Spectra of graphs and the spectral criterion for property (T)

Alain Valette

Institut de Mathématiques
Université de Neuchâtel
Unimail
11 Rue Emile Argand
CH-2000 Neuchâtel, Switzerland

alain.valette@unine.ch

Abstract

For a finite connected graph $X$, we consider the graph $RX$ obtained from $X$ by associating a new vertex to every edge of $X$ and joining by edges the extremities of each edge of $X$ to the corresponding new vertex. We express the spectrum of the Laplace operator on $RX$ as a function of the corresponding spectrum on $X$. As a corollary, we show that $X$ is a complete graph if and only if $\lambda_1(RX) > \frac{1}{2}$. We give a re-interpretation of the correspondence $X \mapsto RX$ in terms of the right-angled Coxeter group defined by $X$.

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1. Introduction

Let $X = (V, E)$ be a finite, connected graph. Denote by $\sim$ the adjacency relation on $V$; that is, $x \sim y$ if and only if $\{x, y\} \in E$. Endow the space $\mathbb{R}V$ of real-valued functions on $V$ with the scalar product $\langle f | g \rangle = \sum_{x \in V} f(x)g(x) \deg(x)$, where $\deg(x)$ is the number of neighbors of $x$. 
The combinatorial Laplace operator of $X$ is the operator $\Delta_X$ on $\mathbb{R}V$, defined by

$$(\Delta_X f)(x) = f(x) - \frac{1}{\deg(x)} \sum_{y \sim x} f(y)$$

($f \in \mathbb{R}V, \ x \in V$). It is classical that $\Delta_X$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$ (that is, $\langle \Delta_X f | g \rangle = \langle f | \Delta_X g \rangle$ for every $f, g \in \mathbb{R}V$), and has spectrum contained in $[0, 2]$; the associated quadratic form is given by:

$$\langle \Delta_X f | f \rangle = \frac{1}{2} \sum_{x,y: x \sim y} (f(x) - f(y))^2$$

($f \in \mathbb{R}V$); see [4] for all this. Then 0 is a multiplicity 1 eigenvalue of $\Delta_X$, and we denote by $\lambda_1(X)$ the smallest non-zero eigenvalue of $X$.

We denote by $RX$ the graph with vertex set $V \sqcup E$ (the disjoint union of $V$ and $E$) and adjacency relation given by:

- if $x, y \in V : x \sim y \Leftrightarrow \{x, y\} \in E$;
- if $x \in V, e \in E : x \sim e \Leftrightarrow x \in e$;
- if $e, e' \in E$, then $e, e'$ are not adjacent in $RX$.

Graphically, this means that every edge $e = \{x, y\}$ in $X$ gets replaced in $RX$ by a triangle $\{x, y, e\}$ (with $\deg e = 2$). This operation on graphs was considered by Cvetkovic [5], who computed, in case $X$ is regular, the spectrum of the adjacency operator of $RX$ as a function of the corresponding spectrum for $X$ (see Theorem 3 in [5]).

The purpose of this note is twofold. First, we explain the relevance of the transformation $X \mapsto RX$ in terms of Cayley graphs for the right-angled Coxeter group associated with $X$. Second, we compute the spectrum $Sp \Delta_{RX}$ of $\Delta_{RX}$ in terms of the spectrum $Sp \Delta_X$ of $\Delta_X$, without regularity assumption on $X$. Observe that, for $f \in \mathbb{R}(V \sqcup E)$:

$$(\Delta_{RX} f)(y) = \begin{cases} f(y) - \frac{1}{2 \deg(y)} \left[ \sum_{x \in V, x \sim y} f(x) + \sum_{e \in E, y \in E} f(e) \right] & \text{if } y \in V \\ f(y) - \frac{1}{2} \sum_{x \in y} f(x) & \text{if } y \in E \end{cases}$$

(1)

The following result will be proved in Section 3:

**Proposition 1.1.** Let $X$ be a finite connected graph with $n$ vertices and $m$ edges. A real number $\lambda \in [0, 2]$ is an eigenvalue of $\Delta_{RX}$ if and only some of the following cases occurs:

- $\lambda = 1$ (this case occurs only if $m > n$);
- $\lambda = \frac{3}{2}$;

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1The graph $RX$ should NOT be confused with the total graph $TX$, whose set of vertices is also $V \sqcup E$ but the 3rd condition above gets replaced by: there is an edge between $e, e' \in E$ if and only if $e$ and $e'$ are incident in $X$. So $RX$ is a spanning subgraph of $TX$. 

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• $2\lambda$ is an eigenvalue of $\Delta_X$.

Taking into account the fact that, for the complete graph $K_n$ on $n$ vertices, we have $Sp(\Delta_{K_n}) = \{0, \frac{n}{n-1}\}$, and that $\lambda_1 > 1$ characterizes complete graphs (see Lemma 1.7 in [4]), we get as an immediate corollary:

**Corollary 1.1.** Let $X$ be a finite connected graph. The following are equivalent:

i) $\lambda_1(RX) > \frac{1}{2}$;

ii) $X$ is a complete graph.

However, it is possible to give a direct, group-theoretic proof of the implication $(i) \Rightarrow (ii)$ in Corollary 1.1: this will be done in Section 2.

2. Cayley graphs and property (T)

Recall that a finitely generated group $\Gamma$ has property (T) if every affine isometric action of $\Gamma$ on a Hilbert space, has a fixed point. We refer to [2] for examples, characterizations and applications of property (T).

