Discretization of Riemannian manifolds applied to the Hodge Laplacian

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Abstract

For $\kappa \geq 0$ and $r_0 > 0$, let $\mathbb{M}(n, \kappa, r_0)$ be the set of all connected compact n-dimensional Riemannian manifolds such that $|K_g| \leq \kappa$ and $Inj(M, g) \geq r_0$. We study the relation between the k^{th} positive eigenvalue of the Hodge Laplacian on differential forms and the k^{th} positive eigenvalue of the combinatorial Laplacian associated to an open cover (acting on Čech cochains). We show that for a fixed sufficiently small $\varepsilon > 0$ there exist positive constants c_1 and c_2 depending only on n, κ and ε such that for any $M \in \mathbb{M}(n, \kappa, r_0)$ and for any ε-discretization X of M we have $c_1 \lambda_{k,p}(X) \leq \lambda_{k,p}(M) \leq c_2 \lambda_{k,p}(X)$ for any $k \leq K$ (K depends on X). Moreover, we find a lower bound for the spectrum of the combinatorial Laplacian and a lower bound for the spectrum of the Hodge Laplacian.

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1 Introduction

Several works like [3], [4], [5] and more recently [24] have shown that discretizing a Riemannian manifold may be really powerful in order to study the spectrum of the Laplacian acting on functions. The question we want to answer here is "Is there a similar tool for understanding the spectrum of the Hodge Laplacian ($\Delta = dd^* + d^*d$) acting on differential forms?". Part of an answer is given by the de Rham Theorem (saying that the de Rham cohomology of a compact manifold is isomorphic to the singular cohomology and to the Čech cohomology) and several authors have been more or less

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inspired by this theorem to study the spectrum of Δ. For instance, in [16], Dodziuk and Patodi show that for a fixed compact Riemannian manifold, we can approximate the spectrum of the Hodge Laplacian with the spectrum of a combinatorial Laplacian associated to finer and finer triangulations of the manifold. The main idea in their proof is to associate Čech cochains to smooth forms and vice versa via the integration on simplices and via the Whitney map. Both tools are really crucial in the proof of the de Rham Theorem as they induce the isomorphism between de Rham cohomology and singular cohomology. In [7] and in [25], the authors use another proof of de Rham Theorem due to A. Weil and based on the Čech - de Rham double complexe (see [18]). In [7], Chanillo and Trèves bound from below the smallest non-zero eigenvalue of the Hodge Laplacian on p-forms for a compact Riemannian manifold with bounded sectional curvature, while the purpose of [25] is to study the spectrum of Δ on compact hyperbolic 3-dimensional manifolds. In particular, McGowan develops in [25] a quite general method to bound from below "small" eigenvalues of Δ on compact manifolds (Lemma 2.3 in [25]).

The purpose of this paper is in some sense to improve or to unify these results in the context given by the discretization. More precisely, if M is a compact Riemannian manifold and if X is a discretization of M (in the sense of [8]), we obtain naturally from X a finite open cover $\mathcal{U}_X$ which will be contractible if the mesh of the discretization is sufficiently small. To such an open cover we can associate the complex of Čech cochains naturally endowed with a coboundary operator δ. Moreover, with an inner product on Čech cochains, we can construct the adjoint of δ, namely δ^* and define the following combinatorial Laplacian $\check{\Delta} = \delta\delta^* + \delta^*\delta$.

The main result consists in establishing a uniform comparison between the spectrum of the Hodge Laplacian and the spectrum of such a combinatorial Laplacian. That is to say, if $\mathbb{M}(n, \kappa, r_0)$ denotes the set of compact connected Riemannian manifolds with bounded (by κ) sectional curvature and injectivity radius bounded from below by r_0, we show that there exists a positive constant ρ_0 depending only on n, κ and r_0 such that if we fix $0 < 3\varepsilon < \rho_0$, there exist positive constants c_1 and c_2 depending only on n, p, κ and ε such that for any $M \in \mathbb{M}(n, \kappa, r_0)$ and for any ε-discretization X of M we can compare the k^{th} eigenvalue of Δ on p-forms to the k^{th} eigenvalue of $\check{\Delta}$ on Čech p-cochains (for $1 \leq p \leq n-1$) in the following way

$$c_1\lambda_{k,p}(X) \leq \lambda_{k,p}(M) \leq c_2\lambda_{k,p}(X)$$

for any $k \leq K$ and K depends on X (see Theorem 3.1 for the precise statement).

As an application of Theorem 3.1, we obtain a lower bound for the first non-zero eigenvalue of Δ (see Theorem 4.1) in terms of the volume of the manifold. This result has to be compared with the result obtained by Chanillo and Trèves (Theorem 1.1, in [7]). In their proof, the authors use in a crucial manner a lemma due to Trèves (Lemma A.5 in [31]) which turns out to be false (see Remark 4.3). In Lemma 4.2, we state and prove a "weaker" version of Trèves' lemma. A direct corollary of this lemma is a lower bound for the spectrum of the combinatorial Laplacian (see Theorem 4.4) and so, thanks to Theorem 3.1, a lower bound for the spectrum of Δ (see Theorem 4.1).

As another consequence of the proof of Theorem 3.1, we obtain a version of McGowan's lemma (Lemma 2.3 in [25]) slightly more general as it is concerned with p-forms on compact Riemannian manifolds with bounded sectional curvature, but not so general as it is valid only for contractible open covers (see Lemma 4.5). Finally, another interesting application of the method developed here concerns Whitney forms. Indeed, Whitney forms come out in [16] as a natural way to smooth Čech cochains. Nevertheless, in order to keep a uniform comparison of the spectra, the results given in [16] on Whitney forms are not useful to our purpose. Hence, we obtain as a corollary of the method, the appropriate results to show that Whitney forms are even so a suitable tool to smooth Čech cochains (see Section 4.2).

The paper is organized as follows. In Section 2, we begin by recalling several definitions and properties of differential forms and Čech cochains. In particular, in Section 2.3, we sketch the proof of the de Rham Theorem due to A. Weil as it will be the starting point of the proof of Theorem 3.1. Finally, we recall the definition of a discretization and its main properties.

Section 3 is devoted to the proof of Theorem 3.1. The basic idea of the proof is to associate a Čech cochain to a differential form via a discretizing operator and vice versa via a smoothing operator, in order to compare "small" eigenvalues. These operators are essentially constructed as in the proof (of A. Weil) of the de Rham Theorem thanks to the Čech - de Rham double complexe. To that aim, we need a few technical results. In particular, we need a normed version of the Poincaré Lemma and a similar result for Čech cochains. This is done in Lemma 3.2 and in Lemma 3.5. Moreover, as in [25], it is necessary to bound from below the spectrum of Δ with absolute boundary conditions on finite intersections of open sets of the open cover. To that aim, we show that for a sufficiently small ε, the intersection of balls of radius ε is convex and is quasi-isometric to a Euclidean convex. Thanks to a result of Guerini ([19]) we can then bound from below the spectrum of such intersections (this appears in Section 2 as properties of the discretization, see Lemma 2.9 and Lemma 2.10). Note that Chanillo and Trèves met also this problem and they solve it using a (finite) sequence of open covers and with Lemma 2.2 in [7] (which is a consequence of a normed version of the Poincaré Lemma in the Euclidean setting). For "large" eigenvalues, it suffices to have an upper bound for the k^{th} eigenvalue of Δ and of $\check{\Delta}$ to have the claim.

In Section 4, we present the consequences of Theorem 3.1 mentioned above.

Finally, in the appendix we recall the (more or less classical) definition and the properties of Whitney forms. At the end of the appendix, we give the proof of the technical lemma about the Euclidean convexity of the intersection of small balls.

2 Settings

In this section, we recall some definitions and basic facts on the Laplacian acting on differential forms and on the Laplacian acting on Čech cochains. For the convenience of the reader and as it is a key tool for the paper, a paragraph is also devoted to the sketch of a classical proof due to A. Weil of the de Rham Theorem (for contractible open covers) relying on the Čech - de Rham double complexe (see for instance Appendix A of [18] or Chapter 3 of [28]). Finally, we define the discretization of a manifold and discuss some of its properties.

2.1 Laplacian acting on differential forms

Let (M^n, g) be a compact connected n-dimensional Riemannian manifold without boundary. Denote by $\Lambda^p(M)$ the vector space of smooth differential p-forms, for $0 \leq p \leq n$. Let $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$ be the exterior differential and $d^* : \Lambda^{p+1}(M) \to \Lambda^p(M)$ its formal adjoint (with respect to the L^2-inner product) the codifferential. Then the Laplacian acting on p-forms is defined by $\Delta : \Lambda^p(M) \to \Lambda^p(M)$, $\Delta = dd^* + d^*d$. The spectrum of Δ is discrete and will be denoted by

$$0 < \lambda_{1,p}(M) \leq \lambda_{2,p}(M) \leq \ldots \leq \lambda_{k,p}(M) \leq \ldots$$

where 0 is of multiplicity $b_p(M)$ and the positive eigenvalues are repeated as many times as their multiplicity. Let us recall that half of the spectrum is redundant. That is to say, if $\lambda > 0$ is an eigenvalue of Δ on p-forms and if $E_p(\lambda)$ denotes the λ-eigenspace, then $E_p(\lambda)$ splits as follows $E_p(\lambda) = E_p^{d^*}(\lambda) \oplus E_p^d(\lambda)$ where $E_p^{d^*}(\lambda) = \{\omega \in E_p(\lambda) : d^*\omega = 0\} \subseteq d^*\Lambda^{p+1}(M)$ is the λ-eigenspace of d^*d and $E_p^d(\lambda) = \{\omega \in E_p(\lambda) : d\omega = 0\} \subseteq d\Lambda^{p-1}(M)$ is the λ-eigenspace of dd^*. Moreover, d^* maps $E_p^d(\lambda)$ isomorphically onto

$E^{d^*}_{p-1}(\lambda)$ and d maps $E^{d^*}_p(\lambda)$ isomorphically onto $E^d_{p+1}(\lambda)$. Hence, $E_p(\lambda) = E^{d^*}_p(\lambda) \oplus dE^{d^*}_{p-1}(\lambda)$. So for our purpose it will be sufficient to study the spectrum of d^*d on coexact forms.

Let $\lambda^{d^*}_{k,p}(M)$ the k^{th} (positive) eigenvalue of $d^*d : d^*\Lambda^{p+1}(M) \to d^*\Lambda^{p+1}(M)$. The following variational characterization of the spectrum of d^*d holds

$$\lambda^{d^*}_{k,p}(M) = \min_{\Sigma^k} \max \left\{ \frac{\|d\omega\|^2}{\|\omega\|^2} : \omega \in \Sigma^k \setminus \{0\} \right\}$$

where Σ^k ranges over all k-dimensional vector subspaces of $d^*\Lambda^{p+1}(M)$ and $\|\cdot\|$ denotes the L^2-norm for differential forms.

2.2 Čech cohomology and combinatorial Laplacian

Let M^n be a compact connected n-dimensional manifold. Let $\mathcal{U} = \{U_i\}_{1\le i\le N}$ be a finite open cover of M. The nerve of $\mathcal{U}$, denoted by $N(\mathcal{U})$, is the simplicial complex whose set of q-simplices is given by

$$S_q(\mathcal{U}) = \{(i_0, \dots, i_q) : i_0 < \dots < i_q \text{ and } U_{i_0} \cap \dots \cap U_{i_q} \neq \emptyset\}$$

for any $q \ge 0$. A Čech q-cochain is an application $c : S_q(\mathcal{U}) \to \mathbb{R}$. Denote by $\mathcal{C}^q(\mathcal{U})$ the set of Čech q-cochains. Let us remark that $\mathcal{C}^q(\mathcal{U})$ is naturally endowed with a vector space structure and let us define a coboundary operator $\delta : \mathcal{C}^q(\mathcal{U}) \to \mathcal{C}^{q+1}(\mathcal{U})$ by

$$\delta c(i_0, \dots, i_{q+1}) = \sum_{j=0}^{q+1} (-1)^j c(i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_{q+1})$$

for any $(i_0, \dots, i_{q+1}) \in S_{q+1}(\mathcal{U})$. Then $\delta \circ \delta = 0$ and the cochain complex $\{\mathcal{C}^q(\mathcal{U}), \delta\}$ gives rise to the Čech cohomology groups of the cover $\mathcal{U}$, $\check{H}^*(\mathcal{U})$. Endow then $\mathcal{C}^q(\mathcal{U})$ with the following scalar product, for any c_1, $c_2 \in \mathcal{C}^q(\mathcal{U})$

$$(c_1, c_2) = \sum_{I \in S_q(\mathcal{U})} c_1(I)c_2(I)$$

and consider $\delta^* : \mathcal{C}^{q+1}(\mathcal{U}) \to \mathcal{C}^q(\mathcal{U})$ the adjoint of δ with respect to $(\cdot, \cdot)$.

Definition 2.1 *The* ***combinatorial Laplacian*** $\check{\Delta} : \mathcal{C}^q(\mathcal{U}) \to \mathcal{C}^q(\mathcal{U})$ *is defined by* $\check{\Delta} = \delta\delta^* + \delta^*\delta$.

