L^p-distortion and p-spectral gap of finite graphs

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Abstract

We give a lower bound for the L^p-distortion $c_p(X)$ of finite graphs X, depending on the first eigenvalue $\lambda_1^{(p)}(X)$ of the p-Laplacian and the maximal displacement of permutations of vertices. For a k-regular vertex-transitive graph it takes the form $c_p(X)^p \geq diam(X)^p \lambda_1^{(p)}(X)/2^{p-1}k$. This bound is optimal for expander families and, for $p = 2$, it gives the exact value for cycles and hypercubes. As new applications we give non-trivial lower bounds for the L^2-distortion for families of Cayley graphs of the finite lamplighter groups $C_2 \wr C_n^d$ ($d \geq 2$ fixed), and for a family of Cayley graphs of $SL_n(q)$ (q fixed, $n \geq 2$) with respect to a standard two-element generating set.

1 Introduction

Let (X, d) and (Y, δ) be two metric spaces. Let $F : X \to Y$ be an imbedding of X into Y. We define the *distortion* of F as

$$dist(F) = \sup_{x,y \in X, x \neq y} \frac{\delta(F(x), F(y))}{d(x, y)} \cdot \sup_{x,y \in X, x \neq y} \frac{d(x, y)}{\delta(F(x), F(y))},$$

where the first supremum is the Lipschitz constant $\|F\|_{Lip}$ of F, and the second supremum is the Lipschitz constant $\|F^{-1}\|_{Lip}$ of F^{-1}. As we will only consider the case where X is finite, supremum can be changed into maximum. The least distortion with which X can be embedded into Y is denoted $c_Y(X)$, namely

$$c_Y(X) := \inf\{dist(F) : F : X \hookrightarrow Y\}.$$

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As target space, we will consider only $L^p = L^p([0,1])$. In this case, we write $c_p(X) = c_{L^p}(X)$. The quantity $c_2(X)$ is also known as the Euclidean distortion of X. As source space, we will take the underlying metric space of a finite, connected graph $X = (V, E)$, where d is then the graph metric. Note that, denoting by $diam(X)$ the diameter of X, we have $c_p(X) \leq diam(X)$, as shown by the embedding $F : V \to \ell^p(V) : x \mapsto \delta_x$. It is a fundamental result of Bourgain [Bou] that $c_p(X) = O(\log |V|)$.

Our aim in this paper is to obtain lower bounds for the distortion c_p of finite graphs. To state our results, we introduce two invariants of graphs. The p-Laplacian $\Delta_p : \ell^p(V) \to \ell^p(V)$ is an operator defined by the formula

$$\Delta_p f(x) = \sum_{x \sim y} (f(x) - f(y))^{[p]},$$

($f \in \ell^p(V), x \in V$), where $a^{[p]} = |a|^{p-1} sign(a)$ and $\sim$ denotes the adjacency relation on V. It is worth noting that for $p = 2$, it corresponds to the standard linear discrete Laplacian. We say that λ is an eigenvalue of Δ_p if we can find $f \in \ell^p(V)$ such that $\Delta_p f = \lambda f^{[p]}$. We define the *p-spectral gap* of X by

$$\lambda_1^{(p)}(X) := \inf \left\{ \frac{\frac{1}{2} \sum_{x \in V} \sum_{y:\, y \sim x} |f(x) - f(y)|^p}{\inf_{\alpha \in \mathbb{R}} \sum_{x \in V} |f(x) - \alpha|^p} \right\},$$

where the infimum is taken over all $f \in \ell^p(V)$ such that f is not constant. It is known that the p-spectral gap is the smallest positive eigenvalue of Δ_p (see [GN]).

For α a permutation of the vertex set V (not necessarily a graph automorphism!), we introduce the *displacement* of α:

$$\rho(\alpha) = \min_{x \in V} d(\alpha(v), v);$$

then the *maximal displacement* of X is $D(X) =: \max_{\alpha \in Sym(V)} \rho(\alpha)$. (Note that this definition makes sense for every finite metric space).

Our main result is:

Theorem 1 *Let X be a finite, connected graph of average degree k. Then*

$$D(X) \left(\frac{\lambda_1^{(p)}(X)}{k\ 2^{p-1}} \right)^{\frac{1}{p}} \leq c_p(X),$$

for $1 < p < \infty$.

