathcal{H}$ or $\mathcal{H}_\pi$), we will assume that π is unitary and strongly continuous. These and similar conventions will be used throughout the notes without further explanation.

1 Affine actions

1.1 1-cohomology. In this subsection, which is completely algebraic, we set up the cohomological framework we need.

Let G be a group and let V be a vector space (over some field k). An *affine action* of G on V is a homomorphism $\alpha : G \to Aff(V)$, where $Aff(V)$ is the group of affine bijections $V \to V$. We have the split exact sequence

$$0 \to V \to Aff(V) \to GL(V) \to 1$$

[*]Based on notes taken by Piotr Mikołaj Sołtan

where V is identified with its group of translations. Composing α with the quotient map $Aff(V) \to GL(V)$, we get a representation $\pi : G \to GL(V)$ called the *linear part* of α.

Let us ask a converse question: if $\pi : G \to GL(V)$ is a representation, what are affine actions α with linear part π? Such α must be of the form

$$\alpha(g)v = \pi(g)v + b(g)$$

for any $v \in V$. The vector $b(g)$ is called the *translation part* of α.

Expressing that α is multiplicative (i.e. $\alpha(gh) = \alpha(g)\alpha(h)$), it follows that $b : G \to V$ must satisfy the 1-*cocycle relation*:

$$b(gh) = \pi(g)b(h) + b(g) \tag{1}$$

for all $g, h \in G$.

Example 1. *If π is the trivial representation of G on V then b is nothing but a homomorphism from G, to the additive group V.*

By $Z^1(G,\pi)$ we shall denote the set of all 1-*cocycles* $G \to V$, i.e. all maps b satisfying (1). It is easy to see that $Z^1(G,\pi)$ is a vector space under pointwise operations. By $B^1(G,\pi)$ we will denote the subset of 1-*coboundaries*, i.e. those $b \in Z^1(G,\pi)$ for which there exists a vector $v \in V$ such that

$$b(g) = \pi(g)v - v$$

for all $g \in G$. Clearly $B^1(G,\pi)$ is a subspace of $Z^1(G,\pi)$. Finally we define

$$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi)$$

and call $H^1(G,\pi)$ the *first cohomology group of G with coefficients in the G-module V.*

We can write down a dictionary between concepts of geometric and algebraic nature, in which each line represents a bijection:

Affine actions with linear part π	$Z^1(G,\pi)$
Affine actions with linear part π and with a globally fixed point (i.e. conjugate to π via a translation)	$B^1(G,\pi)$
Affine actions with linear part π, up to conjugation by a translation	$H^1(G,\pi)$

1.2 Affine isometric actions on Hilbert spaces.

1.2.1 Generalities. Let $\mathcal{H}$ be a real Hilbert space and let $Isom(\mathcal{H})$ denote the group of affine isometries of $\mathcal{H}$. Let α be a homomorphism $G \to Isom(\mathcal{H})$.

Remark: The Mazur-Ulam theorem says that if E is a real Banach space then any isometry of E is affine.[1] For strictly convex Banach spaces (e.g. Hilbert spaces) this result is quite easy because we then have a metric characterization of segments: for $x, y \in E$ the segment $[x, y]$ between x and y is

$$[x, y] = \big\{ z \in E : \|x - z\| + \|z - y\| = \|x - y\| \big\}.$$

In particular any isometry must preserve segments, and it is a classical exercise that a segment-preserving bijection has to be affine.

For a topological group G we will always assume that affine actions are continuous in the sense that the map

$$G \times \mathcal{H} \ni (g, v) \longmapsto \alpha(g)v \in \mathcal{H}$$

is continuous. The linear part of an isometric affine action is then a strongly continuous unitary representation. We will stick to this setting for the rest of these notes.

Definition 1.1. An affine action α of G on $\mathcal{H}$ *almost has fixed points* if

$$\forall \epsilon > 0, \forall K \Subset G, \exists v \in \mathcal{H} : \sup_{g \in K} \big\|\alpha(g)v - v\big\| < \epsilon.$$

We endow $Z^1(G, \pi)$ with the topology of uniform convergence on compact subsets and add one more line to the above dictionary:

Affine actions with linear part π almost having a fixed point	Closure of $B^1(G, \pi)$ in $Z^1(G, \pi)$

We define the *reduced cohomology group* $\overline{H}^1(G, \pi)$ as the quotient

$$\overline{H}^1(G, \pi) = Z^1(G, \pi)/\overline{B^1(G, \pi)}. \tag{2}$$

Let us now give a useful characterization of coboundaries. Remember that any 1-cocycle is, in particular, a function $G \to \mathcal{H}$, so we can speak about bounded cocycles.

[1] For complex Banach spaces we might have to compose with complex conjugation.

Proposition 1.2. *Let $b \in Z^1(G,\pi)$. Then*

$$\Big(b \in B^1(G,\pi)\Big) \iff \Big(b \text{ is bounded}\Big).$$

Proof. "⇒" If $b(g) = \pi(g)v - v$ for some fixed $v \in \mathcal{H}$ and all $g \in G$, we have $\big\|b(g)\big\| \leq 2\|v\|$.

"⇐" We appeal to the *lemma of the center* (see e.g. lemma 2.2.7 in [2]): every nonempty bounded set B in a Hilbert space has a unique circumball, i.e. a closed ball with minimal radius containing B. Thus if B is invariant under some group of isometries then so is its circumball. It follows that the circumcenter (i.e. the center of the circumball) is also invariant under the group.

Let α be the affine action associated to b (now assumed to be bounded). Then for any $g \in G$ and $v \in \mathcal{H}$ we have $\alpha(g)v = \pi(g)v + b(g)$. The set $b(G)$ is the orbit of $0 \in \mathcal{H}$ under α. As this set is bounded and α-invariant, the circumcenter of $b(G)$ is α-fixed. Thus $b \in B^1(G,\pi)$. □

1.2.2 Remarks and comments. Let us begin with the following theorem:

Theorem 1.3 (Delorme [13], Guichardet [17]). *Let G be a locally compact group. Then*

(1) *If G has Property (T) then every affine isometric action of G on a Hilbert space has a fixed point. In particular $H^1(G,\pi) = \{0\}$ for any unitary representation π.*

(2) *If G is σ-compact then the converse of (1) is true.* □

It is now known that the converse of (1) in the above theorem is not true without the σ-compactness assumption (de Cornulier [8]).