The combinatorial Laplacian is self-adjoint and non-negative by definition. Its spectrum will be denoted by

$$0 < \lambda_{1,q}(\mathcal{U}) \le \lambda_{2,q}(\mathcal{U}) \le \dots \le \lambda_{L,q}(\mathcal{U})$$

where 0 is of multiplicity $\check{b}_q(\mathcal{U})$ and $L + \check{b}_q(\mathcal{U}) = dim(\mathcal{C}^q(\mathcal{U})) = |S_q(\mathcal{U})|$. As for the Laplacian on differential forms, half of the spectrum is redundant i.e. if $\lambda > 0$ is an eigenvalue of $\check{\Delta}$ on Čech q-cochains and if $\check{E}_q(\lambda)$ denotes the λ-eigenspace, then $\check{E}_q(\lambda) = \check{E}_q^{\delta^*}(\lambda) \oplus \delta \check{E}_{q-1}^{\delta^*}(\lambda)$ where $\check{E}_q^{\delta^*}(\lambda)$ is the λ-eigenspace of $\delta^*\delta$ acting on $\delta^*\mathcal{C}^{q+1}(\mathcal{U})$. So for our purpose it will be sufficient to study the spectrum of $\delta^*\delta$ on $\delta^*\mathcal{C}^{q+1}(\mathcal{U})$ i.e. on coexact Čech cochains. In the sequel, $\lambda_{k,q}^{\delta^*}(\mathcal{U})$ denotes the k^{th} (positive) eigenvalue of $\delta^*\delta : \delta^*\mathcal{C}^{q+1}(\mathcal{U}) \to \delta^*\mathcal{C}^{q+1}(\mathcal{U})$. The following variational characterization holds

$$\lambda_{k,q}^{\delta^*}(\mathcal{U}) = \min_{V^k} \max \left\{ \frac{\|\delta c\|^2}{\|c\|^2} : c \in V^k \setminus \{0\} \right\}$$

where V^k ranges over all k-dimensional vector subspaces of $\delta^*\mathcal{C}^{q+1}(\mathcal{U})$.

2.3 De Rham Theorem

Recall that an open cover $\mathcal{U}$ is called contractible if for any $I \in S_q(\mathcal{U})$, $U_I = \bigcap_{i \in I} U_i$ is contractible. The following theorem is due to de Rham.

Theorem 2.2 *Let (M^n, g) be a compact connected n-dimensional Riemannian manifold without boundary. Let $\mathcal{U}$ be a contractible finite open cover of M. Then the p^{th} group of de Rham's cohomology $H^p(M)$ is isomorphic to the p^{th} group of Čech cohomology $\check{H}^p(\mathcal{U})$.*

Remark 2.3 *Note that a consequence of the de Rham Theorem is that if $\mathcal{U}$ is a contractible cover, then $b_p(M) = \check{b}_p(\mathcal{U})$.*

Let us introduce now the vector spaces $\mathcal{C}^q(\mathcal{U}, \Lambda^p)$ of q-cochains of p-forms i.e. c is in $\mathcal{C}^q(\mathcal{U}, \Lambda^p)$ if $c(I)$ is a p-form on U_I for any I in $S_q(\mathcal{U})$. Define then the following coboundary operators

$$\begin{aligned}
\delta :\ & \mathcal{C}^q(\mathcal{U}, \Lambda^p) \to \mathcal{C}^{q+1}(\mathcal{U}, \Lambda^p) \text{ such that for any } (i_0, \ldots, i_{q+1}) \in S_{q+1}(\mathcal{U}) \\
& \delta c(i_0, \ldots, i_{q+1}) = \sum_{j=0}^{q+1} (-1)^j c(i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{q+1}), \\
d :\ & \mathcal{C}^q(\mathcal{U}, \Lambda^p) \to \mathcal{C}^q(\mathcal{U}, \Lambda^{p+1}) \text{ such that for any } I \in S_q(\mathcal{U}) \\
& dc(I) = d(c(I)).
\end{aligned}$$

Then $d \circ d = 0$, $\delta \circ \delta = 0$ and $d \circ \delta = \delta \circ d$. The Čech - de Rham double complex is the following commutative diagram, where r denotes the restriction map to each open of the cover and i the natural injection. The first step in the proof of the de Rham Theorem is to show that the rows (except the first) and the columns (except the first) of this diagram are exact. This is a direct consequence of the Poincaré Lemma (Lemma 2.4) and Lemma 2.5.

$$
\begin{tikzcd}
 & \mathcal{C}^0(\mathcal{U}) \arrow[r, "\delta"] \arrow[d, "i"] & \mathcal{C}^1(\mathcal{U}) \arrow[r, "\delta"] \arrow[d, "i"] & \dots \arrow[r, "\delta"] & \mathcal{C}^{q-1}(\mathcal{U}) \arrow[r, "\delta"] \arrow[d, "i"] & \mathcal{C}^q(\mathcal{U}) \arrow[r, "\delta"] \arrow[d, "i"] & \dots \\
\Lambda^0(M) \arrow[r, "r"] \arrow[d, "d"] & \mathcal{C}^0(\mathcal{U},\Lambda^0) \arrow[r, "\delta"] \arrow[d, "d"] & \mathcal{C}^1(\mathcal{U},\Lambda^0) \arrow[r, "\delta"] \arrow[d, "d"] & \dots \arrow[r, "\delta"] & \mathcal{C}^{q-1}(\mathcal{U},\Lambda^0) \arrow[r, "\delta"] \arrow[d, "d"] & \mathcal{C}^q(\mathcal{U},\Lambda^0) \arrow[r, "\delta"] \arrow[d, "d"] & \dots \\
\Lambda^1(M) \arrow[r, "r"] \arrow[d, "d"] & \mathcal{C}^0(\mathcal{U},\Lambda^1) \arrow[r, "\delta"] \arrow[d, "d"] & \mathcal{C}^1(\mathcal{U},\Lambda^1) \arrow[r, "\delta"] \arrow[d, "d"] & \dots \arrow[r, "\delta"] & \mathcal{C}^{q-1}(\mathcal{U},\Lambda^1) \arrow[r, "\delta"] \arrow[d, "d"] & \mathcal{C}^q(\mathcal{U},\Lambda^1) \arrow[r, "\delta"] \arrow[d, "d"] & \dots \\
\vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \dots & \vdots \arrow[d, "d"] & \vdots \arrow[d, "d"] & \\
\Lambda^{p-1}(M) \arrow[r, "r"] \arrow[d, "d"] & \mathcal{C}^0(\mathcal{U},\Lambda^{p-1}) \arrow[r, "\delta"] \arrow[d, "d"] & \mathcal{C}^1(\mathcal{U},\Lambda^{p-1}) \arrow[r, "\delta"] \arrow[d, "d"] & \dots \arrow[r, "\delta"] & \mathcal{C}^{q-1}(\mathcal{U},\Lambda^{p-1}) \arrow[r, "\delta"] \arrow[d, "d"] & \mathcal{C}^q(\mathcal{U},\Lambda^{p-1}) \arrow[r, "\delta"] \arrow[d, "d"] & \dots \\
\Lambda^p(M) \arrow[r, "r"] \arrow[d, "d"] & \mathcal{C}^0(\mathcal{U},\Lambda^p) \arrow[r, "\delta"] \arrow[d, "d"] & \mathcal{C}^1(\mathcal{U},\Lambda^p) \arrow[r, "\delta"] \arrow[d, "d"] & \dots \arrow[r, "\delta"] & \mathcal{C}^{q-1}(\mathcal{U},\Lambda^p) \arrow[r, "\delta"] \arrow[d, "d"] & \mathcal{C}^q(\mathcal{U},\Lambda^p) \arrow[r, "\delta"] \arrow[d, "d"] & \dots \\
\vdots & \vdots & \vdots & & \vdots & \vdots &
\end{tikzcd}
$$

Figure 1: The Čech - de Rham double complexe.

Lemma 2.4 *Let $p > 0$. Let $\mathcal{U}$ be a contractible cover. Let $\omega \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$ such that $d\omega = 0$. Then there exists $\eta \in \mathcal{C}^q(\mathcal{U}, \Lambda^{p-1})$ such that $d\eta = \omega$.*

Proof: see [18], A.6. □

Lemma 2.5 *Let $q > 0$. Let $c \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$ such that $\delta c = 0$. Then there exists $b \in \mathcal{C}^{q-1}(\mathcal{U}, \Lambda^p)$ such that $\delta b = c$.*

Proof: see [18], proof of Lemma A.4.1. □

The proof of the de Rham Theorem goes then as follows. Let $\omega \in \Lambda^p(M)$ such that $d\omega = 0$. Let $f_0 = r(\omega) \in \mathcal{C}^0(\mathcal{U}, \Lambda^p)$, then $df_0 = 0 = \delta f_0$ and the system of equations

$$f_0 = df_1 \ , \ \delta f_1 = df_2 \ , \ \delta f_2 = df_3 \ , \ \dots \ , \ \delta f_{p-1} = df_p$$

has a solution with $f_j \in \mathcal{C}^{j-1}(\mathcal{U}, \Lambda^{p-j})$ for $j \geq 1$. Moreover, $d(\delta f_p) = 0$, hence $\delta f_p \in \mathcal{C}^p(\mathcal{U})$. The application $\Psi : \{\omega \in \Lambda^p(M) : d\omega = 0\} \to \{c \in \mathcal{C}^p(\mathcal{U}) : \delta c = 0\}$ given by $\Psi(\omega) = \delta f_p$, where f_p is constructed as above, induces an isomorphism in cohomology. In particular, if ω is exact, $\Psi(\omega)$ is also exact i.e. there exists $c \in \mathcal{C}^{p-1}(\mathcal{U})$ such that $\delta c = \Psi(\omega)$ (note that in general $f_p \notin \mathcal{C}^{p-1}(\mathcal{U})$). Naturally, we can construct another application going from closed Čech p-cochains to closed p-forms exactly in the same way and obtain also an isomorphism in cohomology. □

2.4 Discretization of a manifold

Let (M^n, g) be a connected compact n-dimensional Riemannian manifold without boundary. Let $\varepsilon > 0$.

Definition 2.6 *An ε-discretization X of M is a maximal ε-separated subset of M i.e. X is a subset of M satisfying*

(i) $\forall p \neq q \in X$, $d(p,q) \geq \varepsilon$,

(ii) $\mathcal{U}_X = \{B(p,\varepsilon)\}_{p\in X}$ *is an open cover of M.*

Note that as M is compact, X is finite of cardinality $|X|$. So we can number the elements of $X = \{p_1, \ldots, p_{|X|}\}$ and denote $U_i = B(p_i, \varepsilon)$, for $i = 1, \ldots, |X|$. In particular, any discretization of M gives rise to a combinatorial Laplacian $\check{\Delta}$ as defined in Section 2.2. In the sequel, $\lambda_{k,q}(X)$ will denote the k^{th} eigenvalue of the combinatorial Laplacian associated to the open cover $\mathcal{U}_X$ acting on Čech q-cochains i.e. $\lambda_{k,q}(X) = \lambda_{k,q}(\mathcal{U}_X)$.
Note also that if ε (the mesh of the discretization) is smaller than the convexity radius of M, then $\mathcal{U}_X$ is a contractible open cover and $\check{b}_p(\mathcal{U}_X) = b_p(M)$.

Definition 2.7 *For $\kappa \geq 0$, $r_0 > 0$ and $n \in \mathbb{N}^*$, we define $\mathbb{M}(n, \kappa, r_0)$ as the set of all connected compact n-dimensional Riemannian manifold (M^n, g) without boundary with uniformly bounded sectional curvature i.e. $|K_g| \leq \kappa$ and injectivity radius bounded below i.e. $Inj(M, g) \geq r_0$.*

Remark 2.8 *For $n \in \mathbb{N}^*$, $\kappa \geq 0$, $r_0 > 0$ and $0 < 2\varepsilon < r_0$, there exists $\nu(n, \kappa) > 0$ such that, for any $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ and any ε-discretization X of M, the cardinality of $\{j : U_j \cap U_I \neq \emptyset\}$ is bounded above by ν, for any $I \in S_q(\mathcal{U}_X)$. This is a direct consequence of the Bishop-Gromov volume comparison Theorem (see for instance [8], Lemma V.3.1, p.147). Furthermore, by Croke's Inequality and Bishop's comparison Theorem (see [8] p.126 and p.136) we can assert that there exist positive constants c_1, c_2 depending only on n, κ and ε such that $c_1 Vol(M) \leq |X| \leq c_2 Vol(M)$. In particular, we obtain that $|\mathcal{S}_q(\mathcal{U}_X)| \leq \frac{\nu^q}{(q+1)!}|X| \leq \frac{\nu^q}{(q+1)!} c_2 Vol(M)$.*

The following lemma shows that in general a sufficiently small ball is quasi-isometric (in the sense of [14], (3.2)) to a Euclidean convex. In particular, this will imply that on intersections of sufficiently small balls we can find a lower bound for the first positive eigenvalue of Δ with absolute boundary condition (see Lemma 2.10). This is an essential result for the discretization as we will see later.