Let $\Gamma$ be a finitely generated group and let $S$ be a finite generating subset such that $S = S^{-1}$ and $1 \notin S$. Let $G(\Gamma, S)$ be the Cayley graph of $\Gamma$ with respect to $S$; that is, the vertex set is $\Gamma$, and two vertices $x, y \in \Gamma$ are adjacent if $x^{-1}y \in S$. Let $X_S$ be the graph induced by $G(\Gamma, S)$ on $S$; that is, the vertex set of $X_S$ is $S$, and two elements $s, t \in S$ are adjacent if $s^{-1}t \in S$. The spectral criterion for property (T) (see [1], [6], [7]; see also [2], Theorem 5.5.2) is the statement that, if $X_S$ is connected and $\lambda_1(X_S) > \frac{1}{2}$, then $\Gamma$ has property (T).

**Proof of $(i) \Rightarrow (ii)$ in Corollary 1.1:** Let $X = (V, E)$ be a finite connected graph and let $W_X$ be the right-angled Coxeter group associated with $X$; this is the group defined by the presentation:

$$W_X = \langle s \in V | s^2 = 1 \ (s \in V); \ st = ts \ (\{s, t\} \in E) \rangle.$$

We will need two standard facts about Coxeter groups:

(a) An infinite Coxeter group does not have property (T) (see [3]);

(b) If $\{s, t\} \notin E$, then $st$ has infinite order in $W_X$.

We define a new generating set of $W_X$ by:

$$S =: X \cup \{st = ts : \{s, t\} \in E\}.$$

Observe that, if $st = ts$, then for any two distinct $x, y \in \{s, t, st\}$, the quotient $x^{-1}y$ is still in $\{s, t, st\}$. In other words, the graph $X_S$ induced by $G(W_X, S)$ on $S$, is isomorphic to $RX$.

So, if we assume $\lambda_1(RX) > \frac{1}{2}$, then $W_X$ has property (T) by the spectral criterion. By fact (a) above, $W_X$ is a finite group, which of course implies that $st$ has finite order for every $s, t \in V$. By fact (b), we must have $s \sim t$ for every $s, t \in V$; that is, $X$ is a complete graph. \qed
3. The Laplace operator on $RX$

Recall that the Laplace operator on $X$, as a matrix indexed by $V \times V$, is:

$$(\Delta_X)_{xy} = \begin{cases} 
1 & \text{if } x = y \\
-\frac{1}{\deg(x)} & \text{if } x \sim y \\
0 & \text{if } x \neq y, x \sim y
\end{cases}$$

Set $|V| = n$ and $|E| = m$. Turning to $RX$ with vertex set $E \sqcup V$, recall that $\deg_{RX}(e) = 2$ for $e \in E$ and $\deg_{RX}(x) = 2 \deg(x)$ for $x \in V$. So, from (1), the Laplace operator $\Delta_{RX}$ on $RX$ is a $(m + n) \times (m + n)$ matrix:

$$\Delta_{RX} = \begin{pmatrix} 1_m & B \\
A & 1_n - \frac{1}{2}M_X \end{pmatrix}$$

where $(M_Xf)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y)$ is the Markov operator on $X$ (with $f \in \mathbb{R}^V$) and, for $x \in V, e \in E$:

$$A_{xe} = \begin{cases} 
-\frac{1}{2\deg(x)} & \text{if } x \in e \\
0 & \text{if } x \not\in e
\end{cases}$$

$$B_{ex} = \begin{cases} 
-\frac{1}{2} & \text{if } x \in e \\
0 & \text{if } x \not\in e
\end{cases}$$

Observe that

$$(AB)_{xy} = \begin{cases} 
0 & \text{if } x \neq y, x \sim y \\
\frac{1}{4\deg(x)} & \text{if } x \sim y \\
\frac{1}{4} & \text{if } x = y
\end{cases}$$

So that

$$AB = \frac{1}{4}(1_n + M_X) = \frac{1}{4}(2 \cdot 1_n - \Delta_X). \quad (2)$$

The characteristic polynomial of $\Delta_{RX}$ is

$$P_{RX}(\lambda) = \det(\Delta_{RX} - \lambda \cdot 1_{m+n}) = \det\left( \begin{pmatrix} 1_m & B \\
A & \frac{1}{2} - \lambda \end{pmatrix} - \lambda \cdot \frac{1}{2}M_X \right)$$

For $\lambda \neq 1$, multiply on the left by the unimodular matrix $\begin{pmatrix} 1_m & 0 \\
-(1 - \lambda)^{-1}A & 1_n \end{pmatrix}$ to get

$$P_{RX}(\lambda) = \det\left( \begin{pmatrix} 1_m & B \\
0 & \frac{1}{2} - \lambda \end{pmatrix} - \lambda \cdot \frac{1}{2}M_X \right)$$

$$= (1 - \lambda)^m \det\left[ (\frac{1}{2} - \lambda)1_n + \frac{\Delta_X}{2} - (1 - \lambda)^{-1}AB \right]$$

$$= (1 - \lambda)^{m-n} \det\left[ (1 - \lambda)(\frac{1}{2} - \lambda)1_n + \frac{(1 - \lambda)\Delta_X}{2} - AB \right]$$
By Equation (2):

\[ P_{RX}(\lambda) = (1 - \lambda)^{m-n} \det[(1 - \lambda)(\frac{1}{2} - \lambda) 1_n + \frac{(1 - \lambda)\Delta X}{2} - \frac{1}{4}(2 \cdot 1_n - \Delta X)] \]

\[ = (1 - \lambda)^{m-n} \det[(\lambda - \frac{3}{2})(\lambda - \frac{\Delta X}{2})] \]

\[ = 2^{-n}(1 - \lambda)^{m-n}(\lambda - \frac{3}{2})^n \det(2\lambda - \Delta X) \]

So

\[ P_{RX}(\lambda) = 2^{-n}(1 - \lambda)^{m-n}(\frac{3}{2} - \lambda)^nP_X(2\lambda) \] (3)

Proposition 1.1 immediately follows from equation (3).

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References