For vertex-transitive graphs, this takes the form:

Corollary 1 *Let X be a finite, connected, vertex-transitive graph. Then for $1 < p < \infty$:*

$$diam(X) \left(\frac{\lambda_1^{(p)}(X)}{k\ 2^{p-1}} \right)^{\frac{1}{p}} \leq c_p(X),$$

where k is the degree of each vertex.

Recall that a countable family of finite, connected graphs is a *family of expanders* if they have bounded degree, their Cheeger constants (measuring edge expansion) are bounded away from 0, while the number of their vertices goes to infinity. Expanders were used by Linial-London-Rabinovich [LLR] for $p = 2$, and by Matoušek [Mat] for arbitrary $p \geq 1$, to show that Bourgain's upper bound on c_p is optimal for every p. Thus, using Theorem 1, we give a short proof of:

Theorem 2 *(see [LLR, Mat]) For every $p > 1$, families of expanders X, satisfy $c_p(X) = \Omega(\log |X|)$.*

Of particular interest is the case $p = 2$, and from Theorem 1 we deduce new proofs of the following results (compare with [LM]):

1) (Linial-Magen [LM]) For even n: the cycle C_n satisfies $c_2(C_n) = \frac{n}{2} \sin \frac{\pi}{n}$.

2) (Enflo [Enf]) The d-dimensional hypercube H_d satisfies $c_2(H_d) = \sqrt{d}$.

As new applications, we provide distortion estimates for certain families of k-regular Cayley graphs (k fixed) which are known NOT to be expander families.

As a first application, we consider lamplighter groups over discrete tori. Recall that, if G is a finite group, the lamplighter group of G is the wreath product $C_2 \wr G$, i.e. the semi-direct product of the additive group of all subsets of G (endowed with symmetric difference) with G acting by shifting indices. Take $G = C_n^d$ and denote by $\{\pm e_j \ : \ 1 \leq j \leq d\}$ the standard symmetric generating set for C_n^d, and denote by W_n^d the Cayley graph of the lamplighter group $C_2 \wr C_n^d$, with respect to the generating set

$$S = \{(\{0\}, 0)\} \cup \{(\emptyset, \pm e_j) : 1 \leq j \leq d\}.$$

(so that W_n^d is $(2d+1)$-regular). We will prove the following:

Proposition 1 $c_2(W_n^d) = \begin{cases} \Omega(\frac{n}{\sqrt{\log(n)}}), & \textit{for } d = 2, \\ \Omega(n^{\frac{d}{2}}), & \textit{for } d \geq 3. \end{cases}$

However, the method we will use does not give a good estimate for the case $d = 1$ as we will see in section 5.

As a second application, let q be a fixed prime, and let Y_n be the Cayley graph of $SL_n(q)$ (where $n \geq 2$) with respect to the following set of 4 generators: $S_n = \{A_n^{\pm 1}, B_n^{\pm 1}\}$ and

$$A_n = \begin{pmatrix} 1 & 1 & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}; \quad B_n = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ (-1)^{n-1} & & & & 0 \end{pmatrix}.$$

Proposition 2 $c_2(Y_n) = \Omega(n^{1/2}) = \Omega((\log |Y_n|)^{1/4})$.

The interest of the family $(Y_n)_{n \geq 2}$ comes from the fact that it is known NOT to be an expander family: see Proposition 3.3.3 in [Lub].

The paper is organized as follows: Theorem 1 is proved in section 2, and Corollary 1 in section. Expanders are discussed in section 4, where asymptotic bounds on the maximal displacement are also given. Examples arising from Cayley graphs in section 5; that section also presents examples where the inequality in Corollary 1 is *not* sharp. Finally section 6 contains a discussion of other published results similar to our Theorem 1, and a comparison of the corresponding inequalities.

In this paper, Landau's notations O, Ω, Θ will be used freely.

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2 Proof of Theorem 1

We start with an easy lemma.