Lemma 2.9 *Let $n \in \mathbb{N}^*$, $\kappa \geq 0$ and $r_0 > 0$. There exists a constant $0 < \rho_0 < r_0$ depending only on n, κ and r_0 such that for any $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ and for any $p \in M$, there exist a Euclidean convex $C_p \subseteq \mathbb{R}^n$ and a diffeomorphism $\varphi : C_p \to B(p, \rho_0)$ such that for any $B(q, \rho) \subseteq B(p, \rho_0)$, the ball $B(q, \rho)$ is convex and $\varphi^{-1}(B(q, \rho))$ is a Euclidean convex. Moreover, $(B(q, \rho), g)$ is quasi-isometric to $B(q, \rho)$ endowed with the Euclidean metric induced by φ^{-1} and the constants of quasi-isometry depend only on n, κ and $d(p, q) + \rho$.*

Proof: see Appendix A.2. □

Note that the intersection of small balls is a convex with not necessarily smooth boundary. So that it is not obvious that in this case the spectrum of the Laplacian with absolute boundary condition is discrete. In [26], the authors show that the spectrum of the Laplacian with absolute (or relative) boundary condition is discrete even if the boundary is only given by a Lipschitz function (Proposition 5.3 in [26]). Moreover, Theorem 5.1 of [27] implies that the following classical variational characterization of the spectrum is still valid for bounded convex domains i.e. if Ω is a bounded convex domain of M, then the k^{th} eigenvalue of the Laplacian for p-forms on Ω with absolute boundary condition is given by

$$\lambda_{k,p}^{abs}(\Omega) = \min_{\Sigma^k} \max \left\{ \frac{\|d\omega\|^2 + \|\delta\omega\|^2}{\|\omega\|^2} : \omega \in \Sigma^k \setminus \{0\} \text{ such that } i_\nu(\omega) = 0 \right\}$$

where Σ^k ranges over all k-dimensional vector subspaces of $\Lambda^p(\Omega)$ and i_ν is the interior product by ν the outward pointing normal unit vector to the boundary (defined almost everywhere). In particular, the result on quasi-isometric metrics of Dodziuk (Proposition 3.3 of [14]) is valid in this context.

Lemma 2.10 *Let $n \geq 2$, $\kappa \geq 0$, $r_0 > 0$ and let ρ_0 given by Lemma 2.9. Let $0 < 3\varepsilon < \rho_0$. Then there exists a positive constant $\mu(n, \kappa, \varepsilon)$ depending only on n, κ and ε such that for any $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ and for any ε-discretization X of M*

$$\lambda_{1,p}^{abs}(U_I) \geq \mu(n, \kappa, \varepsilon)$$

for any $p = 0, \ldots, n$ and any $I \in S_q(\mathcal{U}_X)$, $q \geq 0$.

Proof: let $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ and X an ε-discretization of M with $0 < 3\varepsilon < \rho_0$. Fix $p \in X$ and let $q \in X$ such that $B(p, \varepsilon) \cap B(q, \varepsilon) \neq \emptyset$. Then $B(q, \varepsilon) \subseteq B(p, 3\varepsilon) \subseteq B(p, \rho_0)$. By Lemma 2.9, there exists a diffeomorphism φ such that $\varphi^{-1}(B(q, \varepsilon))$ is a Euclidean convex for any $q \in X$ such that $B(q, \varepsilon) \cap B(p, \varepsilon) \neq \emptyset$. In particular, $\varphi^{-1}(B(p, \varepsilon) \cap B(q, \varepsilon))$ is an intersection of Euclidean convexes and as such it is a Euclidean convex. Moreover, φ^{-1} restricted to $B(p, 3\varepsilon)$ is a quasi-isometry with constants of quasi-isometry depending only on n, κ and ε. Let U_I a non-empty finite intersection of elements of $\mathcal{U}_X$ and $V_I = \varphi^{-1}(U_I)$ the Euclidean convex which is quasi-isometric to U_I via φ i.e. $(\varphi(V_I), (\varphi^{-1})^*(eucl))$ is quasi-isometric to (U_I, g) with constants of quasi-isometry α depending only on n, κ and ε (i.e. $\alpha^{-1}(\varphi^{-1})^*(eucl) \leq g \leq \alpha(\varphi^{-1})^*(eucl)$). Then by Proposition 3.3 of [14], there exist positive constants c_1 and c_2 depending only on α and n such that

$$c_1\lambda_{1,p}^{abs}(U_I, (\varphi^{-1})^*(eucl)) \leq \lambda_{1,p}^{abs}(U_I, g) \leq c_2\lambda_{1,p}^{abs}(U_I, (\varphi^{-1})^*(eucl)). \tag{2.2}$$

Note that $(U_I, (\varphi^{-1})^*(eucl))$ is a Euclidean convex of diameter bounded above by $d(n, \kappa, \varepsilon)$. Finally, Guerini shows in [19], that the first eigenvalue of the Laplacian with absolute boundary condition on a Euclidean convex with smooth boundary is bounded below by a constant depending on the diameter of the convex. Note that Guerini's proof can be adapted straightforward to obtain the same result for convexes with piecewise smooth boundary. Hence, we obtain that there exists a positive constant $c(n, p)$ such that

$$\lambda_{1,p}^{abs}(U_I, (\varphi^{-1})^*(eucl)) \geq \frac{c(n,p)}{diam(U_I, (\varphi^{-1})^*(eucl))^2} \geq \frac{c(n,p)}{d(n,\kappa,\varepsilon)^2} \tag{2.3}$$

Finally, (2.2) and (2.3) imply the claim. $\square$

3 Comparison of spectra

This section is devoted to the proof of the main theorem of the paper. Let us state the result.

Theorem 3.1 *Let $n \geq 2$, $\kappa \geq 0$, $r_0 > 0$. Let $\rho_0(n, \kappa, r_0)$ be given by Lemma 2.9 and $0 < 3\varepsilon < \rho_0$. Let $1 \leq p \leq n-1$. Then there exist positive constants c_1, c_2 depending only on n, p, κ and ε such that for any $M \in \mathbb{M}(n, \kappa, r_0)$ and for any ε-discretization X of M, we have*

$$c_1\lambda_{k,p}(X) \leq \lambda_{k,p}(M) \leq c_2\lambda_{k,p}(X)$$

for any $1 \leq k \leq |\mathcal{C}^p(\mathcal{U}_X)| - \check{b}_p(\mathcal{U}_X) = |\mathcal{C}^p(\mathcal{U}_X)| - b_p(M)$.

As we have seen before (in Section 2.1), it will be sufficient to establish the result for the spectrum of d^*d on coexact p-forms and for the spectrum of $\delta^*\delta$ on coexact Čech p-cochains. The proof goes in two steps. First step consists in comparing "small" eigenvalues. We need to construct a discretizing operator that associates to a coexact p-form a coexact Čech p-cochain (see Section 3.1) and a smoothing operator that goes in the opposite direction (see Section 3.2), in order to compare their respective Rayleigh quotients. The idea is to proceed as in the proof of the de Rham Theorem and use the Čech - de Rham double complexe. But as we need a control of the norms involved, we have to establish versions of the Poincaré Lemma (Lemma 2.4) and of Lemma 2.5 with a suitable control of the norms (see Lemma 3.2 and Lemma 3.5). The second step of the proof deals with "large" eigenvalues and is reduced to find upper bounds for the k^{th} eigenvalues involved depending only on the parameters of the problem (see Section 3.3).

In the sequel, we consider (M,g) in $\mathbb{M}(n,\kappa,r_0)$ and X an ε-discretization with $0<3\varepsilon<\rho_0$. Denote by $\mathcal{U}$ the open cover induced by X i.e. $\mathcal{U}=\{U_i=B(p_i,\varepsilon):i=1,\dots,|X|\}$ and fix $1\leq p\leq n-1$.

3.1 From smooth forms to Čech cochains

In this section, we are going to construct

$$\mathcal{D}:d^*\Lambda^{p+1}(M)\to\delta^*\mathcal{C}^{p+1}(\mathcal{U})$$

such that there exist positive constants c_1, c_2 and Λ depending only on n, p, κ and ε such that

$(i)_{\mathcal{D}}$ $\|\delta\mathcal{D}(\omega)\|^2\leq c_1\|d\omega\|^2$, for any $\omega\in d^*\Lambda^{p+1}(M)$,

$(ii)_{\mathcal{D}}$ $\|\mathcal{D}\omega\|^2\geq c_2\|\omega\|^2$, for any $\omega\in d^*\Lambda^{p+1}(M)$ satisfying $\|d\omega\|^2\leq\Lambda\|\omega\|^2$.

To that aim, we need the following version of the Poincaré Lemma. Note that this lemma will be verified in particular by any non-empty intersection of open sets in $\mathcal{U}$ thanks to Lemma 2.10 (where μ depends on n, κ, ε).

Lemma 3.2 *Let U be a contractible open set such that $\lambda_{1,p}^{abs,d}(U)\geq\mu>0$, $(1\leq p\leq n)$. Let ω be a closed L^2-integrable p-form on U i.e. $d\omega=0$. Then there exists $\eta\in\Lambda^{p-1}(U)$ such that $d\eta=\omega$ and $\|\eta\|^2_{L^2(U)}\leq\frac{2}{\mu}\|\omega\|^2_{L^2(U)}$.*

Proof: we have the following characterization of the first eigenvalue of the Laplacian on exact p-forms (see Proposition 3.1. of [14] or Proposition 2.1. of [25]),

$$\lambda_{1,p}^{abs,d}(U)=\inf_V\sup\left\{\frac{\|\omega\|^2_{L^2(U)}}{\|\eta\|^2_{L^2(U)}}:\omega\in V\setminus\{0\}\ ,\ d\eta=\omega\right\}$$

where V ranges over all 1-dimensional vector subspaces of exact p-forms. If $\omega\in\Lambda^p(U)$ is closed, by the Poincaré Lemma ω is exact. So that we get

$$\mu\leq\lambda_{1,p}^{abs,d}(U)\leq\sup\left\{\frac{\|\omega\|^2_{L^2(U)}}{\|\eta\|^2_{L^2(U)}}:d\eta=\omega\right\}$$

and hence there exists $\eta \in \Lambda^{p-1}(U)$ such that $d\eta = \omega$ and $\frac{1}{2}\mu \leq \frac{\|\omega\|^2_{L^2(U)}}{\|\eta\|^2_{L^2(U)}}$ which is the claim. $\square$

Remark 3.3 *Let us introduce the following norm. If $c \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$ let*

$$\|c\|^2 = \sum_{I \in S_q(\mathcal{U})} \|c(I)\|^2_{L^2(U_I)}$$

where $\|\cdot\|_{L^2(U_I)}$ denotes the L^2-norm for p-forms on U_I. In particular, if ω is a p-form on M and r is the restriction to each open of $\mathcal{U}$, then there exist positive constants c_1 and c_2 depending only on n, κ and ε such that $c_1\|r(\omega)\|^2 \leq \|\omega\|^2 \leq c_2\|r(\omega)\|^2$.

Construction by induction of $\mathcal{D}$

Let $\omega \in d^*\Lambda^{p+1}(M)$. The goal is to construct $\mathcal{D}(\omega) \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$. The idea is to consider $d\omega$ which is an exact $(p+1)$-form and to construct an exact Čech $(p+1)$-cochain $\delta\mathcal{D}(\omega)$ such that $(i)_{\mathcal{D}}$ holds. A suitable candidate for $\delta\mathcal{D}(\omega)$ is the Čech cochain given by the proof of the de Rham Theorem and the double complexe. Moreover, the double complexe and the normed version of the Poincaré Lemma give almost directly the inequality $(i)_{\mathcal{D}}$, whereas $(ii)_{\mathcal{D}}$ is not a so direct consequence of the construction. Hence, as suggested in [7], we construct an auxiliary p-form thanks to Whitney forms to obtain $(ii)_{\mathcal{D}}$. We proceed by induction.

First step of induction: define $c_{p+1,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^{p+1})$ by $c_{p+1,0} = r(d\omega)$ i.e. $c_{p+1,0}(i) = d\omega_{|U_i}$. Then $dc_{p+1,0} = 0 = \delta c_{p+1,0}$ and $W(c_{p+1,0}) = d\omega$, where W is the Whitney map defined in Appendix A.1. Then there exist positive constants c_1, c_2 and c_3 depending only on n, p, κ and ε such that the three following assertions hold.

- $(a)_1$ There exists $c_{p,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^p)$ such that $dc_{p,0} = c_{p+1,0}$ and $\|c_{p,0}\|^2 \leq c_1\|d\omega\|^2$.
- $(b)_1$ Let $c_{p,1} = \delta c_{p,0}$. We have $\delta c_{p,1} = 0 = dc_{p,1}$ and $\|c_{p,1}\|^2 \leq c_2\|d\omega\|^2$.
- $(c)_1$ Let $v^{(1)} = W(c_{p,0}) \in \Lambda^p(M)$. We have $dv^{(1)} = d\omega + W(c_{p,1})$ and $\|v^{(1)}\|^2 \leq c_3\|d\omega\|^2$.