For the second remark we need a definition:

Definition 1.4. A locally compact group G has the *Haagerup property* (or is *a-T-menable*) if G admits a metrically proper affine isometric action α on a Hilbert space $\mathcal{H}$, i.e. such that

$$\forall v \in \mathcal{H} : \lim_{g\to\infty} \big\|\alpha(g)v\big\| = +\infty.$$

Let us remark that an affine isometric action is proper if and only if the norm of the associated cocycle is a proper function (in the sense that the inverse image of a compact set is compact). Indeed, taking the special case of $v = 0$ in Definition 1.4 we see that $\lim_{g\to\infty} \big\|b(g)\big\| = +\infty$. Therefore $g \mapsto \big\|b(g)\big\|$ is a proper function. Conversely if $g \mapsto \big\|b(g)\big\|$ is proper then for any $v \in \mathcal{H}$

$$\big\|\alpha(g)v\big\| = \|\pi(g)v + b(g)\| \geq \Big|\big\|b(g)\big\| - \big\|\pi(g)v\big\|\Big| \to +\infty.$$

for $g \to \infty$.

The class of a-T-menable groups contains σ-compact amenable groups, free groups, Coxeter groups, every closed subgroup of $SO(n,1)$, $SU(n,1)$, etc. ... The interest of this class stems from the following deep result:

Theorem 1.5 (Higson, Kasparov [19]). *A-T-menable groups satisfy the strongest form of the Baum-Connes conjecture, namely the Baum-Connes conjecture with coefficients.* □

1.3 Examples.

1.3.1 Finite-dimensional Hilbert spaces. Let $\mathbb{E}^n$ be n-dimensional Euclidean space. Up to conjugation by a translation, any isometry of $\mathbb{E}^n$ either is a linear isometry (i.e. it has a fixed point) or it is a *helix*, i.e. the composition of a linear isometry and a translation by a vector fixed by the linear isometry.

Let λ be an isometry of $\mathbb{E}^n$ without a fixed point. Then the associated action of $\mathbb{Z}$ (by powers of λ) is proper (of course, if λ had a fixed point the action would not be proper). Moreover there is the following result:

Theorem 1.6 (Bieberbach [5]). *A finitely generated group with a proper isometric action on $\mathbb{E}^n$ is virtually Abelian.* □

1.3.2 Constructing affine actions. Let (X,d) be a metric space with an action of G by isometries. Suppose we have

- a Hilbert space $\mathcal{H}$ with a unitary representation π of G,
- a continuous map $c : X \times X \to \mathcal{H}$ such that
  - $\forall x,y \in X,\ g \in G : c(gx,gy) = \pi(g)c(x,y)$ (equivariance),
  - $\forall x,y,z \in X : c(x,y) + c(y,z) = c(x,z)$ (Chasles' relation),
  - there exists a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|c(x,y)\|^2 = \phi(d(x,y))$ for all $x,y \in X$ (i.e. the norm of $c(x,y)$ depends only on $d(x,y)$).

Then to any $x_0 \in X$ we can associate an affine action α of G on $\mathcal{H}$ with linear part π such that $\|b(g)\|^2 = \phi(d(gx_0,x_0))$ for all $g \in G$. Indeed, we can put $b(g) = c(gx_0,x_0)$. By Chasles' relation and equivariance: $b \in Z^1(G,\pi)$. Moreover the cocycles associated with two different choices of x_0 are cohomologous, i.e. they define the same class in $H^1(G,\pi)$.

If ϕ is a proper function (i.e. $\lim_{t\to\infty} \phi(t) = +\infty$) and G acts properly on X, then b is a proper cocycle and so G is a-T-menable.

Now we give a concrete example of the situation described above. Let $X = (V,E)$ be a tree, i.e. a connected graph without circuit. Let $\mathbb{E}$ be the set of oriented edges in X (each edge appears in $\mathbb{E}$ twice - with both orientations). Let $\mathcal{H} = \ell^2(\mathbb{E})$

and let π be the permutation representation (we assume that a group G acts on X).

For any $x, y \in X$ (or more precisely $x, y \in V$) we have to define the vector $c(x,y) \in \ell^2(\mathbb{E})$. Let $e \in \mathbb{E}$ and let $[x,y]$ be the unique geodesic path from x to y. We let

$$c(x,y)(e) = \begin{cases} 0 & \text{if } e \text{ is not in } [x,y]; \\ +1 & \text{if } e \in [x,y] \text{ and } e \text{ points from } x \text{ to } y; \\ -1 & \text{if } e \in [x,y] \text{ and } e \text{ points from } y \text{ to } x. \end{cases}$$

Chasles' relation follows from the fact that all triangles in a tree are degenerate, i.e. they are *tripods*: if x, y, z are vertices and we take geodesic paths $[x,y]$ and $[y,z]$, then the common part of $[x,y]$ and $[y,z]$ will have to be travelled in both directions, so if e is an oriented edge in this common part then $c(x,y)(e)$ and $c(y,z)(e)$ will cancel out.

Moreover we have $\|c(x,y)\|^2 = 2d(x,y)$.

It follows that groups acting properly on a tree are a-T-menable (such groups are e.g. $\mathbb{F}_n$, $SL(\mathbb{Q}_p)$, etc.).

The above construction extends to groups acting on spaces with walls, CAT(0) cube complexes, spaces with measured walls,...

1.3.3 Infinite-dimensional Hilbert spaces. Contrary to what happens with Euclidean spaces, in infinite-dimensional Hilbert space we can have an "almost recurrent" isometry, i.e. one with unbounded orbits, but such that orbits come back infinitely often within bounded distance from the origin. We shall exhibit one on $\ell^2(\mathbb{N})$, where $\mathbb{N} = \{1, 2, 3, ...\}$.

Let $\mathcal{F}(\mathbb{N}) = \mathbb{C}^{\mathbb{N}}$ (all functions $\mathbb{N} \to \mathbb{C}$). Define a linear operator on $\mathcal{F}(\mathbb{N})$ by

$$(Ua)_n = e^{\frac{2\pi i}{2^n}} a_n$$

for any $(a_n) \in \mathcal{F}(\mathbb{N})$. Note that U has no non-zero fixed vector.

Now let $w = (1, 1, \ldots) \in \mathcal{F}(\mathbb{N})$ and let $\alpha = T_w \circ U \circ T_w^{-1}$, where T_w is the translation by w. This means that

$$\big(\alpha(a)\big)_n = e^{\frac{2\pi i}{2^n}} a_n + \big(1 - e^{\frac{2\pi i}{2^n}}\big)$$

for any $a = (a_n) \in \mathcal{F}(\mathbb{N})$.

The first claim is that $\alpha\big(\ell^2(\mathbb{N})\big) \subset \ell^2(\mathbb{N})$. Indeed, this is the case because the sequence (b_n) with $b_n = 1 - e^{\frac{2\pi i}{2^n}}$, belongs to $\ell^2(\mathbb{N})$.