Lemma 1 *Let $X = (V, E)$ be a finite, connected graph.*

1. *Let α be any permutation of V. For $F : V \to \ell^p(\mathbb{N})$:*

$$\sum_{x \in V} \|F(x) - F(\alpha(x))\|_p^p \leq 2^p \sum_{x \in V} \|F(x)\|_p^p.$$

2. Fix an arbitrary orientation on the edges. Then, for every $F : V \to \ell^p(\mathbb{N})$, there exists $G : V \to \ell^p(\mathbb{N})$ such that $dist(G) = dist(F)$ and

$$\sum_{x\in V} \|G(x)\|_p^p \leq \frac{1}{\lambda_1^{(p)}(X)} \sum_{e\in E} \|G(e^+) - G(e^-)\|_p^p.$$

Proof: 1) Define a linear operator T on $\ell^p(V, \ell^p(\mathbb{N}))$ by setting $(TF)(x) := F(\alpha(x))$. Clearly, $\|T\| = 1$. Then, in the formula to be proved, the LHS is $\|(I - T)F\|_p^p$. Hence, the result immediately follows from the fact that the operator norm of $T - I$ is at most 2, by the triangle inequality.
2) We proceed as in the proof of Theorem 3 in [GN]. Let $\{u_n\}_{n\in\mathbb{N}}$ be the standard basis vectors in $\ell^p(\mathbb{N})$. Write $F(x) = \sum_{n\in\mathbb{N}} F_n(x)u_n$, for all $x \in V$; denote by $\alpha_n \in \mathbb{R}$ the projection of F_n on the subspace of constant functions in $\ell^p(V)$. It satisfies:

$$\inf_{\alpha\in\mathbb{R}} \|F_n - \alpha\|_p = \|F_n - \alpha_n\|_p.$$

By the proof of Theorem 3 in [GN], the sum $w := \sum_{n\in\mathbb{N}} \alpha_n u_n$ belongs to $\ell^p(\mathbb{N})$.

Defining $G(x) := F(x) - w$, so that $G_n(x) = F_n(x) - \alpha_n$, we have $dist(G) = dist(F)$. Recalling the definition of $\lambda_1^{(p)}(X)$, we have for every n:

$$\sum_{x\in V} |G_n(x)|^p \leq \frac{1}{\lambda_1^{(p)}(X)} \sum_{e\in E} |G_n(e^+) - G_n(e^-)|^p.$$

Taking the sum over n, we get the result. □

Let k be the average degree of X. Combining both statements of lemma 1 with the fact that $|E| = \frac{k|V|}{2}$, we deduce the following Poincaré-type inequality:

Proposition 3 *Let $X = (V, E)$ be a finite, connected graph with average degree k. For any permutation α of V and any embedding $G : V \to \ell^p(\mathbb{N})$ as in lemma 1, we have:*

$$\frac{1}{|V|2^p} \sum_{x\in V} \|G(x) - G(\alpha(x))\|_p^p \leq \frac{k}{2|E|\lambda_1^{(p)}(X)} \sum_{e\in E} \|G(e^+) - G(e^-)\|_p^p.$$

□

Proposition 4 *Let $X = (V, E)$ be a finite connected graph with average degree k. For any permutation α of V and any embedding $G : V \to \ell^p(\mathbb{N})$ as in lemma 1, we have:*

$$\rho(\alpha)\left(\frac{\lambda_1^{(p)}(X)}{k\ 2^{p-1}}\right)^{\frac{1}{p}} \leq dist(G).$$

Proof: Clearly, we may assume that α has no fixed point. Then:

$$\frac{1}{\|G^{-1}\|_{Lip}^p} = \min_{x \neq y} \frac{\|G(x) - G(y)\|_p^p}{d(x, y)^p} \leq \min_{x \in V} \frac{\|G(x) - G(\alpha(x))\|_p^p}{d(x, \alpha(x))^p}$$

$$\leq \frac{1}{\rho(\alpha)^p} \min_{x \in V} \|G(x) - G(\alpha(x))\|_p^p \leq \frac{1}{\rho(\alpha)^p |V|} \sum_{x \in V} \|G(x) - G(\alpha(x))\|_p^p$$

$$\leq \frac{2^{p-1}k}{\lambda_1^{(p)}(X)\rho(\alpha)^p |E|} \sum_{e \in E} \|G(e^+) - G(e^-)\|_p^p \text{ (by Proposition 3)}$$

$$\leq \frac{2^{p-1}k}{\lambda_1^{(p)}(X)\rho(\alpha)^p} \max_{x \sim y} \|G(x) - G(y)\|_p^p = \frac{2^{p-1}k}{\lambda_1^{(p)}(X)\rho(\alpha)^p} \|G\|_{Lip}^p,$$

where the last equality comes from the fact that the above maximum is attained for adjacent points in the graph (see for instance Claim 3.2 in [LM]). Re-arranging and taking p-th roots, we get the result. □