Indeed, $(a)_1$ is a direct consequence of Lemma 3.2, of the definition of $c_{p+1,0}$ and of Remark 3.3. Then, clearly $\delta c_{p,1} = 0$ and $dc_{p,1} = \delta dc_{p,0} = \delta c_{p+1,0} = 0$. Moreover, there exists $c(n, \kappa, \varepsilon)$ such that for any cochain $\|\delta b\|^2 \leq c\|b\|^2$ (see (3.3)) and combined with $(a)_1$ this implies $(b)_1$. Finally, by Lemma A.4 $dv^{(1)} = W(c_{p,1}) + W(c_{p+1,0}) = d\omega + W(c_{p,1})$. Moreover, by Lemma A.5 and by $(a)_1$, we get $\|v^{(1)}\|^2 \leq cst\|c_{p,0}\|^2 \leq c_3\|d\omega\|^2$.

Induction hypothesis: (for $1 \leq q < p+1$) there exist positive constants c_1, c_2 and c_3 depending only on n, p, κ and ε such that the three following assertions hold.

$(a)_q$ There exists $c_{p+1-q,q-1} \in \mathcal{C}^{q-1}(\mathcal{U}, \Lambda^{p+1-q})$ such that $dc_{p+1-q,q-1} = c_{p+1-(q-1),q-1}$ and $\|c_{p+1-q,q-1}\|^2 \leq c_1 \|d\omega\|^2$.

$(b)_q$ Let $c_{p+1-q,q} = (-1)^{q+1} q \cdot \delta c_{p+1-q,q-1}$. We have $\delta c_{p+1-q,q} = 0 = dc_{p+1-q,q}$ and $\|c_{p+1-q,q}\|^2 \leq c_2 \|d\omega\|^2$.

$(c)_q$ Let $v^{(q)} = v^{(q-1)} + W(c_{p+1-q,q-1}) \in \Lambda^p(M)$. We have $d\omega = dv^{(q)} + (-1)^q W(c_{p+1-q,q})$ and $\|v^{(q)}\|^2 \leq c_3 \|d\omega\|^2$.

Proof: suppose the hypothesis of induction is satisfied for some $1 \leq q \leq p$ and let us show it holds for $q+1$. By $(b)_q$, Lemma 3.2 and Lemma 2.10, there exists $c_{p-q,q} \in \mathcal{C}^q(\mathcal{U}, \Lambda^{p-q})$ and $\mu > 0$ such that $dc_{p-q,q} = c_{p+1-q,q}$ and $\|c_{p-q,q}(I)\|^2_{L^2(U_I)} \leq \frac{2}{\mu} \|c_{p+1-q,q}(I)\|^2_{L^2(U_I)}$. Combined with $(b)_q$ this implies that $\|c_{p-q,q}\|^2 \leq \frac{2}{\mu} \|c_{p+1-q,q}\|^2 \leq c_1 \|d\omega\|^2$ which is $(a)_{q+1}$. Let us consider now

$$c_{p-q,q+1} = (-1)^q (q+1) \delta c_{p-q,q}$$

then clearly $\delta c_{p-q,q+1} = 0$ and $dc_{p-q,q+1} = (-1)^q (q+1) \delta c_{p+1-q,q} = 0$ by $(b)_q$. Moreover, $\|c_{p-q,q+1}\|^2 \leq cst \|c_{p-q,q}\|^2 \leq c_2 \|d\omega\|^2$ by $(a)_{q+1}$. This concludes the proof of $(b)_{q+1}$. Finally, if $v^{(q+1)} = v^{(q)} + W(c_{p-q,q})$ we obtain with $(c)_q$ and Lemma A.4 that

$$\begin{aligned} d\omega &= dv^{(q+1)} - d(W(c_{p-q,q})) + (-1)^q W(c_{p+1-q,q}) \\ &= dv^{(q+1)} - (q+1) W(\delta c_{p-q,q}) - (-1)^q W(dc_{p-q,q}) + (-1)^q W(c_{p+1-q,q}) \\ &= dv^{(q+1)} + (-1)^{q+1} W(c_{p-q,q+1}). \end{aligned}$$

Finally, thanks to Lemma A.5, $(c)_q$ and $(a)_{q+1}$ we obtain that $\|v^{(q+1)}\|^2 \leq cst(\|v^{(q)}\|^2 + \|c_{p-q,q}\|^2) \leq c_3 \|d\omega\|^2$. This concludes the induction.

End of the induction: (for $q = p+1$) we get $c_{0,p+1} \in \mathcal{C}^{p+1}(\mathcal{U}, \Lambda^0)$ such that $dc_{0,p+1} = 0$. This implies in particular that $c_{0,p+1} \in i(\mathcal{C}^{p+1}(\mathcal{U}))$. Moreover by the proof of the de Rham Theorem seen in Section 2.3, the cochain $c_{0,p+1}$ represents the same cohomology class as $d\omega$ i.e. there exists $\gamma \in \mathcal{C}^p(\mathcal{U})$ such that $i(\delta\gamma) = c_{0,p+1}$.

Definition 3.4 *We define $\mathcal{D}\omega$ as the unique Čech p-cochain in $\delta^* \mathcal{C}^{p+1}(\mathcal{U})$ such that $i(\delta \mathcal{D}(\omega)) = c_{0,p+1}$.*

We prove now $(i)_{\mathcal{D}}$ and $(ii)_{\mathcal{D}}$. Firstly, by $(b)_{p+1}$ of the induction we get that there exists a constant c_1 depending only on n, p, κ and ε such that

$$\|\delta \mathcal{D}(\omega)\|^2 \leq cst \|c_{0,p+1}\|^2 \leq c_1 \|d\omega\|^2$$

and this proves $(i)_{\mathcal{D}}$. Secondly, by $(c)_{p+1}$ we can write

$$d\omega = dv^{(p+1)} + (-1)^{p+1} W(\delta\mathcal{D}(\omega)) = dv^{(p+1)} + \frac{(-1)^{p+1}}{p+1} dW(\mathcal{D}(\omega)) \quad (3.1)$$

where we used Lemma A.4 and the fact that $d(i(\mathcal{D}(\omega))) = 0$ in the last equality. Moreover, as ω is coexact, and if $coex(\cdot)$ denotes the coexact part of a form given by the Hodge decomposition, we deduce that

$$\omega = coex(v^{(p+1)}) + \frac{(-1)^{p+1}}{p+1} coex\left(W(\mathcal{D}(\omega))\right).$$

Therefore, by Lemma A.5 and using this last equality we obtain

$$\|\mathcal{D}(\omega)\| \geq cst\|W(\mathcal{D}(\omega))\| \geq cst(\|\omega\| - \|v^{(p+1)}\|). \quad (3.2)$$

Finally, by $(c)_{p+1}$ there exists C' depending only on n, p, κ and ε such that $\|\mathcal{D}(\omega)\| \geq cst(\|\omega\| - C'\|d\omega\|)$. Let then $\Lambda = \frac{1}{4C'^2}$ so that if $\|d\omega\|^2 \leq \Lambda\|\omega\|^2$ then $\|\mathcal{D}(\omega)\| \geq c_2\|\omega\|$ which is the requested inequality in $(ii)_{\mathcal{D}}$. $\square$

3.2 From Čech cochains to smooth forms

In this section, we are going to construct

$$\mathcal{S} : \delta^*\mathcal{C}^{p+1}(\mathcal{U}) \to d^*\Lambda^{p+1}(M)$$

such that there exist positive constants c_1', c_2' and Λ' depending only on n, p, κ and ε such that

$(i)_{\mathcal{S}}$ $\|d\mathcal{S}(c)\|^2 \leq c_1'\|\delta c\|^2$, for any $c \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$,

$(ii)_{\mathcal{S}}$ $\|\mathcal{S}c\|^2 \geq c_2'\|c\|^2$, for any $c \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$ satisfying $\|\delta c\|^2 \leq \Lambda'\|c\|^2$.

The construction of $\mathcal{S}$ is similar to the construction of $\mathcal{D}$. The main difference is that the Whitney map is not the suitable tool to obtain $(ii)_{\mathcal{S}}$. So we have to do a first induction to construct $\mathcal{S}$ and a second induction (slightly different) to prove $(ii)_{\mathcal{S}}$. We begin by adjusting Lemma 2.5 to our purpose.

Lemma 3.5 *Let $\mathcal{U}$ be a contractible cover and $\{\varphi_j\}$ a partition of unity subordinated to $\mathcal{U}$. Let $\nu > 0$ such that $|\{j : U_j \cap U_I \neq \emptyset\}| \leq \nu$ for any $I \in S_k(\mathcal{U})$ and any $k = 0, \ldots, n$. Let $c \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$ $(q \geq 1)$ such that $\delta c = 0$. Then there exists $b \in \mathcal{C}^{q-1}(\mathcal{U}, \Lambda^p)$ such that $\delta b = c$ and there exist positive constants c_1, c_2 depending only on ν and on a bound on $\|d\varphi_j\|_\infty$ such that*

(i) $\|b\|^2 \leq c_1\|c\|^2$

(ii) $||db||^2 \leq c_2(||c||^2 + ||dc||^2)$

Proof: a suitable b is given by Lemma A.4.1 in [18] and defined by

$$b(I) = \sum_{j \text{ s.t. } U_j \cap U_I \neq \emptyset} \varphi_j \cdot c(\{j\} \cup I)$$

so that b verifies already $\delta b = c$. Then (i) is an immediate consequence of the definition of b and ν. It remains to show (ii). We have $||db||^2 = \sum_{I \in S_{q-1}(\mathcal{U})} ||db(I)||^2$. Moreover

$$\begin{aligned} ||db(I)||^2 &= \left\| \sum_{j \text{ s.t. } U_j \cap U_I \neq \emptyset} d\varphi_j \wedge c(\{j\} \cup I) + \varphi_j dc(\{j\} \cup I) \right\|^2 \\ &\leq 2\nu \sum_{j \text{ s.t. } U_j \cap U_I \neq \emptyset} ||d\varphi_j \wedge c(\{j\} \cup I)||^2 + ||\varphi_j dc(\{j\} \cup I)||^2 \end{aligned}$$

and this implies the claim. $\square$

Remark 3.6 *In the sequel, we will consider a partition of unity* $\{\varphi_j\}$ *subordinated to an open cover made of balls of radius* ε*, so that we can find a bound on* $||d\varphi_j||_\infty$ *depending only on* ε*. In particular, this bound will be replaced by a constant depending only on* ε*.*

Construction by induction of $\mathcal{S}(\cdot)$

Let us now proceed to the construction of $\mathcal{S}$ and to the proof of $(i)_\mathcal{S}$. Let $c \in \delta^* \mathcal{C}^{p+1}(\mathcal{U})$. Then δc is an exact Čech $(p+1)$-cochain.

First step of induction: define $c_{0,p+1} \in \mathcal{C}^{p+1}(\mathcal{U}, \Lambda^0)$ by $c_{0,p+1} = i(\delta c)$ i.e. $c_{0,p+1}(I) = \delta c(I)$ for any $I \in S_{p+1}(\mathcal{U})$. Clearly, $\delta c_{0,p+1} = 0 = dc_{0,p+1}$. Then there exist positive constants c'_1, c'_2 depending only on n, p, κ and ε such that

$(a')_1$ there exists $c_{0,p} \in \mathcal{C}^p(\mathcal{U}, \Lambda^0)$ such that $\delta c_{0,p} = c_{0,p+1}$ and $||c_{0,p}||^2 \leq c'_1 ||\delta c||^2$.

$(b')_1$ Let $c_{1,p} = dc_{0,p}$. Then $\delta c_{1,p} = 0$ and $||c_{1,p}||^2 \leq c'_2 ||\delta c||^2$.

Indeed, $(a')_1$ is a direct consequence of Lemma 3.5 as $\delta c_{0,p+1} = 0$ and of (3.3). The bound on the norm of $dc_{0,p}$ follows also from Lemma 3.5 as $dc_{0,p+1} = 0$. Finally, we have $\delta c_{1,p} = d\delta c_{o,p} = dc_{0,p+1} = 0$.

Induction hypothesis: (for $1 \leq q < p+1$) there exist positive constants c'_1, c'_2 depending only on n, p, κ and ε such that

$(a')_q$ there exists $c_{q-1,p+1-q} \in \mathcal{C}^{p+1-q}(\mathcal{U}, \Lambda^{q-1})$ such that $\delta c_{q-1,p+1-q} = c_{q-1,p+1-(q-1)}$ and $\|c_{q-1,p+1-q}\|^2 \leq c_1' \|\delta c\|^2$.

$(b')_q$ Let $c_{q,p+1-q} = dc_{q-1,p+1-q}$. Then $\delta c_{q,p+1-q} = 0$ and $\|c_{q,p+1-q}\|^2 \leq c_2' \|\delta c\|^2$.

Proof: suppose the hypothesis of induction is verified for some $1 \leq q \leq p$ and let us show it holds for $q+1$. By $(b')_q$ and by Lemma 3.5 there exists $c_{q,p-q} \in \mathcal{C}^{p-q}(\mathcal{U}, \Lambda^q)$ such that $\delta c_{q,p-q} = c_{q,p+1-q}$ and $\|c_{q,p-q}\|^2 \leq cst \|c_{q,p+1-q}\|^2$. Combined with $(b')_q$, this implies $(a)_{q+1}$. Moreover, let us consider $c_{q+1,p-q} = dc_{q,p-q}$. Then, by definition of $c_{q,p+1-q}$ we have $\delta c_{q+1,p-q} = d\delta c_{q,p-q} = dc_{q,p+1-q} = 0$. Finally, by Lemma 3.5, we have $\|c_{q+1,p-q}\|^2 \leq cst(\|c_{q,p+1-q}\|^2 + \|dc_{q,p+1-q}\|^2)$. As we have $dc_{q,p+1-q} = 0$ and by $(b')_q$, we get $\|c_{q+1,p-q}\|^2 \leq c_2' \|\delta c\|^2$. This concludes the induction.
End of the induction: (for $q = p+1$) we obtain $c_{p+1,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^{p+1})$ such that $\delta c_{p+1,0} = 0$. This implies that $c_{p+1,0}$ is the restriction of a well-defined $(p+1)$-form and by the de Rham Theorem as δc is exact, the 0-cochain $c_{p+1,0}$ is exact and is the restriction of an exact $(p+1)$-form.