Proposition 1.7 (Edelstein [15]). *The map $\alpha\big|_{\ell^2(\mathbb{N})}$ is an isometry with unbounded orbits. Moreover there is a constant $R > 0$ such that*

$$\|\alpha^l(0)\| \leq R$$

for infinitely many l's.

Proof. The only fixed point of U is $0 \in \mathcal{F}(\mathbb{N})$, so the only fixed point of α is w which does not belong to $\ell^2(\mathbb{N})$. Therefore α has no fixed point in $\ell^2(\mathbb{N})$. It follows that α has unbounded orbits (Proposition 1.2 and the above dictionary).

Now $\alpha^l(0) = w - U^l w$, so for $l = 2^k$ we have

$$\left\|\alpha^{2^k}(0)\right\|^2 = \sum_{n=1}^{\infty}\left|1 - e^{\frac{2\pi i 2^k}{2^n}}\right|^2 = \sum_{n=k+1}^{\infty}\left|1 - e^{\frac{2\pi i}{2^{(n-k)}}}\right|^2 = \sum_{t=1}^{\infty}\left|1 - e^{\frac{2\pi i}{2^t}}\right|^2,$$

and we can define R as the square root of the sum of the last series above. □

1.3.4 Minimal actions. An action is called *minimal* if it has dense orbits.

Question 1 (A. Navas). *Which finitely generated groups admit an isometric minimal action on an infinite-dimensional Hilbert space?*

Proposition 1.8. *The wreath product $\mathbb{Z}^2 \wr \mathbb{Z} =: (\bigoplus_{\mathbb{Z}} \mathbb{Z}^2) \rtimes \mathbb{Z}$ admits a minimal action on $\ell^2_{\mathbb{R}}(\mathbb{Z})$.*

Proof. First we identify $\mathbb{Z}^2$ with $\mathbb{Z}[\sqrt{2}]$. The latter acts minimally on $\mathbb{R}$ by translation, so $\bigoplus_{\mathbb{Z}} \mathbb{Z}[\sqrt{2}]$ acts minimally by translations on $\ell^2_{\mathbb{R}}(\mathbb{Z})$ (because $\bigoplus_{\mathbb{Z}} \mathbb{R}$ is dense in $\ell^2_{\mathbb{R}}(\mathbb{Z})$). This action is equivariant with respect to the left regular representation of $\mathbb{Z}$, so it extends to an action of the wreath product. □

Theorem 1.9 (see [11]). *Every minimal isometric action of a finitely generated nilpotent group on a Hilbert space is an action by translations on a finite-dimensional Euclidean space.* □

Let us conclude this section with an open question:

Question 2. *Can polycyclic groups act minimally isometrically on an infinite-dimensional Hilbert space?*

2 Amenability and 1-cohomology

Definition 2.1. Let π be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$. We say that π *almost has invariant vectors* if

$$\forall \epsilon > 0, \forall K \Subset G, \exists \xi \in \mathcal{H} : \|\xi\| = 1,\ \sup_{g \in K}\left\|\pi(g)\xi - \xi\right\| < \epsilon.$$

As an example of the use of this notion let us state the following theorem:

Theorem 2.2 (Reiter's property (P_2)). *A locally compact group G is amenable if and only if the left regular representation λ_G on $L^2(G)$ almost has invariant vectors.* □

Theorem 2.3 (Guichardet [17]). *Let G be a σ-compact group and π a unitary representation of G with no non zero fixed vector. Then*

$$\left(\begin{array}{c}\pi \textit{ does not almost} \\ \textit{have invariant vectors}\end{array}\right) \Longleftrightarrow \left(\begin{array}{c}\textit{The space } B^1(G,\pi) \textit{ is closed} \\ \textit{in the space } Z^1(G,\pi)\end{array}\right)$$

Before proving this theorem let us state an immediate corollary.

Corollary 2.4. *If G is σ-compact and non compact then G is non amenable if and only if $B^1(G,\lambda_G)$ is closed in $Z^1(G,\lambda_G)$. In particular for an amenable, σ-compact, non compact group G we have $H^1(G,\lambda_G) \neq \{0\}$.* □

Proof of Theorem 2.3: Because G is σ-compact, $Z^1(G,\pi)$ is a Fréchet space. Consider the coboundary map $\partial : \mathcal{H}_\pi \to Z^1(G,\pi)$, given by $\partial\xi(g) = \pi(g)\xi - \xi$.

We know that

- ∂ is linear,
- ∂ is continuous,
- ∂ is injective (because π has no non zero fixed vectors),
- the image of ∂ is, of course, $B^1(G,\pi)$.

We have the following chain of equivalences:

$$\Big(B^1(G,\pi) \text{ is closed in } Z^1(G,\pi)\Big)$$
$$\Updownarrow$$
$$\Big(\partial^{-1} \text{ is continuous}\Big)$$
$$\Updownarrow$$
$$\Big(\exists C > 0,\ K \Subset G, \forall \xi \in \mathcal{H}_\pi : \|\xi\| \leq C \sup_{g\in K}\big\|\pi(g)\xi - \xi\big\|\Big)$$
$$\Updownarrow$$
$$\Big(\pi \text{ does not almost have invariant vectors}\Big)$$

The first equivalence follows from the closed graph theorem (the version for Fréchet spaces, here we use σ-compactness [2]). The second equivalence follows from the definition of the seminorms defining the topology of $Z^1(G,\pi)$. □

Exercice 1. *Let $\mathbb{R}_d$ denote the the group of real numbers with discrete topology. Show that $\partial : \ell^2(\mathbb{R}) \to Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$ is a continuous isomorphism with discontinuous inverse (i.e. $H^1(G,\lambda_{\mathbb{R}_d}) = \{0\}$).*

Why does it not contradict the closed graph theorem?

[2] Note that the familiar version of the closed graph theorem for Banach spaces does not suffice to prove Theorem 2.3.

3 Property (BP_0)

Definition 3.1. A unitary representation π of a locally compact group G is a *C_0-representation*, or is *mixing* if

$$\forall \xi, \eta \in \mathcal{H}_\pi : \lim_{g \to \infty} \langle \pi(g)\xi | \eta \rangle = 0.$$

Examples:

(1) Any representation of a compact group is C_0.

(2) The regular representation of any locally compact group is C_0.

(3) If G acts on a probability space $(X, \mathcal{B}, \mu)$ in a measure preserving way, then we may consider the associated unitary representation π_X of G on $L^2_0(X, \mu)$, i.e. the orthogonal complement in $L^2(X, \mu)$ of the space of constant functions. We have

$$\Big(\pi_X \text{ is } C_0\Big) \iff \Big(\text{The action of } G \text{ on } X \text{ is mixing}\Big).$$

Recall that an action is mixing if for any $A, B \in \mathcal{B}$ we have

$$\lim_{g \to \infty} \mu(A \cap gB) = \mu(A)\mu(B),$$

i.e. A and gB are asymptotically independent.