Proof of Theorem 1: Since ℓ^p embeds isometrically in L^p, we clearly have $c_p(X) \leq c_{\ell^p}(X)$. Actually $c_p(X) = c_{\ell^p}(X)$, since for every map $F : V \to L^p$ and every $\varepsilon > 0$, we can find a finite measurable partition $[0, 1] = \bigcup_{j=1}^k \Omega_j$ and, for each $x \in V$, a step function $H(x)$ which is constant on each Ω_j, such that $\|F(x) - H(x)\|_p < \varepsilon$ for $x \in V$. Denoting by m the Lebesgue measure on $[0, 1]$, the embedding $G : V \to \ell^p\{1, ..., k\} : x \mapsto (H(x)|_{\Omega_j} m(\Omega_j)^{1/p})_{1 \leq j \leq k}$ then satisfies $\|G(x) - G(y)\| = \|H(x) - H(y)\|_p$ for every $x, y \in V$, hence the distortion of G is $\delta(\varepsilon)$-close to the one of F, where $\delta(\epsilon) \to 0$ for $\varepsilon \to 0$.

Finally, Theorem 1 for embeddings $V \to \ell^p$ immediately follows from Proposition 4. □

3 Graphs with antipodal maps

From the definition of the invariant $D(X)$, we have $D(X) \leq diam(X)$. The equality holds if and only if the graph X admits an *antipodal map*, i.e. a permutation α of the vertices such that $d(x, \alpha(x)) = diam(X)$ for every $x \in V$.

The existence of an antipodal map is a fairly strong condition. Recall that the *radius* of X is $\min_{x\in V}\max_{y\in V} d(x,y)$, so that the existence of an antipodal map implies that the radius is equal to the diameter of X. The converse is false however, a counter-example was provided by G. Paseman. A necessary and sufficient condition for X to admit an antipodal map was provided by R. Bacher: for $S \subset V$, set $\mathcal{A}(S) = \{v \in V : \exists w \in S, d(v,w) = diam(X)\}$; the graph X admits an antipodal map if and only if $|\mathcal{A}(S)| \geq |S|$ for every $S \subset V$. For all this, see [MO].

The proof of Corollary 1 follows immediately from Theorem 1 and the next lemma:

Lemma 2 *Finite, connected, vertex-transitive graphs admit antipodal maps.*

Proof: For S a finite subset of the vertex set of some graph Y, denote by $\Gamma(S)$ the set of vertices adjacent to at least one vertex of S. It is classical that, if Y is a regular graph, then the inequality $|\Gamma(S)| \geq |S|$ holds[1].

Now, let $X = (V,E)$ be a finite, connected, vertex-transitive graph. Define the *antipodal graph* X^a as the graph with vertex set V, with x adjacent to y whenever the distance between x and y in X, is equal to $diam(X)$. By vertex-transitivity of X, the graph X^a is regular. Now observe that, for $S \subset V$, the set $\Gamma(S)$ in X^a is exactly the set $\mathcal{A}(S)$ defined above. By regularity of X^a and the observation beginning the proof, we therefore have $|\mathcal{A}(S)| \geq |S|$ for every $S \subset V$, and Bacher's result applies. □

Remark 1 *For Cayley graphs, there is a direct proof of the existence of antipodal maps. Indeed, let G be a finite group, and let X be a Cayley graph of G with respect to some symmetric, generating set S; use right multiplications by generators to define X, so that the distance d is left-invariant. Let $g \in G$ be any element of maximal word length with respect to S. Then $\alpha(x) = xg$ (right multiplication by g) is an antipodal map.*

4 Bounds on the maximal displacement

Proposition 5 *For finite, connected graphs X with maximal degree $k \geq 3$:*

$$D(X) = \Omega(\log |X|).$$

[1]Recall the easy argument: assuming that Y is k-regular, count in two ways the edges joining S to $\Gamma(S)$; as edges emanating from S, there are $k|S|$ of them; as edges entering $\Gamma(S)$, there are at most $k|\Gamma(S)|$ of them.