Definition 3.7 *Let $\mathcal{S}(c) \in d^*\Lambda^{p+1}(M)$ be the unique coexact p-form such that $r(d\mathcal{S}(c)) = c_{p+1,0}$.*

An immediate consequence of the induction is $(i)_{\mathcal{S}}$. Indeed, from $(b')_{p+1}$ and Remark 3.3 follows that there exists a positive constant c_1' depending only on n, p, κ and ε such that $\|d\mathcal{S}(c)\|^2 \leq c_1' \|\delta c\|^2$.
Let us now proceed to a second induction in order to prove $(ii)_{\mathcal{S}}$. The goal is to construct $b \in \mathcal{C}^p(\mathcal{U})$ such that $\delta b = \pm \delta c$ and $\|b\| \leq cst(\|\mathcal{S}(c)\| + \|\delta c\|)$ where cst is a positive constant depending only on n, p, κ and ε. These are in fact the corresponding equations for (3.1) and (3.2) in the discretizing part. In the induction, we will use the $c_{r,s}$ appearing in the construction of $\mathcal{S}$.
First step of induction: define $b_{p,0} = r(\mathcal{S}(c)) - c_{p,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^p)$. We have $db_{p,0} = c_{p+1,0} - dc_{p,0} = 0$. Then there exist positive constants c_1'', c_2'' depending only on n, p, κ and ε such that

$(a'')_1$ there exists $b_{p-1,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^{p-1})$ such that $db_{p-1,0} = b_{p,0}$ and $\|b_{p-1,0}\|^2 \leq c_1''(\|\mathcal{S}(c)\| + \|\delta c\|)$.

$(b'')_1$ Let $b_{p-1,1} = \delta b_{p-1,0} + c_{p-1,1}$. Then we have $db_{p-1,1} = 0$ and $\|b_{p-1,1}\| \leq c_2''(\|\mathcal{S}(c)\| + \|\delta c\|)$.

Indeed, as $p \geq 1$ and $db_{p,0} = 0$, by Lemma 3.2 there exists $b_{p-1,0} \in \mathcal{C}^0(\mathcal{U}, \Lambda^{p-1})$ such that $db_{p-1,0} = b_{p,0}$ and $\|b_{p-1,0}\| \leq cst \|b_{p,0}\|$. By definition of $b_{p,0}$ and by $(a')_{p+1}$ of the previous induction we obtain then $(a'')_1$. Let us consider now $b_{p-1,1} = \delta b_{p-1,0} + c_{p-1,1}$. Then we have $db_{p-1,1} = \delta b_{p,0} + c_{p,1} = -\delta c_{p,0} + c_{p,1} = 0$.

Finally, by construction and by (3.3) $\|b_{p-1,1}\| \leq cst(\|b_{p-1,0}\| + \|c_{p-1,1}\|)$. This last inequality combined with $(a'')_1$ and $(a')_p$ leads to $(b'')_1$.

Induction hypothesis: (for $1 \leq q < p-1$) there exist positive constants c''_1, c''_2 depending only on n, p, κ and ε such that

$(a'')_q$ there exists $b_{p-q,q-1} \in \mathcal{C}^{q-1}(\mathcal{U}, \Lambda^{p-q})$ such that $db_{p-q,q-1} = b_{p-(q-1),q-1}$ and $\|b_{p-q,q-1}\|^2 \leq c''_1(\|\mathcal{S}(c)\| + \|\delta c\|)$.

$(b'')_q$ Let $b_{p-q,q} = \delta b_{p-q,q-1} + (-1)^{q+1} c_{p-q,q}$. Then we have $db_{p-q,q} = 0$ and $\|b_{p-q,q}\| \leq c''_2(\|\mathcal{S}(c)\| + \|\delta c\|)$.

Proof: suppose the induction hypothesis holds for some $1 \leq q \leq p-1$ and let us show it holds for $q+1$. By $(b'')_q$ and Lemma 3.2 there exists $b_{p-(q+1),q} \in \mathcal{C}^q(\mathcal{U}, \Lambda^{p-(q+1)})$ such that $db_{p-(q+1),q} = b_{p-q,q}$ and $\|b_{p-(q+1),q}\|^2 \leq cst\|b_{p-q,q}\|^2$ and it suffices to use $(b'')_q$ to obtain $(a'')_{q+1}$. Then consider $b_{p-(q+1),q+1} = \delta b_{p-(q+1),q} + (-1)^q c_{p-(q+1),q+1}$. We have

$$
\begin{aligned}
db_{p-(q+1),q+1} &= \delta b_{p-q,q} + (-1)^q c_{p-q,q+1} \\
&= \delta(\delta b_{p-q,q-1} + (-1)^{q+1} c_{p-q,q}) + (-1)^q \delta c_{p-q,q} \\
&= 0.
\end{aligned}
$$

Finally, by construction of $b_{p-(q+1),q+1}$ we have

$$\|b_{p-(q+1),q+1}\| \leq cst(\|b_{p-(q+1),q}\| + \|c_{p-(q+1),q+1}\|)$$

and with $(a'')_{q+1}$ and $(a')_{p-q}$ we obtain $(b'')_{q+1}$. This ends the induction.

End of the induction: (for $q = p$) we obtain $b_{0,p} \in \mathcal{C}^p(\mathcal{U}, \Lambda^0)$ such that $db_{0,p} = 0$ i.e. $b_{0,p} \in \mathcal{C}^p(\mathcal{U})$ and $\delta b_{0,p} = (-1)^{p+1}\delta c_{0,p} = (-1)^{p+1} c_{0,p+1} = (-1)^{p+1}\delta c$. Hence, $b_{0,p}$ and c have same coexact part and as c is already coexact we obtain by $(b'')_p$, $\|c\| \leq \|b_{0,p}\| \leq cst(\|\mathcal{S}(c)\| + \|\delta c\|)$. In particular,

$$\|\mathcal{S}(c)\| \geq \frac{1}{cst}\|c\| - \|\delta c\|$$

then let $\Lambda' = \frac{1}{4cst^2}$ so that if $\|\delta c\|^2 \leq \Lambda\|c\|^2$ then $\|\mathcal{S}(c)\| \geq c'_2\|c\|$. This ends the proof of $(ii)_{\mathcal{S}}$. □

3.3 Upper bounds on the spectra

Lemma 3.8 *Let (M^n, g) be a compact connected Riemannian manifold and let $\mathcal{U}$ be a finite contractible open cover of M such that there exists $\nu > 0$ such that $|\{j : U_j \cap U_I \neq \emptyset\}| \leq \nu$ for any $I \in S_q(\mathcal{U})$ and any $q \geq 0$. Then there exists a positive constant c depending only on ν and p such that $\lambda_{k,q}(\mathcal{U}) \leq c$ for any $k = 1, \ldots, |S_q(\mathcal{U})| - \check{b}_q(\mathcal{U})$.*

Proof: it suffices to show the result for the spectrum of $\delta^*\delta$ on $\delta^*\mathcal{C}^{p+1}(\mathcal{U})$. We are going to show that there exists a positive constant depending only on ν and p such that for any $b \in \mathcal{C}^p(\mathcal{U})$

$$\|\delta b\|^2 \leq cst\|b\|^2 \tag{3.3}$$

and then the variational characterization of the spectrum of $\delta^*\delta$ will imply the claim. Recall that $\delta b(I) = \sum_{i\in I} \epsilon(i, I\backslash i)b(I\backslash i)$ where $\epsilon(i, I\backslash i)$ denotes the signature of the permutation ordering $\{i\}\cup(I\backslash i)$ to obtain I and $I \in S_{p+1}(\mathcal{U})$. Hence

$$|\delta b(I)|^2 \leq (p+2)\sum_{i\in I} |b(I \setminus i)|^2.$$

This implies that

$$\begin{aligned}\|\delta b\|^2 = \sum_{I\in S_{p+1}(\mathcal{U})} |\delta b(I)|^2 &\leq (p+2) \sum_{I\in S_{p+1}(\mathcal{U})} \sum_{i\in I} |b(I\setminus i)|^2 \\ &\leq (p+2)\nu \sum_{J\in S_p(\mathcal{U})} |b(J)|^2 = (p+2)\nu\|b\|^2\end{aligned}$$

which is the claim. □

Lemma 3.9 *Let $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ and X an ε-discretization with $0 < \varepsilon \leq r_0$. Let $1 \leq p \leq n-1$. Then there exists a positive constant c' depending only on n, p, κ and ε such that $\lambda_{k,p}(M) \leq c'$ for any $k \leq |S_p(\mathcal{U}_X)| - \check{b}_p(\mathcal{U}_X)$.*

Proof: it suffices to show the result for $k = |S_p(\mathcal{U}_X)| - \check{b}_p(\mathcal{U}_X)$. By a theorem of Abresch (see [11], Theorem 1.12) there exists a Riemannian metric $\tilde{g}$ on M such that

(a) $e^{-\frac{1}{4}}g \leq \tilde{g} \leq e^{\frac{1}{4}}g$

(b) $|\nabla^g - \nabla^{\tilde{g}}| \leq \frac{1}{4}$

(c) $|K_{\tilde{g}}| \leq \tilde{\kappa}(n, \kappa)$ and $|\nabla^{\tilde{g}} R_{\tilde{g}}| \leq K(n, \kappa)$

where $\tilde{\kappa}$ and K depend only on n and κ. By Proposition 3.3. of [14], there exist a positive constant c depending only on $e^{\frac{1}{4}}$ such that

$$\lambda_{k,p}(M, g) \leq c\lambda_{k,p}(M, \tilde{g}).$$

Therefore it suffices to show the claim for $(M, \tilde{g})$. By Remark 2.8 and by construction of $\tilde{g}$, there exists a positive constant d depending only on n, p, κ, ε such that $|S_p(\mathcal{U}_X)| \leq dVol(M, \tilde{g})$. Moreover, there exist $\alpha > 0$ depending only on p, n, κ and ε such that if Y is an α-discretization of $(M, \tilde{g})$ then

$|Y| \geq |S_p(\mathcal{U}_X)|$ and $\check{b}_p(\mathcal{U}_Y) = \check{b}_p(\mathcal{U}_X)$. Consider then the disjoint balls (for $\tilde{g}$) centered at $y \in Y$ of radius $\frac{\alpha}{2}$. From Proposition 2.3. of [14], on any of these balls there exists a p-form ω_y which is zero on the boundary of the ball, so that we can extend ω_y by zero to obtain a p-form on M also denoted ω_y such that

$$\frac{\|d\omega_y\|_{\tilde{g}}^2 + \|d_{\tilde{g}}^*\omega_y\|_{\tilde{g}}^2}{\|\omega_y\|_{\tilde{g}}^2} \leq \mu(n, p, \kappa, \varepsilon) \tag{3.4}$$

where $\mu(n, p, \kappa, \varepsilon)$ is a positive constant depending only on n, p, κ and ε. Moreover, we can choose ω_y such that $\|\omega_y\| = 1$.

Let then V the vector subspace of p-forms spanned by $\{\omega_y : y \in Y\}$. By construction, ω_y is orthogonal to ω_x if $x \neq y$. In particular, V is of dimension $|Y|$. Therefore, by the variational characterization of the spectrum, we obtain

$$\lambda_{|Y|-\check{b}_p(\mathcal{U}_Y),p}(M, \tilde{g}) \leq \max\left\{\frac{\|d\omega\|_{\tilde{g}}^2 + \|d_{\tilde{g}}^*\omega\|_{\tilde{g}}^2}{\|\omega\|_{\tilde{g}}^2} : \omega \in V \setminus \{0\}\right\}. \tag{3.5}$$

Furthermore, if $\omega = \sum_{y \in Y} a_y \omega_y$, then as the balls centered on Y of radius $\frac{\alpha}{2}$ are disjoint $\|\omega\|_{\tilde{g}}^2 \geq \sum_{y \in Y} a_y^2$ and combined with (3.4) this implies that

$$\|d\omega\|_{\tilde{g}}^2 \leq \sum_{y \in Y} a_y^2 \|d\omega_y\|_{\tilde{g}}^2 \leq \mu \|\omega\|_{\tilde{g}}^2 \tag{3.6}$$

and

$$\|d_{\tilde{g}}^*\omega\|_{\tilde{g}}^2 \leq \sum_{y \in Y} a_y^2 \|d_{\tilde{g}}^*\omega_y\|_{\tilde{g}}^2 \leq \mu \|\omega\|_{\tilde{g}}^2. \tag{3.7}$$

It suffices then to introduce (3.6) and (3.7) in (3.5) to obtain that

$$\lambda_{|Y|-\check{b}_p(\mathcal{U}_Y),p}(M, \tilde{g}) \leq 2\mu$$

and in particular that $\lambda_{k,p}(M, g) \leq 2c\mu$, for $k \leq |S_p(\mathcal{U}_X)| - \check{b}_p(\mathcal{U}_X)$. $\square$

3.4 Proof of the main result

We prove now Theorem 3.1. We will only proceed to the proof of the inequality $\lambda_{k,p}(M) \leq c_2 \lambda_{k,p}(X)$ as the other inequality can be proved in the same way using the corresponding results. Recall it suffices to prove the result for d^*d on coexact forms and for $\delta^*\delta$ on coexact Čech cochains. We proceed in two steps. Let Λ' given by $(ii)_{\mathcal{S}}$.