Definition 3.2. A locally compact group G has *property* (BP_0) if for every affine isometric action of G on a Hilbert space with C_0 linear part either the action has a fixed point or the action is metrically proper.

Equivalently: G has property (BP_0) if and only if for any C_0-representation π and any $b \in Z^1(G, \pi)$ either b is bounded or b is proper (cf. Proposition 1.2). This explains the origin of the acronym (BP_0): "Bounded", "Proper" and "C_0-representations".

Remark:

(1) Property (T) clearly implies property (BP_0).

(2) The groups $SO(n, 1)$ and $SU(n, 1)$ have property (BP_0) and they do not have property (T) (Shalom [24]).

(3) If H is a closed cocompact subgroup of G and H has property (BP_0) then G has (BP_0). Indeed, cocompactness of H in G guarantees that, if the restriction of $b \in Z^1(G, \pi)$ to H is bounded/proper then b must be bounded/proper.

Theorem 3.3 (see [12]). *Solvable groups have property* (BP_0). □

Corollary 3.4. *Let G be either a connected Lie group or a linear algebraic group over $\mathbb{Q}_p$ (or some other local field of characteristic 0). Then G has property (BP_0).*

Proof. By structure theory, such a group has a cocompact solvable subgroup. □

Theorem 3.3 is proved by induction on the solvability rank of G. The first step is provided by the following Proposition:

Proposition 3.5. *Let G be a locally compact group with non compact center $Z(G)$; then G has (BP_0). In particular every Abelian group has (BP_0).*

Before proving Proposition 3.5, let us mention the following Corollary:

Corollary 3.6 (see [3]). *σ-compact, amenable groups are a-T-menable.*

Proof. If G is σ-compact and amenable then $H = G \times \mathbb{Z}$ is non compact, σ-compact and amenable. Therefore by the second statement in Corollary 2.4 the group $H^1(H, \lambda_H)$ is not trivial. Take $b \in Z^1(H, \lambda_H) \setminus B^1(H, \lambda_H)$. By Proposition 3.5 H has (BP_0), so the cocycle b is proper (because it is not bounded). Thus b remains proper after restriction to G. Therefore G does admit a proper affine isometric action on a Hilbert space. □

Proof of Proposition 3.5: Let π be a C_0-representation of G and let $b \in Z^1(G, \pi)$. Assume that b is not proper. We must prove that b is bounded (cf. Proposition 1.2).
Claim: It is enough to show that $b\big|_{Z(G)}$ is bounded.

Let us first prove that the above claim implies the Proposition. Let α be the action associated to b, so that: $\alpha(g)v = \pi(g)v + b(g)$. If $b\big|_{Z(G)}$ is bounded then the fixed point set $\mathcal{H}^{\alpha\left(Z(G)\right)}$ is not empty. In fact this set consists of one point because if v_0, v_1 are fixed by $\alpha\big(Z(G)\big)$ then $v_0 - v_1$ is fixed under $\pi\big(Z(G)\big)$; thus

$$Z(G) \ni z \longmapsto \langle \pi(z)(v_0 - v_1) | v_0 - v_1 \rangle$$

is a constant C_0 function on the non compact group $Z(G)$. It is therefore identically zero and consequently $v_0 = v_1$ (just evaluate this function at $1 \in Z(G)$).

Moreover, since $Z(G)$ is a normal subgroup of G, we have that $\mathcal{H}^{\alpha\left(Z(G)\right)}$ is α-invariant. Therefore α has a globally fixed point and b is a coboundary. This proves the Proposition.

It remains to prove the Claim, i.e. to show that indeed $b\big|_{Z(G)}$ is bounded. We assumed that b is not proper, so $\liminf_{g \to \infty} \big\| b(g) \big\| = C < +\infty$ in the sense that there is a net in G divergent to infinity (i.e. eventually outside of every compact set) for which the function $g \mapsto \big\| b(g) \big\|$ remains bounded. Now for any $z \in Z(G)$ and $g \in G$ we have

$$\pi(g)b(z) + b(g) = b(gz) = b(zg) = \pi(z)b(g) + b(z)$$

(by the 1-cocycle relation), so that

$$b(z) = \big(1 - \pi(z)\big)b(g) + \pi(g)b(z). \tag{3}$$

Taking scalar product with $b(z)$ of both sides of (3) we obtain

$$\langle b(z)|b(z)\rangle = \langle \big(1 - \pi(z)\big)b(g)|b(z)\rangle + \langle \pi(g)b(z)|b(z)\rangle$$

For fixed z, the absolute value of the first term on the right hand side is smaller than $2\big\|b(g)\big\|\big\|b(z)\big\|$ while the second term tends to 0 when $g \to \infty$. Taking g to infinity of G in such a way that $\big\|b(g)\big\|$ remains bounded we find that

$$\big\|b(z)\big\|^2 \leq 2C\big\|b(z)\big\|$$

and canceling $\big\|b(z)\big\|$ we obtain $\big\|b(z)\big\| \leq 2C$ for any $z \in Z(G)$. □

4 Growth of cocycles

4.1 Generalities. If G is a locally compact compactly generated group and S is a compact and symmetric (i.e. $S = S^{-1}$) generating set for G then we can define the word length function $|\cdot|_S$ on G by

$$|g|_S = \min\big\{n : g = s_1 s_2 \cdots s_n,\ s_i \in S\big\}$$

for any $g \in G$.

Now let π be a unitary representation of G and let $b \in Z^1(G, \pi)$. We consider the following question:

Question 3. *How fast does $\big\|b(g)\big\|$ grow with respect to $|g|_S$?*

We first observe that $\big\|b(g)\big\|$ grows at most linearly:

Lemma 4.1. *We have $\big\|b(g)\big\| = O\big(|g|_S\big)$. More precisely*

$$\big\|b(g)\big\| \leq \Big(\max_{s\in S}\big\|b(s)\big\|\Big) \cdot |g|_S. \tag{4}$$

Proof. Let us first remark that whenever G acts by isometries on a metric space (X, d) then for every $x_0 \in X$:

$$d(gx_0, x_0) \leq \Big(\max_{s\in S} d(sx_0, x_0)\Big) \cdot |g|_S. \tag{5}$$

Indeed, for $g = s_1 s_2 \cdots s_n$ with $n = |g|_S$ we have

$$d(gx_0, x_0) = d(s_1 s_2 \cdots s_n x_0, x_0)$$

$$\leq d(s_1 s_2 \cdots s_n x_0, s_1 s_2 \cdots s_{n-1} x_0) + d(s_1 s_2 \cdots s_{n-1} x_0, x_0)$$

$$= d(s_n x_0, x_0) + d(s_1 s_2 \cdots s_{n-1} x_0, x_0)$$

$$\leq d(s_n x_0, x_0) + d(s_1 s_2 \cdots s_{n-1} x_0, s_1 s_2 \cdots s_{n-2} x_0) + d(s_1 s_2 \cdots s_{n-2} x_0, x_0)$$

$$= d(s_n x_0, x_0) + d(s_{n-1} x_0, x_0) + d(s_1 s_2 \cdots s_{n-2} x_0, x_0)$$

$$\vdots$$

$$\leq d(s_n x_0, x_0) + d(s_{n-1} x_0, x_0) + \cdots + d(s_1 x_0, x_0)$$

$$\leq n\Big(\max_{s \in S} d(s x_0, x_0)\Big)$$

because the action of G is isometric.

