Spectral Theory and Geometry

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Preamble : These are informal notes of a series of 4 talks I gave in Teheran, in the CIMPA-UNESCO-IRAN School "Recent Topics in Geometric Analysis", May 21-June 2, 2006. The goal was to give an introduction to the geometric spectral theory of the Laplacian acting on p-differential forms.

Of course, in a few hours, it is hopeless to be complete, and I had to make some choice for the content of these lectures. Clearly, the main purpose is to give an intuition based on examples about the following question : to what extend is it possible to construct large or small first nonzero eigenvalue for the Laplacian on forms on a compact Riemannian manifold. This point of view is quit reductive, in particular I do not say one word about the asymptotic of the spectrum and about the heat kernel, and the reader interested on such aspects of the theory may look at the book of Rosenberg [Ro]. However, this question of large or small eigenvalues is easy to understand, it allows to make clear the importance of the topology of the manifold, much more crucial as in the case of functions, and also give the opportunity to present a lot of open but accessible questions.

In the introduction (first lecture), I just recall without many explanations the basis of the Laplace operator and of its spectrum. We do it first for functions and then in the case of p-forms. This is of course partially redundant, but I expected the audience more or less familiar with the case of functions and not necessarily with the case of forms. There exists among other three excellent monographs about this question : the very complete book of M. Taylor [Ta] and the lecture note of G. Schwarz [Sc], mainly for the questions related to the boundary conditions, and the above mentioned book of Rosenberg [Ro], mainly concerned with the asymptotic of the spectrum, and whose introduction is very instructive. We also recall the De Rham theory and Hodge decomposition for compact Riemannian manifolds. In the beginning, in the case of functions, I will mainly refer to the books of P. Bérard [Be] and I. Chavel [Ch].

I profited of the introduction to recall some typical and classical results in the case of functions, because what is known in this context is inspiring of what we try to do for forms.

The second lecture is mainly concerned with examples. First I begin with a very simple example (Example 22), where all the calculations may be done explicitly, but which shows the role of the topology in the sense where it shows that two Riemannian manifolds, close in a rough sense, may have a very different spectrum on p-forms, which is not possible for functions, as proved in [Ma1]. I explain then in detail the example of the Berger's sphere (Example 24), which shows that collapsing can produce small eigenvalues, and which is a very special case of a much more elaborated theory we partially describe in the fourth lecture.

In the lecture 3, I construct large eigenvalues on manifolds. To do this, the main ingredient is to adapt a result of J. Mc Gowan [Go]. I treat this aspect in a complete way and this is the only result I prove with all details in these notes. This proof is not so difficult, but mixes the Riemannian aspects, the Hodge decomposition, the variational characterization of the spectrum, the problems related to the boundary conditions, and shows the importance of the topology. In this part, I have to recall the Hodge decomposition for compact Riemannian manifolds with boundary. As in the closed case, I do it without proof, and refer to [Ta] and [Sc] for more details.

The last lecture has the purpose to say some words about collapsing and spectrum. This subject has had a large development these last years, and I choose to explain the simplest case : the collapsing of S^1 bundle. I take also the opportunity of this lecture to ask some open questions about "collapsing and spectrum".

I complete these four lectures with a last chapter describing briefly some recent developments in the direction of our main purpose (small and large eigenvalues on compact manifolds) I did not treat during the meeting. This is mainly the occasion to complete the references and to describe some open questions.

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1 Lecture 1 : Introduction

1.1 The case of functions

Let (M, g) be a smooth, connected and C^∞ Riemannian manifold with boundary ∂M. The boundary is a Riemannian manifold with induced metric $g_{|\partial M}$. We suppose ∂M to be smooth. We refer to the book of Sakai [Sa] for a general introduction to Riemannian Geometry and to Bérard [Be] for an introduction to the spectral theory.

For a function $f \in C^2(M)$, we define the Laplace operator or Laplacian by

$$\Delta f = \delta df = -div\ gradf$$

where d is the exterior derivative and δ the adjoint of d with respect to the usual L^2-inner product

$$(f, h) = \int_M fh\ dV$$

where dV denotes the volume form on (M, g).