First step: assume $\lambda^{\delta^*}_{k,p}(X) \geq \Lambda'$. Then, $\lambda^{d^*}_{k,p}(M) \leq \Lambda'^{-1}\lambda^{\delta^*}_{k,p}(X)\lambda^{d^*}_{k,p}(M)$ and by Lemma 3.9 we obtain $\lambda^{d^*}_{k,p}(M) \leq \Lambda'^{-1}c'\lambda^{\delta^*}_{k,p}(X)$ which is the claim.

Second step: assume now $\lambda^{\delta^*}_{k,p}(X) \leq \Lambda'$. Let us consider $c_1, \dots, c_k \in \delta^*\mathcal{C}^{p+1}(\mathcal{U},)$ the Čech $\lambda^{\delta^*}_{1,p}(X), \dots, \lambda^{\delta^*}_{k,p}(X)$-eigencochains such that $(c_i, c_j) = \delta_{ij}$. Denote by V^k the k-dimensional vector subspace of $\delta^*\mathcal{C}^{p+1}(\mathcal{U})$ they span. By the variational characterization of the spectrum we have

$$\lambda^{\delta^*}_{k,p}(X) = \max\left\{\frac{\|\delta c\|^2}{\|c\|^2} : c \in V^k \setminus \{0\}\right\}.$$

Let us consider now $\mathcal{S}V^k$ the vector subspace of $d^*\Lambda^{p+1}(M)$ spanned by $\{\mathcal{S}(c_1), \dots, \mathcal{S}(c_k)\}$. Then if $\mathcal{S}(c) \in \mathcal{S}V^k$, $\mathcal{S}(c) = \sum_{i=1}^k a_i\mathcal{S}(c_i)$ with $c = \sum_{i=1}^k a_i c_i \in V^k$. So that we have $\|\delta c\|^2 \leq \lambda^{\delta^*}_{k,p}(X)\|c\|^2 \leq \Lambda'\|c\|^2$. Therefore, by $(ii)_{\mathcal{S}}$ we obtain

$$\|\mathcal{S}(c)\|^2 \geq c_2'\|c\|^2 \tag{3.8}$$

and this says in particular that $\mathcal{S}V^k$ is of dimension k. Using the variational characterization of $\lambda^{d^*}_{k,p}(M)$ we get

$$\begin{aligned}\lambda^{d^*}_{k,p}(M) &\leq \max\left\{\frac{\|d\omega\|^2}{\|\omega\|^2} : \omega \in \mathcal{S}V^k \setminus \{0\}\right\}\\ &= \max\left\{\frac{\|d\mathcal{S}(c)\|^2}{\|\mathcal{S}(c)\|^2} : c \in V^k \setminus \{0\}\right\}.\end{aligned}$$

Finally, (3.8) and $(i)_{\mathcal{S}}$ imply that $\frac{\|d\mathcal{S}(c)\|^2}{\|\mathcal{S}(c)\|^2} \leq \frac{c_1'}{c_2'}\frac{\|\delta c\|^2}{\|c\|^2}$ so that we obtain

$$\lambda^{d^*}_{k,p}(M) \leq \frac{c_1'}{c_2'}\max\left\{\frac{\|\delta c\|^2}{\|c\|^2} : c \in V^k \setminus \{0\}\right\} = \frac{c_1'}{c_2'}\lambda^{\delta^*}_{k,p}(X) \tag{3.9}$$

which concludes the proof. □

4 Applications

In this section, we develop several consequences of Theorem 3.1 or of the methods used to prove Theorem 3.1.

4.1 A lower bound for the spectrum of the Laplacian on differential forms

The goal of this section is to prove the following theorem.

Theorem 4.1 *Let $(M, g) \in \mathbb{M}(n, \kappa, r_0)$. Let $1 \leq p \leq n-1$. Then there exist positive constants c, c' depending only on n, p, κ and r_0 such that*

$$\lambda_{1,p}(M) \geq \frac{c}{Vol(M)e^{c'Vol(M)}}$$

where $Vol(M)$ denotes the volume of (M, g).

By Theorem 3.1, it suffices to choose a suitable discretization X of M and prove then a similar result for $\lambda_{1,p}(X)$. To that aim we need the following lemma.

Lemma 4.2 *Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear operator with matrix coefficients (in the canonical bases) in $\{-1, 0, 1\}$. Suppose there exists an integer k such that any column and any row has at most k non-zero coefficients. Then, there exists $B : \mathbb{R}^n \to \mathbb{R}^m$ such that $ABAv = Av$ for any $v \in \mathbb{R}^m$ and*

$$||Bu||^2 \leq nk^{2n}||u||^2$$

for any $u \in \mathbb{R}^n$.

Remark 4.3 *In [31], the author proves a similar result (see Lemma A.5 in [31]) but with a better constant for the matrix norm of B. He asserts that $||Bu||^2 \leq c(k)m||u||^2$. With the following example we will show that the proof of Trèves' result is not correct. Consider the matrix A with m columns and $m-1$ rows given by*

$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & -1 \end{pmatrix}$$

and consider $v = \sum_{i=1}^{m} ie_i$ in $\mathbb{R}^m$. Then $Av = -\sum_{i=1}^{m-1} e_i$ in $\mathbb{R}^{m-1}$. So that $||Av||^2 = m-1$. An easy calculation shows that if we choose the $m-1$ first columns of A to span $Im(A)$ and if we consider the map B defined by Trèves, then $BAv = \sum_{i=1}^{m-1} -(m-i)e_i$ in $\mathbb{R}^m$. Hence $||BAv||^2 = \frac{(m-1)m(2m-1)}{6} = \frac{m(2m-1)}{6}||Av||^2$ which contradicts Lemma A.5 in [31] (here $k = 2$). The assertion A.44 in [31] is wrong. It is not clear to us how we can correct this mistake. We think that we should replace k^{2n} by n^l for a suitable l in Lemma 4.2 but we cannot prove it yet. Let us emphasize that the constant given by Trèves can not be suitable. The result of Trèves would imply a lower bound for the first positive eigenvalue of the combinatorial Laplacian on a graph with n vertices of the kind $\frac{cst}{n}$ (see Theorem 4.4). But it is a well-known fact that the first positive eigenvalue of the combinatorial Laplacian (for functions) on a cyclic graph with n vertices behaves like $\frac{cst}{n^2}$.

Proof of Lemma 4.2: let r be the dimension of $Im(A)$. Without lost of generality we can suppose that the r first columns $\{a_1, \dots a_r\}$ of A span $Im(A)$. Then define B as follows (as in Lemma A.5 of [31]). On the orthogonal complement of $Im(A)$ let $B = 0$. Moreover, if $u = Av$ then write u in the basis $\{a_1, \dots, a_r\}$ of $Im(A)$, $u = \sum_{i=1}^{r} u_i a_i$ and define $Bu = \sum_{i=1}^{r} u_i e_i$ where $\{e_i\}$ denotes the canonical basis of $\mathbb{R}^m$. An immediate consequence of the definition of B is that $ABAv = Av$. Moreover, $\|Bu\|^2 = \sum_{i=1}^{r} u_i^2$. Let us show now that

$$u_i^2 \leq k^{2n} \|u\|^2. \tag{4.1}$$

This will imply $\|Bu\|^2 \leq rk^{2n}\|u\|^2 \leq nk^{2n}\|u\|^2$ which is the claim.
We prove (4.1) for $i = 1$. Let V_1 the vector space spanned by $\{a_2, \dots, a_r\}$ and let $V_1^{\perp}$ its orthogonal complement in $Im(A)$. Consider $P_1 : Im(A) \to V_1^{\perp}$ the orthogonal projection onto $V_1^{\perp}$. We have $P_1(u) = u_1 P_1(a_1)$ so that

$$u_1^2 = \frac{\|P_1(u)\|^2}{\|P_1(a_1)\|^2} \leq \frac{\|u\|^2}{\|P_1(a_1)\|^2}. \tag{4.2}$$

We can write $P_1(a_1) = a_1 + \alpha_2 a_2 + \ldots + \alpha_r a_r$ with $(P_1(a_1)|a_j) = 0$ for $j = 2, \dots, r$ and $(P_1(a_1)|a_1) = \|P_1(a_1)\|^2$. In matrix form we obtain

$$\begin{pmatrix} \|a_1\|^2 & (a_1|a_2) & \dots & (a_1|a_r) \\ (a_1|a_2) & \|a_2\|^2 & \dots & (a_2|a_r) \\ \vdots & \vdots & \ddots & \vdots \\ (a_1|a_r) & (a_2|a_r) & \dots & \|a_r\|^2 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{pmatrix} = \begin{pmatrix} \|P_1(a_1)\|^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and if we call P the matrix $r \times r$ above and Q the submatrix of P obtained by removing the first row and the first column of P we get that

$$\|P_1(a_1)\|^2 = \frac{|\det(P)|}{|\det(Q)|}.$$

As $\{a_1, \dots a_r\}$ are linearly independent, $\det(P) \neq 0$. Moreover, P is a matrix with integer coefficients so that $|\det(P)| \geq 1$. It remains to find an upper bound for $|\det(Q)|$. So, we are going to prove by induction that the minors of P of size $l \times l$ are bounded above by k^{2l-1}.

The first step of induction asserts that the minors of P of size 1×1 are bounded above by k. This is a direct consequence of the assumption that each column of A has at most k non-zero coefficients. Suppose then that the minors of P of size $l \times l$ are bounded above by k^{2l-1}. Consider then D a minor of P of size $(l+1) \times (l+1)$. Then D can be written as

$$D = \sum_{j=1}^{l+1} c_j D_j$$

where $(c_1, \dots, c_{l+1})$ is a part of a line of P and D_j is a minor of P of size $l \times l$. By construction of P, the coefficients c_j can be written as follows. There exists $1 \leq J \leq r$ such that

$$c_j = (a_J | a_{i_j}) \text{ for a suitable } i_j$$

so that

$$|D| \;=\; \left| \sum_{j=1}^{l+1} (a_J | a_{i_j}) D_j \right| = \left| \sum_{i=1}^{n} (a_J | e_i) \sum_{j=1}^{l+1} (e_i | a_{i_j}) D_j \right| .$$

But by assumption, the i^{th} row of A has at most k coefficients of absolute value 1 and by induction hypothesis we get $| \sum_{j=1}^{l+1} (e_i | a_{i_j}) D_j | \leq k \cdot k^{2l-1}$. Moreover, by assumption the J^{th} column of A has at most k coefficients of absolute value 1 and with the previous remark this implies

$$|D| \leq k \cdot k \cdot k^{2l-1}$$

and this ends the induction. We apply then the result to $|\det(Q)|$ and we obtain $|\det(Q)| \leq k^{2r-3} \leq k^{2n}$. Finally, we deduce that

$$||P_1(a_1)||^2 \geq \frac{1}{k^{2n}}$$

and combined with (4.2) this implies (4.1). $\square$

Theorem 4.4 *Let $\mathcal{U}$ be a finite open cover of M compact. Let $p \geq 0$. Assume there exists ν such that $|\{j : U_j \cap U_I \neq \emptyset\}| \leq \nu$ for any $I \in S_q(\mathcal{U})$ and $q \geq 0$. Then there exist positive constants $c(\nu, p)$, $c'(\nu, p)$ depending only on ν and p such that*

$$\lambda_{1,p}(\mathcal{U}) \geq \frac{c(\nu, p)}{|\mathcal{U}| \cdot e^{c'(\nu,p)|\mathcal{U}|}} \; .$$

Proof: it suffices to prove the result for $\lambda_{1,p}^{\delta^*}(\mathcal{U})$. By the variational characterization of the spectrum, we have

$$\lambda_{1,p}^{\delta^*}(\mathcal{U}) = \min_V \max \left\{ \frac{||\delta c||^2}{||c||^2} : c \in V \setminus \{0\} \right\}$$

where V ranges over all 1-dimensional vector subspaces of $\delta^* \mathcal{C}^{p+1}(\mathcal{U})$. As in Proposition 3.1 of [14], we can get from the above characterization the following description

$$\lambda_{1,p}^{\delta^*}(\mathcal{U}) = \min_V \max \left\{ \frac{||\delta c||^2}{||b||^2} : \delta b = \delta c \; , \text{ and } \delta c \in V \right\}$$

where V ranges over all 1-dimensional vector subspaces of $\delta\mathcal{C}^p(\mathcal{U})$. In particular, if we consider V that realizes the minimum, then

$$\lambda_{1,p}^{\delta^*}(\mathcal{U}) = \max\left\{\frac{||\delta c||^2}{||b||^2} : \delta b = \delta c \ , \ \text{and } \delta c \in V\right\}. \tag{4.3}$$