Now let α be the affine isometric action of G on $\mathcal{H}$ associated to b. Using (5) with $X = \mathcal{H}$ and $x_0 = 0$ we obtain precisely (4). $\square$

A 1-coboundary is bounded as a function on G, so a limit of 1-coboundaries should not grow too fast. Next lemma makes this precise.

Lemma 4.2. *If $b \in \overline{B^1(G,\pi)}$ then $\|b(g)\| = o(|g|_S)$, i.e.*

$$\frac{\|b(g)\|}{|g|_S} \to 0$$

for $|g|_S \to \infty$ (in this case, we say that b has sub-linear growth.)

Proof. Fix $\epsilon > 0$. There exists $b' \in B^1(G,\pi)$ such that

$$\max_{s \in S} \|b(s) - b'(s)\| < \frac{\epsilon}{2}$$

(recall that $Z^1(G,\pi)$ carries the topology of uniform convergence on compact sets). Therefore

$$\frac{\|b(g)\|}{|g|_S} \leq \frac{\|b(g) - b'(g)\|}{|g|_S} + \frac{\|b'(g)\|}{|g|_S} \leq \frac{\epsilon}{2} + \frac{\|b'(g)\|}{|g|_S},$$

where in the last inequality we simply used (4) with b replaced by $b - b'$.

Now b' is a coboundary, so by Proposition 1.2 it is bounded and

$$\frac{\|b'(g)\|}{|g|_S} < \frac{\epsilon}{2}$$

for sufficiently large $|g|_S$. $\square$

4.2 Application: a new look at an old proof.

Theorem 4.3 (Von Neumann's mean ergodic theorem). *Let U be a unitary operator on a Hilbert space $\mathcal{H}$. Then for any $v \in \mathcal{H}$ we have*

$$\lim_{n\to\infty} \frac{1}{n}\big(1 + U + U^2 + \cdots + U^{n-1}\big)v = Pv,$$

where P is the orthogonal projection onto $\ker(U-1)$, and the convergence is in the norm of $\mathcal{H}$.

Proof. Let us define a unitary representation π of $\mathbb{Z}$ on $\mathcal{H}$ by $\pi(n) = U^n$. Also let $b \in Z^1(\mathbb{Z}, \pi)$ be the unique cocycle with $b(1) = v$. Using the cocycle relation (1) we find that

$$\begin{aligned} b(n) &= b\big((n-1)+1\big) = U^{n-1}b(1) + b(n-1) \\ &= U^{n-1}b(1) + U^{n-2}b(1) + b(n-2) \\ &\;\;\vdots \\ &= U^{n-1}b(1) + U^{n-2}b(1) + \cdots + Ub(1) + b(1) \\ &= \big(1 + U + U^2 + \cdots + U^{n-1}\big)v. \end{aligned}$$

Let $\mathcal{H}_1 = P\mathcal{H}$ and $\mathcal{H}_0 = \mathcal{H}_1^\perp$. We have $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0$ and both subspaces are invariant under U. Let π_1 and π_0 be corresponding subrepresentations of π. Furthermore let

$$\begin{aligned} b_1(n) &= P\,b(n), \\ b_0(n) &= (1-P)b(n). \end{aligned}$$

Then $b_i \in Z^1(\mathbb{Z}, \pi_i)$ for $i = 1, 0$. On $\mathcal{H}_1$ the operator U acts as the identity, so

$$b_1(n) = P(1 + U + U^2 + \cdots + U^{n-1}\big)v = (1 + U + U^2 + \cdots + U^{n-1}\big)Pv = nPv.$$

Therefore $\frac{1}{n}b_1(n) = Pv$. On the other hand, we have

$$\mathcal{H}_0 = \ker(U-1)^\perp = \ker(U^*-1)^\perp = \overline{\operatorname{ran}(U-1)}$$

(indeed $U\xi = \xi$ if and only if $U^*\xi = \xi$). This means that $b_0(1) = (1-P)v$ is the limit of a sequence $(U-1)\xi_n$ for some $\xi_n \in \mathcal{H}$. It is easy to see that for each fixed $k \in \mathbb{Z}$ the vector $b_0(k)$ is the corresponding limit of $(\partial\xi_n)(k)$ (where ∂ was defined in the proof of Theorem 2.3), so b_0 is in the closure of $B^1(G, \pi_0)$ in the topology of uniform convergence on compact subsets of $\mathbb{Z}$. By Lemma 4.2 we have

$$\frac{\|b_0(n)\|}{n} \to 0.$$

for $n \to \infty$. This proof was originally due to F. Riesz. ∎

Exercice 2. *Let α be an affine isometry of a Hilbert space $\mathcal{H}$. Prove that*

$$\inf_{w\in\mathcal{H}} \|\alpha(w) - w\| = \lim_{n\to\infty} \frac{\|\alpha^n(0)\|}{n}$$

That quantity is called the drift *of α.*

The next exercise is a recap on Sections 1 and 4.

Exercice 3. *Let α be an affine isometry of a Hilbert space $\mathcal{H}$. For any $\xi \in \mathcal{H}$ we have*

$$\alpha(\xi) = U\xi + v$$

where U is a unitary operator and $v \in \mathcal{H}$ is a fixed vector. Let b be the cocycle on $\mathbb{Z}$ with $b(1) = v$. Prove that

(1) *the following are equivalent:*

  (a) *α has a fixed point,*
  (b) *$v \in ran(U - 1)$,*
  (c) *b is bounded;*

(2) *the following are equivalent:*

  (a) *α almost has a fixed point, but no fixed point,*
  (b) *$v \in \overline{ran(U - 1)} \setminus ran(U - 1)$,*
  (c) *b is unbounded with $\big\|b(n)\big\| = o(n)$;*

(3) *the following are equivalent:*

  (a) *α does not almost have a fixed point,*
  (b) *$v \notin \overline{ran(U - 1)}$,*
  (c) *$\exists C > 0 : \big\|b(n)\big\| \geq C|n|$.*

Let us comment that part (3) of Exercise 3 is analogous to the finite-dimensional situation of 1.3.1. Edelstein's example (Proposition 1.7) falls under case (2).