Proof: For a positive integer $r > 0$, the number of vertices in X at distance at most r from a given vertex, is at most the number of vertices in the ball of radius r in the k-regular tree, i.e.

$$1 + k + k(k-1) + k(k-1)^2 + ... + k(k-1)^{r-1} = \frac{k(k-1)^r - 2}{k-2}.$$

For $r = [\log_{k-1}(\frac{|V|}{6})]$, we have $\frac{k(k-1)^r-2}{k-2} < \frac{|V|}{2}$. Let Y be the graph with same vertex set V as X, where two vertices are adjacent if their distance in X is at least $\log_{k-1}(\frac{|V|}{6})$. The preceding computation shows that, in the graph Y, every vertex has degree at least $\frac{|V|}{2}$. By G.A. Dirac's theorem (see e.g. Theorem 2 in Chapter IV of [Bol]), Y admits a Hamiltonian circuit. Let $\alpha \in Sym(V)$ be the cyclic permutation of V defined by this Hamiltonian circuit. Then $\rho(\alpha) \geq \log_{k-1}(\frac{|V|}{6})$, which concludes the proof. $\square$

Proof of Theorem 2: If $(X_n)_n$ is a family of expanders, then by the p-Laplacian version of the Cheeger inequality (see Theorem 3 in [Amg]), the sequence $(\lambda_1^{(p)}(X_n))_n$ is bounded away from 0. So the result follows straight from Theorem 1 together with Proposition 5. $\square$

We now observe that, for families of non-vertex-transitive k-regular graphs, the maximal displacement can be much smaller than the diameter (compare with lemma 2). We thank the referee of a previous version of the paper for suggesting this construction.

Proposition 6 *Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) = \Omega(n)$ and $f(n) = o(8^n)$. There exists a family $(X_n)_{n\geq 1}$ of 3-regular graphs such that:*

a) $|X_n| = \Theta(8^n)$;

b) $diam(X_n) = \Theta(f(n))$;

c) $D(X_n) = \Theta(n)$.

Proof: Let $(Y_n)_{n\geq 1}$ be a family of 3-regular graphs with $|Y_n| = \Theta(8^n)$ and $diam(Y_n) = \Theta(n)$ (such a family is constructed e.g. in Theorem 5.13 of Morgenstern [Mor]). Let Z_n be the product of the cycle $C_{2f(n)}$ with the one-edge graph (so that Z_n is 3-regular on $4f(n)$ vertices). Let $\{y_1, y_2\}$ (resp. $\{z_1, z_2\}$) be an edge in Y_n (resp. Z_n). We "stitch" Y_n and Z_n by replacing the edges $\{y_1, y_2\}$ and $\{z_1, z_2\}$ by edges $\{y_1, z_1\}$ and $\{y_2, z_2\}$, and define X_n as the resulting 3-regular graph. Clearly $|X_n| = \Theta(8^n)$.

Observe that, since every edge in Z_n belongs to some 4-cycle, the distance in X_n between any two vertices in Y_n will differ by at most 5 from the original distance in Y_n; and similarly for vertices in Z_n. So:

$$f(n) = diam(Z_n) \leq diam(X_n) \leq diam(Y_n) + diam(Z_n) + 5,$$

hence $diam(X_n) = \Theta(f(n))$.

Finally, let α be any permutation of the vertices of X_n. Since the overwhelming majority of vertices belongs to Y_n, we find a vertex x such that x and $\alpha(x)$ are both in Y_n. Then

$$\rho(\alpha) \leq d_{X_n}(x, \alpha(x)) \leq d_{Y_n}(x, \alpha(x)) + 5 \leq diam(Y_n) + 5,$$

hence $D(X_n) = O(n)$. The equivalence $D(X_n) = \Theta(n)$ then follows from Proposition 5. $\square$

5 Examples with Cayley graphs

We give a series of consequences of Corollary 1, in case $p = 2$.