Consider then the canonical basis of $\mathcal{C}^q(\mathcal{U})$ given by

$$\{e_I : S_q(\mathcal{U}) \to \mathbb{R}, I \in S_q(\mathcal{U}) \text{ such that } e_I(J) = \delta_{IJ}\}.$$

In this bases, the matrix of $\delta : \mathcal{C}^p(\mathcal{U}) \to \mathcal{C}^{p+1}(\mathcal{U})$ has coefficients in $\{-1, 0, 1\}$ and has at most $K(\nu, p) = \max\{\nu, p+2\}$ non-zero coefficients by row and by column. Hence we can apply Lemma 4.2 to δ to obtain that for any $c \in \mathcal{C}^p(\mathcal{U})$, there exists $b \in \mathcal{C}^p(\mathcal{U})$ such that $\delta b = \delta c$ and

$$||b||^2 \leq |S_{p+1}(\mathcal{U})| K(\nu, p)^{|S_{p+1}(\mathcal{U})|} ||\delta c||^2. \tag{4.4}$$

Finally, if we introduce (4.4) in (4.3) and by Remark 2.8, we obtain

$$\lambda_{1,p}^{\delta^*}(\mathcal{U}) \geq \frac{1}{|S_{p+1}(\mathcal{U})| K(\nu, p)^{|S_{p+1}(\mathcal{U})|}} \geq \frac{c(\nu, p)}{|\mathcal{U}| \cdot e^{c'(\nu,p)|\mathcal{U}|}} \ . \ \square$$

Proof of Theorem 4.1: let $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ and X a $\frac{\rho_0}{4}$-discretization of M (where ρ_0 is given by Lemma 2.9). By Theorem 3.1, there exists $c_1(n, p, \kappa, r_0) > 0$ such that

$$\lambda_{1,p}(M, g) \geq c_1 \lambda_{1,p}(X). \tag{4.5}$$

Moreover, by Theorem 4.4 there exist positive constants c_2, c_3 depending only on n, p, κ and r_0 such that

$$\lambda_{1,p}(X) \geq \frac{c_2}{|\mathcal{U}| \cdot e^{c_3|\mathcal{U}|}}. \tag{4.6}$$

Finally, by Remark 2.8 there exists $c_4(n, p, \kappa, r_0) > 0$ such that

$$|\mathcal{U}| \leq c_4 Vol(M). \tag{4.7}$$

To conclude, put (4.5), (4.6) and (4.7) together to obtain that there exist positive constants c, c' depending only on n, p, κ and r_0 such that

$$\lambda_{1,p}(M, g) \geq \frac{c}{Vol(M) e^{c' Vol(M)}}$$

and this ends the proof. $\square$

4.2 Whitney forms: a natural way of smoothing

As suggested in [16], a candidate for the smoothing operator should be given by Whitney forms in the following way. Let

$$\tilde{\mathcal{S}} : \delta^*\mathcal{C}^{p+1}(\mathcal{U}) \to d^*\Lambda^{p+1}(M)\ ,\ c \mapsto \tilde{\mathcal{S}}(c) = coex(W(c))$$

where W is the Whitney map (see Appendix A.1). The results of Dodziuk and Patodi in [16] concerning Whitney forms can not be used in our context as their approximations (obtained thanks to the heat kernel) involve the manifold itself. More precisely, the constants there depend on the volume of the manifold.

Here, we show that there exist positive constants $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{\Lambda}$ depending only on n, p, κ and ε such that

$(i)_{\tilde{\mathcal{S}}}$ $||d\tilde{\mathcal{S}}(c)||^2 \leq \tilde{c}_1||\delta c||^2$, for any $c \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$,

$(ii)_{\tilde{\mathcal{S}}}$ $||\tilde{\mathcal{S}}c||^2 \geq \tilde{c}_2||c||^2$, for any $c \in \delta^*\mathcal{C}^{p+1}(\mathcal{U})$ satisfying $||\delta c||^2 \leq \tilde{\Lambda}||c||^2$.

The inequality $(i)_{\tilde{\mathcal{S}}}$ is a direct consequence of Lemma A.4 and Lemma A.5. Indeed, as $dc = 0$ we have

$$d\tilde{\mathcal{S}}(c) = dW(c) = (p+1)W(\delta c)$$

and Lemma A.5 leads to $(i)_{\tilde{\mathcal{S}}}$.

The second inequality is less obvious and it can be shown adding a point to the first induction in the construction of $\mathcal{S}$ in Section 3.2. The idea is to construct a p-form $u^{(0)}$ linking $\mathcal{S}(c)$ and $\tilde{\mathcal{S}}(c)$ playing the same role as $v^{(p)}$ in the construction of $\mathcal{D}$ (see Section 3.1). Then the control on the norm of $\mathcal{S}(c)$ (see $(ii)_{\mathcal{S}}$) and a control on the norm of $u^{(0)}$ will imply the desired inequality.

Proof of $(ii)_{\tilde{\mathcal{S}}}$: in the "first step of induction" (of Section 3.2), add

$(c')_1$ there exists a positive constant c_3' depending only on n, p, κ and ε such that if

$$u^{(p)} = (-1)^{p+2}\frac{1}{p+1}W(c_{0,p})$$

then $||u^{(p)}||^2 \leq c_3'||\delta c||^2$ and

$$dW(c) = (-1)^{p+2}(p+1)\left(du^{(p)} + \frac{(-1)^{(p+2)(p+1)}}{p+1}W(c_{1,p})\right).$$

Indeed, by Lemma A.5 and $(a')_1$, $\|u^{(p)}\|^2 \leq cst\|c_{0,p}\|^2 \leq c_3'\|\delta c\|^2$. Moreover, by Lemma A.4 and $(b')_1$

$$\begin{aligned} du^{(p)} &= \frac{(-1)^{p+2}}{p+1}\left((p+1)W(c_{0,p+1}) + (-1)^p W(c_{1,p})\right) \\ &= \frac{(-1)^{p+2}}{p+1}\left(dW(c) - (-1)^{p+1}W(c_{1,p})\right). \end{aligned}$$

The induction hypothesis gets

$(c')_q$ there exists a positive constant c_3' depending only on n, p, κ and ε such that if

$$u^{(p+1-q)} = u^{(p+1-(q-1))} + \frac{(-1)^{p+2}(-1)^{p+1}\dots(-1)^{p+2-(q-1)}}{(p+1)p(p-1)\dots(p+2-q)}W(c_{q-1,p+1-q})$$

then $\|u^{(p+1-q)}\|^2 \leq c_3'\|\delta c\|^2$ and

$$\frac{(-1)^{p+2}}{p+1}dW(c) = du^{(p+1-q)} + \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-q)}W(c_{q,p+1-q}).$$

Then, the proof goes as follows. Let us consider

$$u^{(p-q)} = u^{(p+1-q)} + \frac{(-1)^{p+2}(-1)^{p+1}\dots(-1)^{p+2-q}}{(p+1)p(p-1)\dots(p+2-(q+1))}W(c_{q,p-q}).$$

Then, by $(c')_q$, by Lemma A.5 and by $(a')_{q+1}$, we obtain $\|u^{(p-q)}\|^2 \leq c_3'\|\delta c\|^2$. Moreover, by $(c')_q$ and Lemma A.4 we have

$$\begin{aligned} \frac{(-1)^{p+2}}{p+1}dW(c) &= du^{(p-q)} - \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-(q+1))}dW(c_{q,p-q}) \\ &\quad + \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-q)}W(c_{q,p+1-q}) \\ &= du^{(p-q)} - \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-q)}W(c_{q,p+1-q}) \\ &\quad + \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-(q+1))}(-1)^{p+1-q}W(c_{q+1,p-q}) \\ &\quad + \frac{(-1)^{p+2}\dots(-1)^{p+2-q}}{(p+1)\dots(p+2-q)}W(c_{q,p+1-q}) \end{aligned}$$

and the claim follows.

At the end of the induction (for $q = p+1$), we obtain a p-form $u^{(0)}$ such that $\|u^{(0)}\|^2 \leq c'_3\|\delta c\|^2$ and

$$\begin{aligned} dW(c) &= (-1)^p(p+1)\left(du^{(0)} + k(p)W(c_{p+1,0})\right) \\ &= (-1)^p(p+1)\left(du^{(0)} + k(p)W(r(d\mathcal{S}(c)))\right) \\ &= (-1)^p(p+1)\left(du^{(0)} + k(p)d(\mathcal{S}(c))\right) \end{aligned}$$

where $k(p)$ is a constant depending only on p. Moreover, as $\mathcal{S}(c)$ is a coexact p-form, this implies

$$coex(W(c)) = (-1)^p(p+1)\left(coex(u^{(0)}) + k(p)\mathcal{S}(c)\right)$$

so that

$$\begin{aligned} \|coex(W(c))\| &\geq (p+1)|k(p)|\cdot\|\mathcal{S}(c)\| - (p+1)\|u^{(0)}\| \\ &\geq (p+1)|k(p)|\cdot\|\mathcal{S}(c)\| - (p+1)(c'_3)^{\frac{1}{2}}\|\delta c\|. \end{aligned}$$

But, by $(ii)_{\mathcal{S}}$, if $\|\delta c\|^2 \leq \Lambda'\|c\|^2$ then $\|\mathcal{S}(c)\| \geq (c'_2)^{\frac{1}{2}}\|c\|$. Therefore,

$$\|coex(W(c))\| \geq (p+1)|k(p)|(c'_2)^{\frac{1}{2}}\left(\|c\| - \sqrt{\frac{c'_3}{c'_2 k(p)^2}}\|\delta c\|\right)$$

Finally, if $\|\delta c\|^2 \leq \tilde{\Lambda}\|c\|^2$, with $\tilde{\Lambda} = \min\left\{\Lambda', \frac{k(p)^2 c'_2}{4c'_3}\right\}$, then

$$\|coex(W(c))\| \geq \frac{1}{2}(p+1)|k(p)|(c'_2)^{\frac{1}{2}}\|c\|$$

which is the desired inequality in $(ii)_{\tilde{\mathcal{S}}}$. $\square$

4.3 Another proof of "McGowan lemma"

In [25], the author gives a lower bound for the N^{th} eigenvalue of Δ on exact 2-forms on a compact Riemannian manifold M (see Lemma 2.3 in [25]) where N depends on an open cover of M. In particular, if the open cover is contractible then $N-1$ is the number of non-empty intersections of triples of open sets in the open cover. The lower bound depends then essentially on lower bounds for the smallest positive eigenvalue of Δ on exact forms on the open sets of the cover, on the intersection of pairs of such open sets and on the intersection of triples of such open sets. The proof of McGowan relies also on the double complexe of Čech - de Rham and can be compared to the induction done in Section 3.1 to construct the discretizing operator $\mathcal{D}$. So it is not so surprising that we obtain the following generalization of the lemma. The main difference is that in our technique, if the discretization is of sufficiently small mesh then Lemma 2.10 gives the lower bound for the spectrum on the intersections. But, then N can get quite large as it is comparable to the number of open sets in the open cover. Let us now state and prove the result.

Lemma 4.5 *Let $n \geq 1$, $\kappa \geq 0$ and $r_0 > 0$. Then there exists a positive constant $\lambda(n, \kappa, r_0)$ depending only on n, κ and r_0 such that for any $(M, g) \in \mathbb{M}(n, \kappa, r_0)$ we have*

$$\lambda_{N,p}^{d^*}(M) \geq \lambda(n, \kappa, r_0)$$

where $N \leq c(n, p, \kappa, r_0) Vol(M)$ and $c(n, p, \kappa, r_0)$ is a positive constant.

Proof: let ρ_0 be given by Lemma 2.9 and let X a $\frac{\rho_0}{4}$-discretization of M. Then the discretizing operator

$$\mathcal{D} : d^*\Lambda^{p+1}(M) \to \delta^*\mathcal{C}^{p+1}(\mathcal{U})$$

constructed in Section 3.1 satisfies $(i)_\mathcal{D}$ and $(ii)_\mathcal{D}$. Let then

$$N = dim\left(\delta^*\mathcal{C}^{p+1}(\mathcal{U})\right) + 1.$$

Consider moreover $\phi_1, \ldots, \phi_N$ the N first eigenforms in $d^*\Lambda^{p+1}(M)$. By definition of N, there exist $a_1, \ldots, a_N$ such that $\sum_{i=1}^N a_i \mathcal{D}(\phi_i) = 0$ and $\sum_{i=1}^N a_i \phi_i \neq 0$. In particular, by $(ii)_\mathcal{D}$, we get

$$\left\| d\left(\sum_{i=1}^N a_i\phi_i\right)\right\|^2 \geq \Lambda \left\|\sum_{i=1}^N a_i\phi_i\right\|^2$$

and thanks to the variational characterization of the spectrum

$$\lambda_{N,p}^{d^*}(M) = \max\left\{\frac{\|d\phi\|^2}{\|\phi\|^2} : \phi \in \langle\phi_1, \ldots, \phi_N\rangle \setminus \{0\}\right\} \geq \Lambda.$$

Note that by Remark 2.8, we have $N \leq |\mathcal{S}_p(\mathcal{U}_X)| \leq c_2 \frac{\nu^p}{(p+1)!} Vol(M)$ where c_2 and ν depend only on n, p, κ and r_0. $\square$

A Appendix

A.1 Whitney forms

Let (M^n, g) be a compact connected n-dimensional Riemannian manifold without boundary. Let $\mathcal{U}$ be a finite contractible open cover of M. Let $\{\varphi_j\}$ be a partition of unity subordinated to $\mathcal{U}$. Let ν be a bound on the cardinality of the sets $\{j : U_j \cap U_I \neq \emptyset\}$, $I \in S_q(\mathcal{U})$, $q \geq 0$.