5 Applications to geometric group theory

5.1 Uniform embeddings.

Definition 5.1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a map.

(1) f is a *uniform embedding* if there exist *control functions* $\rho_+, \rho_- : \mathbb{R}^+ \to \mathbb{R}$ such that $\lim_{r\to+\infty} \rho_\pm(r) = +\infty$ and

$$\forall x_1, x_2 \in X : \rho_-\big(d_X(x_1, x_2)\big) \leq d_Y\big(f((x_1), f(x_2)\big) \leq \rho_+\big(d_X(x_1, x_2)\big).$$

(2) f is a *quasi-isometric embedding* if f is a uniform embedding for which the functions $\rho_\pm$ can be chosen to be affine functions.

(3) f is a *quasi-isometry* if f is a quasi-isometric embedding and there exists a quasi-isometric embedding $g : Y \to X$ such that $f \circ g$ is a bounded distance away from Id_X and $g \circ f$ is a bounded distance away from Id_Y.

An example of a quasi-isometry is the map $\mathbb{R} \to \mathbb{Z} : x \mapsto [x]$. Thus a quasi-isometry need not be an isometry nor even a continuous map. Similarly a quasi-isometric or uniform embedding need not be continuous nor an embedding.

It is not difficult to see that, among finitely generated groups, being of polynomial growth is an invariant of quasi-isometry. By a deep result of Gromov [18], a finitely generated group has polynomial growth if and only if it is virtually nilpotent; from this we deduce immediately:

Theorem 5.2. *For finitely generated groups, being virtually nilpotent is an invariant of quasi-isometry.* □

The following corollary of Gromov's theorem is due to Gersten [16]; we will give a proof below.

Corollary 5.3 (quasi-isometric rigidity of $\mathbb{Z}^n$). *If G is a finitely generated group quasi-isometric to $\mathbb{Z}^n$ then G contains $\mathbb{Z}^n$ as a finite index subgroup.*

The next result is at first sight unrelated to previous statements, but we will see that in fact it is!

Theorem 5.4 (Bourgain [6]). *The 3-regular tree T_3 does not embed quasi-isometrically into a Hilbert space.*

Other results on quasi-isometry invariants for finitely generated groups include:

Theorem 5.5 (Dyubina [14]). *Being virtually solvable is not a quasi-isometry invariant property.* □

Question 4. *Is being virtually polycyclic a quasi-isometry invariant?*

Question 4 is open. The following definition is due to Shalom [25]:

Definition 5.6. Let G be a locally compact group. We say that G belongs to the class $(AmenH_{FD})$ if

(1) G is amenable,

(2) if a unitary representation π of G satisfies $\overline{H^1}(G, \pi) \neq \{0\}$ then π contains a finite-dimensional subrepresentation.

The acronym $(AmenH_{FD})$ stands for "Amenable", "coHomology" and "Finite Dimension".

Theorem 5.7 (Shalom [25]).

(1) *The following groups are in the class ($AmenH_{FD}$):*

- *connected solvable Lie groups,*
- *virtually polycyclic groups,*
- *semi direct products $\mathbb{Q}_p \rtimes \mathbb{Z}$ (where $\mathbb{Q}_p$ is the field of p-adic numbers and $\mathbb{Z}$ acts on its additive group by multiplication by powers of p),*
- *lamplighter groups, i.e. groups of the form $F \wr \mathbb{Z}$, where F is a finite group.*

(2) *For finitely generated groups, being in ($AmenH_{FD}$) is a quasi-isometry invariant.*

(3) *A finitely generated infinite group in ($AmenH_{FD}$) admits a finite index subgroup which surjects onto $\mathbb{Z}$.* □

An immediate consequence of Theorem 5.7 (3) is:

Corollary 5.8. *A group quasi-isometric to a polycyclic group virtually surjects onto $\mathbb{Z}$.* □

We address the following question:

Question 5. *Which compactly generated groups admit a quasi-isometric embedding into a Hilbert space?*

The group $\mathbb{Z}^n$ acts by translations on $\mathbb{E}^n$. The choice of any orbit gives a quasi-isometric embedding of $\mathbb{Z}^n$ into $\mathbb{E}^n$. More generally any closed subgroup of $Isom(\mathbb{E}^n)$ embeds quasi-isometrically into $\mathbb{E}^n$. It does not seem to be easy to find other examples.

Remark: There are some negative results concerning quasi-isometric embeddings. For example the following:

Theorem 5.9 (Cheeger-Kleiner [7]). *The discrete Heisenberg group does not embed quasi-isometrically into ℓ^1.*

Of course ℓ^1 is not a Hilbert space, but we mention this result here because, in conjunction with a result of Lee-Naor [20] it solved negatively the *Goemans-Linial conjecture*, a conjecture coming from theoretical computer science. In passing, non-embeddings results for ℓ^1 are usually harder, as they imply non-embedding results in ℓ^2 (Reason: ℓ^2 embeds linearly isometrically into ℓ^1).

The following conjecture appears in [10].

Conjecture 5.10. *A compactly generated group which embeds quasi-isometrically into a Hilbert space admits a proper isometric action on a finite-dimensional Euclidean space. In particular, because of Bieberbach's theorem (Theorem 1.6), if G is finitely generated then it should be virtually Abelian.*

Remark:

(1) A non amenable finitely generated group cannot embed quasi-isometrically into a Hilbert space. This is because of a deep result of Benjamini-Schramm [4] which says that the Cayley graphs of such a group contains a quasi-isometrically embedded copy of the 3-regular tree, and Bourgain's theorem (Theorem 5.4).

(2) A finitely generated solvable group which is not virtually nilpotent cannot be embedded quasi-isometrically into a Hilbert space. The reason for this is a result of de Cornulier-Tessera [9] that such a group contains a quasi-isometrically embedded copy of the free semigroup on two generators, together with Bourgain's result.

Theorem 5.11 (see [10]). *Conjecture 5.10 holds for compactly generated groups in $(AmenH_{FD})$.*

In particular we have

Corollary 5.12. *A virtually polycyclic group embeds quasi-isometrically into a Hilbert space if and only if it is virtually Abelian.* □

Compare this with the following result:

Theorem 5.13 (Pauls [21]). *A virtually nilpotent group embeds quasi-isometrically into a* CAT(0) *space if and only if it is virtually Abelian.* □

Compared to Corollary 5.12, Theorem 5.13 holds for a smaller class of groups, but for a larger class of actions (observe that Hilbert spaces are CAT(0), in fact they are prototypical examples of such spaces). The proofs are quite different.