5.1 Cycles

Corollary 2 *(Linial-Magen [LM], 3.1) For n even: $c_2(C_n) = \frac{n}{2} \sin \frac{\pi}{n}$.*

Proof: We apply Corollary 1 with $k = 2$, and $D = \frac{n}{2}$, and $\lambda_1^{(2)}(C_n) = 4 \sin^2 \frac{\pi}{n}$ (see Example 1.5 in [Chu]): so $c_2(C_n) \geq \frac{n}{2} \sin \frac{\pi}{n}$. For the converse inequality, it is an easy computation that the embedding of C_n as a regular n-gon in $\mathbb{R}^2$, has distortion $\frac{n}{2} \sin \frac{\pi}{n}$. $\square$

5.2 The hypercube H_d

The hypercube H_d is the set of d-tuples of 0's and 1's, endowed with the Hamming distance. It is the Cayley graph of $\mathbb{F}_2^d$ with respect to the standard basis.

Corollary 3 *(Enflo [Enf]) $c_2(H_d) = \sqrt{d}$*

Proof: For H_d, we have $k = d$, and $diam(H_d) = d$, and $\lambda_1^{(2)}(H_d) = 2$ (see Example 1.6 in [Chu] for the latter): so $c_2(H_d) \geq \sqrt{d}$ by Corollary 1. For the converse inequality, it is easy to see that the canonical embedding of H_d into $\mathbb{R}^d$, has distortion $\sqrt{d}$. $\square$

5.3 Lamplighters over discrete tori

Once again we apply Corollary 1 in order to prove Proposition 1. Let us define the matrix M on $C_2 \wr C_n^d$ given by

$$M_{[(f,a),(g,b)]} = \begin{cases} \frac{1}{4} & \text{if } (f,a) = (g,b); \\ \frac{1}{4} & \text{if } a = b \text{ and } f = g + \delta_a; \\ \frac{1}{16d} & \text{if } a = b \pm e_j \text{ and } f(z) = g(z), \forall z \notin \{a,b\}; \\ 0 & \text{otherwise.} \end{cases}$$

($a, b \in C_n^d$ and $f, g : C_2 \wr C_n^d \to \{0,1\}$). Then M is the transition matrix of the lazy random walk on $C_2 \wr C_n^d$ analysed by Peres and Revelle in Theorem 1.1 of [PR]. Using their estimation of the relaxation time of M, we deduce that the spectral gap of M behaves as $\Theta(\frac{1}{n^d})$ for $d \geq 3$ and as $\Theta(\frac{1}{n^2 \log(n)})$ for the case $d = 2$. By standard comparison theorems (see e.g. Theorems 3.1 and 3.2 in [Woe]), the Dirichlet forms for M and for the Laplace operator on W_n^d are bi-Lipschitz equivalent; moreover the Lipschitz constants do not depend on n (since the comparison can be made on the group $C_2 \wr \mathbb{Z}^d$, of which our lamplighters are quotients). So, we find $\lambda_1^{(2)}(W_n^2) = \Theta(n^{-2} \log(n)^{-1})$ and $\lambda_1^{(2)}(W_n^d) = \Theta(n^{-d})$ for $d \geq 3$. Furthermore, since the diameter of a regular graph is at least logarithmic in the number of vertices, we have $diam(W_n^d) = \Omega(n^d)$, so we apply Corollary 1 to get:

$$c_2(W_n^d) = \begin{cases} \Omega(\frac{n}{\sqrt{\log(n)}}) & \text{for } d = 2, \\ \Omega(n^{\frac{d}{2}}) & \text{for } d \geq 3. \end{cases}$$

□

5.4 Cayley graphs of $SL_n(q)$

We now prove Proposition 2. Since $|SL_n(q)| \approx q^{n^2-1}$, we have $diam(Y_n) = \Omega(n^2)$ (actually it is a result by Kassabov and Riley [KR] that $diam(Y_n) = \Theta(n^2)$). On the other hand, from Kassabov's estimates for the Kazhdan constant $\kappa(SL_n(\mathbb{Z}), S_n)$ (see [Kas], and also the Introduction of [KR]), we have: $\kappa(SL_n(\mathbb{Z}), S_n) = \Omega(n^{-3/2})$.

If X is a Cayley graph of a finite quotient of a Kazhdan group G, with respect to a finite generating set $S \subset G$, then $\lambda_1^{(2)}(X) \geq \frac{\kappa(G,S)^2}{2}$ (see [Lub], Proposition 3.3.1 and its proof). From this we get: $\sqrt{\lambda_1^{(2)}(Y_n)} = \Omega(n^{-3/2})$ and therefore $c_2(Y_n) = \Omega(n^{1/2})$ by Corollary 1. □

5.5 The limits of the method

We give examples of Cayley graphs for which the lower bound of the Euclidean distortion given by Corollary 1 is not tight.