Definition A.1 *For any $I = \{i_0, \ldots, i_q\} \in S_q(\mathcal{U})$, we define the* ***Whitney form*** *$W_I \in \Lambda^q(M)$ by*

$$W_I = \sum_{j=0}^{q} (-1)^j \varphi_{i_j} d\varphi_{i_0} \wedge \ldots \wedge d\varphi_{i_{j-1}} \wedge d\varphi_{i_{j+1}} \wedge \ldots \wedge d\varphi_{i_q}$$

Remark A.2 *Note that W_I has support in U_I. Moreover, we have $dW_I = (q+1) d\varphi_{i_0} \wedge \ldots \wedge d\varphi_{i_q}$, for $I = \{i_0, \ldots, i_q\}$. In the sequel, we will write $d\varphi_I = d\varphi_{i_0} \wedge \ldots \wedge d\varphi_{i_q}$.*

We can extend the definition of Whitney forms to q-cochains as follows.

Definition A.3 *Let $W : \mathcal{C}^q(\mathcal{U}, \Lambda^p) \to \Lambda^{p+q}(M)$ the application defined by*

$$W(c) = \sum_{I \in S_q(\mathcal{U})} W_I \wedge c(I).$$

The application W restricted to Čech cochains is the Whitney map introduced by Whitney (see [32]) (up to a constant). The following lemma generalizes the well-known fact that the Whitney map commutes with the exterior differential and the coboundary.

Lemma A.4 *For any $c \in \mathcal{C}^q(\mathcal{U}, \Lambda^p)$, we have*

$$dW(c) = (q+1) W(\delta c) + (-1)^q W(dc).$$

Proof: we have

$$\begin{aligned} dW(c) &= \sum_{I \in S_q(\mathcal{U})} d(W_I \wedge c(I)) \\ &= \sum_{I \in S_q(\mathcal{U})} dW_I \wedge c(I) + (-1)^q \sum_{I \in S_q(\mathcal{U})} W_I \wedge dc(I) \\ &= (q+1) \sum_{I \in S_q(\mathcal{U})} d\varphi_I \wedge c(I) + (-1)^q W(dc). \end{aligned}$$

Let us now compute $W(\delta c)$. We have

$$W(\delta c) = \sum_{J \in S_{q+1}(\mathcal{U})} W_J \wedge \left(\sum_{j \in J} \epsilon(j, J \setminus j) c(J \setminus j) \right)$$

where $\epsilon(j, J \setminus j)$ is ± 1 according to the signature of the permutation ordering the set $\{j\} \cup (J \setminus j)$ in J. If we let $I = J \setminus j$, we can write

$$W(\delta c) = \sum_{I \in S_q(\mathcal{U})} \sum_{j : U_j \cap U_I \neq \emptyset} W_{\{j, I\}} \wedge c(I)$$

so that it suffices to show that

$$\sum_{j:U_j\cap U_I\neq\emptyset} W_{\{j,I\}} = d\varphi_I \tag{A.1}$$

to conclude the proof. Let us rewrite the expression as follows

$$\sum_{j:U_j\cap U_I\neq\emptyset} W_{\{j,I\}} = \sum_{j:U_j\cap U_I\neq\emptyset} \varphi_j d\varphi_I - d\varphi_j \wedge W_I. \tag{A.2}$$

But as $\{\varphi_j\}$ is a partition of unity $\sum_{j:U_j\cap U_I\neq\emptyset} \varphi_j = 1$ and $\sum_{j:U_j\cap U_I\neq\emptyset} d\varphi_j = 0$, hence (A.2) implies (A.1). $\square$

Lemma A.5 *There exists a positive constant k depending only on n, ν and on $\|d\varphi_j\|_\infty$ such that for any Čech cochain c, $\|W(c)\|^2 \leq k\|c\|^2$.*

Proof: it follows from the definition of W and from a direct calculation. $\square$

A.2 About the convexity of balls

Proof of Lemma 2.9: the main idea to prove this lemma is to smooth g to obtain a more regular metric $\tilde{g}$ and then compare $\tilde{g}$ to a Euclidean metric $\tilde{e}$. We do not compare directly g with a Euclidean metric as we need to control the difference between the several connections involved. So let $(M,g) \in \mathbb{M}(n,\kappa,r_0)$. It follows from a result of Abresch (see [11], Theorem 1.12) that there exists a Riemannian metric $\tilde{g}$ on M such that

(a) $e^{-\frac{1}{4}}g \leq \tilde{g} \leq e^{\frac{1}{4}}g$

(b) $|\nabla^g - \nabla^{\tilde{g}}| \leq \frac{1}{4}$

(c) $|K_{\tilde{g}}| \leq \tilde{\kappa}(n,\kappa)$ and $|\nabla^{\tilde{g}} R_{\tilde{g}}| \leq k(n,\kappa)$

where $\tilde{\kappa}$ and k depend only on n and κ. In particular, (a) implies that, the length of the curves, the distances and the volumes are comparable within a ratio depending only on n. Moreover, if B denotes a ball for g and $\tilde{B}$ a ball for $\tilde{g}$, we get $B(p, e^{-\frac{1}{8}}r) \subseteq \tilde{B}(p,r) \subseteq B(p, e^{\frac{1}{8}}r)$. Another consequence of the properties of $\tilde{g}$ is the existence of $\tilde{r}_0 > 0$ depending only on n, κ, r_0 such that

$$inj(M,\tilde{g}) \geq \tilde{r}_0. \tag{A.3}$$

Indeed, this is a direct consequence of Theorem 4.7 of Cheeger, Gromov and Taylor in [12] and Croke's Inequality. Then, a suitable candidate to be the diffeomorphism cited in the claim is the exponential map with respect to the metric $\tilde{g}$. Let then

$$\varphi = \widetilde{\exp}_p : B(0,\tilde{r}_0) \to \tilde{B}(p,\tilde{r}_0)$$

and $\tilde{e}$ the Euclidean metric on $\tilde{B}(p,\tilde{r}_0)$ induced by φ^{-1} and the normal coordinates. As soon as $e^{\frac{1}{8}}r \leq \tilde{r}_0$, we have $B(p,r) \subseteq \tilde{B}(p,\tilde{r}_0)$ and then $\varphi^{-1}(B(p,r))$ is well-defined. We are going to show now that there exists a positive constant $0 < \rho_0(n,\kappa,r_0) \leq e^{-\frac{1}{8}}\tilde{r}_0$ such that for any $B(q,\rho) \subseteq B(p,\rho_0) \subseteq \tilde{B}(p,\tilde{r}_0)$ we have

$$\varphi^{-1}(B(q,\rho)) \text{ is a Euclidean convex.} \tag{A.4}$$

This is equivalent to showing that the application

$$f : (B(q,\rho),\tilde{e}) \to \mathbb{R} \ , \ x \mapsto \frac{1}{2}d(q,x)^2 \tag{A.5}$$

is convex (w.r.t. $\tilde{e}$), in other words that the Hessian of f with respect to $\tilde{e}$ is non-negative i.e. $D^2_{\tilde{e}}f(U,U) \geq 0$ on $B(q,\rho)$, for ρ and ρ_0 well-chosen. Let us recall the following definition of the Hessian

$$D^2 f(U,V) = U \cdot df(V) - df(\nabla_U V)$$

where ∇ is the Levi-Civita connection. Using this definition of the Hessian for $\tilde{e}$ and g, we get

$$\begin{aligned} D^2_{\tilde{e}}f(U,U) &= D^2_g f(U,U) + df(\nabla^g_U U - \nabla^{\tilde{e}}_U U) \\ &= D^2_g f(U,U) + df(\nabla^g_U U - \nabla^{\tilde{g}}_U U) + df(\nabla^{\tilde{g}}_U U - \nabla^{\tilde{e}}_U U) \end{aligned} \tag{A.6}$$

Proposition 6.4.6. of Buser and Karcher in [6] says that

$$D^2_g f(U,U) \geq \rho \frac{s'_\kappa(\rho)}{s_\kappa(\rho)} g(U,U)$$

where $s_\kappa(\rho) = \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}\rho)$. So that $\frac{s'_\kappa(\rho)}{s_\kappa(\rho)} = \sqrt{\kappa}\cot(\sqrt{\kappa}\rho)$ and hence there exists $\rho_1(\kappa) > 0$ such that for any $0 < \rho < \rho_1$, $\frac{s'_\kappa(\rho)}{s_\kappa(\rho)} \geq 1$. Therefore, on $B(q,\rho)$ with $\rho \leq \rho_1$ we have

$$D^2_g f(U,U) \geq \rho g(U,U) \tag{A.7}$$

and this shows also that for such ρ's, $B(q,\rho)$ is convex (w.r.t. g). Also as a consequence of Proposition 6.4.6. of [6], we get

$$g(\nabla^g f, \nabla^g f) \leq \rho^2 \tag{A.8}$$

where $\nabla^g f$ is the gradient of f with respect to g.
Moreover, by construction of $\tilde{g}$ and by (b) in the result of Abresch, we have

$$|\nabla^g_U U - \nabla^{\tilde{g}}_U U|_g \leq \frac{1}{4} g(U,U). \tag{A.9}$$

By construction of $\tilde{e}$ and as the $\nabla^{\tilde{g}} R_{\tilde{g}}$ is uniformly bounded, Corollary 1 of Kaul in [22] asserts the existence of an application $h \geq 0$ such that

$$|\nabla^{\tilde{g}}_U U - \nabla^{\tilde{e}}_U U|_{\tilde{g}}(y) \leq h(\tilde{d}(p,y))\tilde{g}(U,U)$$

with $h(0) = 0$ and h depends only on bounds on $K_{\tilde{g}}$ and $\nabla^{\tilde{g}} R_{\tilde{g}}$. Hence, there exists $R(n,\kappa,r_0) > 0$ such that for any $r \leq R$, $h(r) \leq \frac{1}{4} e^{-\frac{3}{8}}$. So that we obtain on $\tilde{B}(p,r)$ with $r \leq R$

$$|\nabla^{\tilde{g}}_U U - \nabla^{\tilde{e}}_U U|_g \leq e^{\frac{1}{8}} |\nabla^{\tilde{g}}_U U - \nabla^{\tilde{e}}_U U|_{\tilde{g}} \leq \frac{1}{4} e^{-\frac{1}{4}} \tilde{g}(U,U) \leq \frac{1}{4} g(U,U). \tag{A.10}$$

Finally, introduce (A.7), (A.8), (A.9) and (A.10) in (A.6) and let us define $\rho_0 = \min\{e^{-\frac{1}{8}}\tilde{r}_0, \rho_1, e^{-\frac{1}{8}} R\}$ to obtain the following. We have $B(p,\rho_0) \subseteq \tilde{B}(p,\tilde{r}_0)$, $B(p,\rho_0) \subseteq \tilde{B}(p,R)$ and for any $B(q,\rho) \subseteq B(p,\rho_0)$, $\rho \leq \rho_1$ holds. Hence on $B(p,\rho_0)$ and for any $B(q,\rho) \subseteq B(p,\rho_0)$ we have

$$D^2_{\tilde{e}} f(U,U) \geq \rho g(U,U) - \frac{1}{4}\rho g(U,U) - \frac{1}{4}\rho g(U,U) = \frac{1}{2}\rho g(U,U) \geq 0 \tag{A.11}$$

i.e. f is convex. To conclude the proof, we remark that

$$B(q,\rho) \subseteq B(p, d(p,q)+\rho) \subseteq \tilde{B}(p, e^{\frac{1}{8}}(d(p,q)+\rho)) \subseteq \tilde{B}(p,\tilde{r}_0)$$

so that φ^{-1} restricted to $\tilde{B}(p, e^{\frac{1}{8}}(d(p,q)+\rho))$ is a quasi-isometry with constants of quasi-isometry depending only on n, κ and $d(p,q)+\rho$. More precisely, $(\tilde{B}(p, e^{\frac{1}{8}}(d(p,q)+\rho)), \tilde{e})$ is quasi-isometric to $(\tilde{B}(p, e^{\frac{1}{8}}(d(p,q)+\rho)), \tilde{g})$ with constants of quasi-isometry depending only on $d(p,q)+\rho$ and $\tilde{\kappa}(n,\kappa)$ and by construction of $\tilde{g}$ we can deduce that $(B(q,\rho), g)$ is quasi-isometric to $(B(q,\rho), \tilde{e})$ with constants of quasi-isometry depending only on n, κ and $d(p,q)+\rho$. This ends the proof of the lemma. $\square$

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