Let us show that Theorem 5.11 implies both Corollary 5.3 and Theorem 5.4.

Proof of Corollary 5.3: $\mathbb{Z}^n$ is in the class $(AmenH_{FD})$. Therefore so is G by Theorem 5.7 (2). Also $\mathbb{Z}^n$ embeds quasi-isometrically into a Hilbert space, thus so does G. By Theorem 5.11, the group G is virtually Abelian, so G has $\mathbb{Z}^m$ as a finite-index subgroup. To conclude that $m = n$ we consider growth which on one hand is a quasi-isometry invariant and on the other hand detects the rank of $\mathbb{Z}^k$. □

Observe that the latter proof is independent of Gromov's Theorem 5.2.

Proof of Theorem 5.4: The idea behind the proof is the following: first view $\mathbb{Q}_2 \rtimes \mathbb{Z}$ as a subgroup of the affine group $\mathbb{Q}_2 \rtimes \mathbb{Q}_2^\times$, and embed the latter in $GL_2(\mathbb{Q}_2)$ in the standard way. It is known (see Serre's book on trees [23]) that there is an action of $GL_2(\mathbb{Q}_2)$ on the 3-regular tree T_3. One can show that the action of $\mathbb{Q}_2 \rtimes \mathbb{Z}$ on T_3 is proper and co-compact, so that T_3 is quasi-isometric to $\mathbb{Q}_2 \rtimes \mathbb{Z}$. This last group is in $(AmenH_{FD})$. By Theorem 5.11, all we need to do is show that $\mathbb{Q}_2 \rtimes \mathbb{Z}$ cannot act properly and isometrically on a finite dimensional Euclidean space.

Such an action would be a homomorphism $\mathbb{Q}_2 \rtimes \mathbb{Z} \to Isom(\mathbb{E}^n)$ and by properness it would have a compact kernel. But the only compact normal subgroup of $\mathbb{Q}_2 \rtimes \mathbb{Z}$ is $\{1\}$, so $\mathbb{Q}_2 \rtimes \mathbb{Z}$ would have to embed into the Lie group $Isom(\mathbb{E}^n)$. But Lie groups don't have small subgroups (i.e. there exists a neighborhood of the identity not containing any non-trivial subgroup), and an embedding of $\mathbb{Q}_2 \rtimes \mathbb{Z}$ would contradict that. □

5.2 Ideas on how to prove Theorem 5.11.

Theorem 5.14 (Schönberg [22]). *Let X be a set and let $\psi : X \times X \to \mathbb{R}^+$ be a kernel, symmetric and vanishing on the diagonal. Let $\mathcal{H}$ be a Hilbert space. There exists a map $f : X \to \mathcal{H}$ such that $\psi(x,y) = \|f(x) - f(y)\|^2$ if and only if ψ is conditionally negative definite, i.e. for any $n \in \mathbb{N}$, any $x_1, \dots, x_n \in X$ and any $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^n \lambda_i = 0$ we have*

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \psi(x_i, x_j) \leq 0.$$

Moreover if a group G acts on X and ψ is G-invariant then f can be taken to be G-equivariant with respect to some isometric affine action of G on $\mathcal{H}$. □

The following remarkable result allows, in the amenable case, to convert a purely metric information into a very strong algebraic information.

Lemma 5.15 (Aharoni-Maurey-Mityagin [1], see also Proposition 4.4 in [10]). *Let G be a compactly generated and amenable group. Let f be a uniform embedding of G into a Hilbert space $\mathcal{H}$ with control functions $\rho_\pm$. Then there exists a constant $A \geq 0$ (which can be taken equal to 0 if G is discrete) and an equivariant uniform embedding $\widetilde{f}$ of G into a Hilbert space $\widetilde{\mathcal{H}}$ endowed with an affine isometric action of G, such that $\tilde{f}$ has control functions $\rho_- - A$ and $\rho_+ + A$.*

Proof for G discrete: Set $\psi(x,y) = \|f(x) - x(y)\|^2$. We have

$$\rho_-\big(|x^{-1}y|_S\big)^2 \leq \psi(x,y) \leq \rho_+\big(|x^{-1}y|_S\big)^2. \tag{6}$$

Fix $x, y \in G$ and consider the function

$$u_{xy} : G \ni g \longmapsto \psi(gx, gy).$$

It is bounded by the second inequality of (6). Let m be an invariant mean on $\ell^\infty(G)$ and define

$$\widetilde{\psi}(x,y) = m(u_{xy}).$$

The function $\widetilde{\psi} : G \times G \to \mathbb{R}^+$ is then G-invariant and we have

$$\rho_-\big(|x^{-1}y|_S\big)^2 \leq \widetilde{\psi}(x,y) \leq \rho_+\big(|x^{-1}y|_S\big)^2.$$

Moreover $\widetilde{\psi}$ is conditionally negative definite: for $x_1, ..., x_n \in G$, $\lambda_1, ..., \lambda_n \in \mathbb{R}$, $\sum_{i=1}^n \lambda_i = 0$:

$$\sum_{i,j} \lambda_i \lambda_j \widetilde{\psi}(x_i, x_j) = \sum_{i,j} \lambda_i \lambda_j m(u_{x_i x_j}) = m(\sum_{i,j} \lambda_i \lambda_j u_{x_i, x_j});$$

now $\sum_{i,j} \lambda_i \lambda_j u_{x_i,x_j}$ is a non-positive function, as ψ is conditionally negative definite, and therefore $m(\sum_{i,j} \lambda_i \lambda_j u_{x_i,x_j}) \leq 0$. It remains to apply Schönberg's theorem 5.14.

□

Theorem 5.11 is then proved along the following steps:

- Let G be a compactly generated group in $(AmenH_{FD})$ and let f be a quasi-isometric embedding of G into a Hilbert space. By lemma 5.15, we may assume that f is equivariant with respect to an affine isometric action α.

- Write $\alpha(g)v = \pi(g)v + b(g)$. Up to conjugating α by a translation, we may assume $f(1) = 0$ so that, by equivariance, we get $f = b$; observe that b has *linear growth*, i.e. there exists $C > 0$ such that $\|b(g)\| \geq C|g|_S$ for $|g|_S$ large enough. In particular the 1-cocycle b is not in $\overline{B}^1(G, \pi)$, by lemma 4.2. By the definition of $(AmenH_{FD})$, the representation π has a finite-dimensional invariant subspace. Projecting b orthogonally onto this subspace provides an affine isometric action of G on a finite-dimensional Euclidean space. This action may not be proper however: we may have to enlarge the space to make it proper. This is achieved as follows.