5.5.1 Products of cycles

Let us consider the product of 2 cycles $C_n \times C_N$, where n, N are even integers such that $n < N$. It is clear that it corresponds to the Cayley graph of the additive group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ with generating set $S = \{(\pm 1, 0), (0, \pm 1)\}$. It is well-known from representation theory of finite abelian groups G that, if $X = \mathcal{G}(G, S)$ is a Cayley graph of G and S is symmetric, then the spectrum of the Laplace operator on X is given by $\{\sum_{s\in S}(1-\chi) \ : \ \chi \in \hat{G}\}$. Since for the product of finite abelian groups G, H, we can identify the dual of $G \times H$ as $\{\chi \cdot \eta \ : \ \chi \in \hat{G}, \eta \in \hat{H}\}$, it is easy to see that $\lambda_1(C_n \times C_N) = 4 \sin^2 \frac{\pi}{N}$. As the diameter is equal to $\frac{n+N}{2}$, we get the lower bound

$$c_2(C_n \times C_N) \geq \frac{(n+N)\sin\frac{\pi}{N}}{2\sqrt{2}}.$$

On the other hand, it is known from [LM] that the normalized trivial embedding of $C_n \times C_N$ into $\mathbb{C}^2$ gives the optimal embedding. Namely, defining

$$\phi : C_n \times C_N \to \mathbb{C}^2 : (k,l) \mapsto \left(\frac{\exp \frac{2\pi i k}{n}}{2\sin\frac{\pi}{n}}, \frac{\exp\frac{2\pi i l}{N}}{2\sin\frac{\pi}{N}} \right)$$

we have

$$c_2(C_n \times C_N) = dist(\phi).$$

Since $\|\phi(x) - \phi(y)\| \leq 1$ for every $x, y \in C_n \times C_N$, we have to estimate

$$\|\phi^{-1}\|_{Lip} = \max_{k\leq \frac{n}{2}, l \leq \frac{N}{2}} \frac{k+l}{\sqrt{\frac{sin^2\frac{\pi k}{n}}{sin^2\frac{\pi}{n}} + \frac{sin^2\frac{\pi l}{N}}{sin^2\frac{\pi}{N}}}}.$$

By taking $k = \frac{n}{2}$ and $l = \frac{N}{2}$, we get

$$dist(\phi) \geq \frac{n+N}{2\sqrt{\sin^{-2}\frac{\pi}{n} + \sin^{-2}\frac{\pi}{N}}}.$$

Since it is always the case that

$$\sqrt{\frac{1}{\sin^{-2}\frac{\pi}{n} + \sin^{-2}\frac{\pi}{N}}} > \frac{\sin\frac{\pi}{N}}{\sqrt{2}},$$

we conclude that the lower bound given by Corollary1 is not sharp in this case.

5.5.2 Lamplighter groups over the discrete circle

Here we consider the graphs W_n^1 associated with the lamplighter groups $C_2 \wr C_n$, associated with the generating S described in the Introduction. It is known from [ANV] that $c_2(W_n^1) = \Theta(\sqrt{\log(n)})$.

By way of contrast, let us check that $diam(W_n^1)\sqrt{\lambda_1^{(2)}(W_n^1)} = O(1)$. Let us first estimate $\lambda_1^{(2)}$. For every homomorphism $\chi : C_2 \wr C_n \to \mathbb{C}^\times$, the quantity $\sum_{s\in S}(1-\chi(s))$ is an eigenvalue of the Laplace operator (see the previous example). Let us consider the homomorphism χ given by $\chi(A,k) = e^{2\pi i k/n}$ (it factors through the epimorphism $C_2 \wr C_n \to C_n$). Here we get $\lambda_1^{(2)}(W_n^1) \leq \sum_{s\in S}(1-\chi(s)) = 2 - 2\cos(2\pi/n) = 4\sin^2(\pi/n)$, hence $\lambda_1^{(2)}(W_n^1) = O(\frac{1}{n^2})$. On the other hand, by Theorem 1.2 in [Par], the word length of $(A,k) \in C_2 \wr C_n$ is equal to $|A| + \ell(A,k)$, where $\ell(A,k)$ is the length of the shortest path in the cycle C_n, going from 0 to k and containing A. From this it is clear that $diam(W_n^1) \leq 2n$.

6 Comparison with similar inequalities

Lower bounds of spectral nature on $c_2(X)$, can be traced back to [LLR]. At least two other inequalities (see [GN, NR]) linking the distortion, the p-spectral gap and other graph invariants have been published. In this section, we compare them to Theorem 1. We start with the Grigorchuk-Nowak inequality [GN].

Definition 1 *Let X be a finite metric space. Given $0 < \epsilon < 1$ define the constant $\rho_\epsilon(X) \in [0,1]$, called the volume distribution, by the relation*

$$\rho_\epsilon(X) = \min\left\{\frac{diam(A)}{diam(X)} \ : \ A \subset X \text{ such that } |A| \geq \epsilon|X|\right\}.$$

Theorem 3 *([GN] Theorem 3) Let X be a connected graph of degree bounded by k and let $1 \leq p < +\infty$. Then, for every $0 < \epsilon < 1$,*

$$\frac{(1-\epsilon)^{\frac{1}{p}}\rho_\epsilon(X)}{2^{\frac{1}{p}}}\, diam(X)\left(\frac{\lambda_1^{(p)}(X)}{k\ 2^{p-1}}\right)^{\frac{1}{p}} \leq c_p(X).$$

It is easy to see that, when the graph satisfies $D(X) = diam(X)$ (this is the case for vertex-transitive graphs, by lemma 2), then this result is weaker than our Theorem 1, since the factor $\frac{(1-\epsilon)^{\frac{1}{p}}\rho_\epsilon(X)}{2^{\frac{1}{p}}}$ is strictly smaller than 1.

The second result, due to Newman-Rabinovich [NR], holds for $p = 2$:

Proposition 7 *([NR] Proposition 3.2) Let $X = (V, E)$ be a k-regular graph. Then,*

$$\sqrt{\frac{(|V|-1)\lambda_1^{(2)}(X)}{|V|\ k}\ avg(d^2)} \leq c_2(X),$$

where $avg(d^2) := \frac{1}{|V|(|V|-1)} \sum_{x,y \in V} d(x,y)^2$.

In the following, we will compute the term $avg(d^2)$ for the cycle C_n and for the hypercube H_d in order to give explicitly the LHS term of the inequality due to Newman and Rabinovich. First, it is true that for a vertex-transitive graph $X = (V, E)$, we have

$$\sum_{y,x \in V} d(x,y)^2 = |V| \sum_{j=1}^{diam(X)} j^2 |S(x_0, j)|,$$

where x_0 is an arbitrary point in X and $S(x_0, j)$ is the sphere of radius j, centered in x_0. By taking $n \geq 4$ and even, we clearly have

$$\sum_{x,y \in C_n} d(x,y)^2 = n \left(2 \sum_{j=1}^{\frac{n}{2}-1} j^2 + \frac{n^2}{4} \right) = \frac{n^2(n^2+2)}{12}.$$

Therefore, we get $\sqrt{\frac{n^2+2}{6}}\ \sin\frac{\pi}{n}$ as lower bound for $c_2(C_n)$, which is strictly weaker than Corollary 2. On the other hand, for the hypercube H_d, by the same argument, we have

$$avg(d^2) = \frac{1}{2^d(2^d-1)} \sum_{x,y \in H_d} d(x,y)^2 = \frac{1}{2^d-1} \sum_{j=1}^{d} j^2 \binom{d}{j}.$$

Since $\sum_{j=1}^{d} j^2 \binom{d}{j} < d^2 2^{d-1}$ for $d \geq 2$, we conclude that Corollary 3 gives a better lower bound for $c_2(H_d)$.

Finally, we mention for completeness a remarkable result, of a different nature, due to Linial, Magen and Naor [LMN]:

Theorem 4 *([LMN], Theorem 1.3) There is a universal constant $C > 0$ such that, for every k-regular graph X with girth g:*

$$c_2(X) \geq \frac{Cg}{\sqrt{\min\{g, \frac{k}{\lambda_1^{(2)}(X)}\}}}.$$

Observe however that, for the family $(H_d)_{d \geq 2}$ of hypercubes, the right-hand side of the inequality remains bounded, while $c_2(H_d) = \sqrt{d}$ by Corollary 3.

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