- Let σ be a sub-representation of π which decomposes as a direct sum of finite-dimensional invariant subspaces, and is maximal with respect to that property; let $\sigma^\perp$ be the representation on the orthogonal subspace, so that $\pi = \sigma \oplus \sigma^\perp$. The representation $\sigma^\perp$ has no finite-dimensional invariant subspace, by maximality of σ. By property $(AmenH_{FD})$, this implies $\overline{H}^1(G, \sigma^\perp) = 0$. If $b = b_\infty + b^\perp$ is the decomposition of b corresponding to $\pi = \sigma \oplus \sigma^\perp$, then $b^\perp \in Z^1(G, \sigma^\perp)$ has sub-linear growth, by lemma 4.2. Since b has linear growth, so has b_∞. At this point we have shown that G admits a 1-cocycle b_∞ with linear growth, with respect to a representation σ which is a direct sum of finite-dimensional representations.

- Write $\sigma = \bigoplus_{i=1}^{\infty} \tau_i$, with τ_i a finite-dimensional representation, and $\sigma_N = \bigoplus_{i=1}^{N} \tau_i$. Let b_N be the orthogonal projection of b_∞ on the subspace of σ_N. Then $\lim_{N\to\infty} b_N = b_\infty$, uniformly on compact subsets of G. Observe that the set of 1-cocycles with linear growth is *open* in $Z^1(G, \sigma)$ (for the topology of uniform convergence on compact subsets). So b_N has linear growth for N large enough. The corresponding affine action $\alpha_N(g)v = \sigma_N(g)v + b_N(g)$ defines an affine isometric action on a finite-dimensional Euclidean space; for N large enough this action is proper, because a 1-cocycle with linear growth is clearly proper. □

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Solutions of exercises

Solution of Exercise 1: The regular representation of a non compact group does not have non zero fixed vectors, so ∂ is injective and continuous by the reasoning in the proof of Theorem 2.3. It remains to show that ∂ maps $\ell^2(\mathbb{R})$ onto $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$.

Let us skip ahead to the result that every Abelian group has property (BP_0) (it follows from Proposition 3.5). This means that $\mathbb{R}_d$ must have (BP_0). So if b is in $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$ then it must be either bounded (i.e. lie in $B^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$) or

$$\mathbb{R}_d \ni t \longmapsto \|b(t)\|$$

must be a proper function (preimage of a compact set is compact). Observe that existence of a proper continuous function on a locally compact space implies σ-compactness. Therefore there are no non zero proper cocycles (ones whose norm is a proper function). Therefore, by property (BP_0), there are no nontrivial cocycles in $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$. This means that ∂ maps $\ell^2(\mathbb{R})$ onto $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$.

To see that ∂^{-1} is not continuous, look at the proof of Theorem 2.3 and observe that, since every Abelian group is amenable, the regular representation $\lambda_{\mathbb{R}_d}$ almost has invariant vectors in $\ell^2(\mathbb{R}_d)$.

This does not contradict the closed graph theorem because $Z^1(\mathbb{R}_d, \lambda_{\mathbb{R}_d})$ is not a Fréchet space: indeed uncountably many seminorms are needed to define its topology. □

Solution of Exercise 2: For $w \in \mathcal{H}$, write $\alpha(w) = Uw + v$. Then, as in the proof of von Neumann's ergodic theorem, $\alpha^n(0) = (1 + U + ... + U^{n-1})v$, so that by this theorem $\lim_{n\to\infty} \frac{\|\alpha^n(0)\|}{n} = \|P(v)\|$, where P is the orthogonal projection onto $\ker(U-1)$. On the other hand

$$\inf_{w\in\mathcal{H}} \|\alpha(w) - w\| = \inf_{w\in\mathcal{H}} \|(U-1)w + v\| = \inf_{z\in \operatorname{ran}(U-1)} \|z + w\|$$

$$= dist(\operatorname{ran}(U-1), v) = \|P(v)\|$$

because the orthogonal of $\operatorname{ran}(U-1)$ is $\ker(U-1)$. □

Solution of Exercise 3: As in the proof of Theorem 4.3 the isometry α defines a representation π of $\mathbb{Z}$ by $\pi(n) = U^n$, where U is the linear part of α.

Now let us turn to the following observation: the map

$$\Psi : Z^1(\mathbb{Z}, \pi) \ni b \longmapsto b(1) \in \mathcal{H}$$

is an isomorphism of topological vector spaces. Indeed, any vector can be a value of a cocycle at the point $1 \in \mathbb{Z}$ and this value determines the cocycle uniquely (cf. proof of Theorem 4.3). This shows that Ψ is an isomorphism. Moreover the topology on $Z^1(\mathbb{Z}, \pi)$ is the topology of pointwise convergence (and value of a cocycle at any point $n \in Z$ is given by applying a fixed bounded operater to its value at $1 \in \mathbb{Z}$). This shows that Ψ is a homeomorphism.

It is easy to see that $\Psi\big(B^1(\mathbb{Z}, \pi)\big) = \operatorname{ran}(U-1)$. Thus also $\Psi\big(\overline{B^1(\mathbb{Z}, \pi)}\big) = \overline{\operatorname{ran}(U-1)}$.

Now recall the dictionary presented in Section 1 to see that we have the equivalences

$$(1a) \Longleftrightarrow (1b), \qquad (2a) \Longleftrightarrow (2b), \qquad (3a) \Longleftrightarrow (3b).$$

In order to have the whole exercise wrapped up we need one more remark, namely that if $v \notin \overline{\operatorname{ran}(U-1)}$ then we have $Pv \neq 0$, where P is the projection onto $\ker(U-1)$. Moreover by von Neumann's mean ergodic theorem we have

$$\frac{1}{n} b(n) \to Pv,$$

for $n \to \infty$, so $\big\|b(v)\big\| \geq Cn$ for some constant $C > 0$ (e.g. $C = \frac{1}{2}\|Pv\|$).

Now we can finish the solution of our exercise. Equivalence between (1c) and (1a) is the content of Proposition 1.2.

From Lemma 4.2 we see that (3c) implies (3a) and (3b), and by the remark above (3b) implies (3c).

Finally by Proposition 1.2 and Lemma 4.2 we know that (2c) follows from (2a) and/or (2b). Conversely if (3c) is satisfied then b cannot be a coboundary (because it is unbounded), but $v = b(1)$ cannot at the same time lie outside $\overline{\mathrm{ran}(U-1)}$ (again by the remark above). □

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