In local coordinates $\{x_i\}$, the laplacian reads

$$\Delta f = -\frac{1}{\sqrt{det(g)}} \sum_{i,j} \frac{\partial}{\partial x_j}(g^{ij}\sqrt{det(g)}\frac{\partial}{\partial x_i} f).$$

In particular, in the euclidean case, we recover the usual expression

$$\Delta f = -\sum_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} f.$$

Let $f \in C^2(M)$ and $h \in C^1(M)$ such that hdf has compact support on M. Then we have Green's Formula

$$(\Delta f, h) = \int_M < df, dh > dV - \int_{\partial M} h\frac{df}{dn} dA$$

where $\frac{df}{dn}$ denotes the derivative of f in the direction of the outward unit normal vector field n on ∂M and dA the volume form on ∂M.

In particular, if one of the following conditions $\partial M = \emptyset$, $h_{|\partial M} = 0$ or $(\frac{df}{dn})_{|\partial M} = 0$, then we have the relation

$$(\Delta f, h) = (df, dh).$$

In the sequel, we will study the following eigenvalue problems when M is compact :

– Closed Problem :

$$\Delta f = \lambda f \ in \ M; \ \partial M = \emptyset;$$

– Dirichlet Problem

$$\Delta f = \lambda f \ in \ M; \ f_{|\partial M=0};$$

– Neumann Problem :

$$\Delta f = \lambda f \ in \ M; \ (\frac{df}{dn})_{|\partial M} = 0.$$

We have the following standard result about the spectrum, see [Be] p. 53.

Theorem 1. *Let M be a compact manifold with boundary ∂M (eventually empty), and consider one of the above mentioned eigenvalue problems. Then :*

1. *The set of eigenvalue consists of an infinite sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \to \infty$, where 0 is not an eigenvalue in the Dirichlet problem ;*
2. *Each eigenvalue has finite multiplicity and the eigenspaces corresponding to distinct eigenvalues are $L^2(M)$-orthogonal ;*
3. *The direct sum of the eigenspaces $E(\lambda_i)$ is dense in $L^2(M)$ for the L^2-norm. Futhermore, each eigenfunction is C^∞-smooth and analytic.*

Remark 2. *The Laplace operator depends only on the given Riemannian metric. If*

$$F : (M, g) \to (N, h)$$

is an isometry, then (M, g) and (N, h) have the same spectrum, and if f is an eigenfunction on (N, h), then $F \circ f$ is an eigenfunction on (M, g) for the same eigenvalue.

It turns out that in general, the spectrum cannot be computed explicitly. The very few exceptions are manifolds like round sphere, flat tori, ball. In general, it is only possible to get estimate of the spectrum, and these estimation are related to the geometry of the manifold (M, g) we consider.

The Dirichlet problem has a physical interpretation in terms of "how sounds a drum", see Courant-Hilbert [CoHi] Vol.1, Ch. V. And this gives a good intuition of the two types of problems we are dealing with : there is clearly a relation between the shape of a drum and how it sounds. The shape is the geometry of the manifold, that is all the Riemannian invariants depending on the Riemannian metric g : curvature, diameter, volume, injectivity radius, systole, etc... How the drum sounds corresponds to the spectrum of the manifold.

- **The "direct" problem :** we see a drum and try to imagine how it sounds. The mathematical translation is : some of the geometry of the manifold is given, in general in term of bounds on the Riemannian invariants. Then, the goal is to estimate part of the spectrum with respect to these bounds.
- **The "inverse" problem :** we hear a drum without seeing it and try to deduce informations on its shape. The mathematical question associate to this is to decide if the spectrum determines the geometry. The most famous question is about the isospectrality : if two Riemannian manifolds have the same spectrum (with multiplicity), are they isometric ?

In these lectures, I will focus only in the direct problem (for an introduction to the inverse problem, see the survey of C. Gordon, [Go], and the short paper of R. Brook [Br]) and indeed on a very particular part of it : the main problem I will investigate is "can λ_1 be very large or very small ?". The question seems trivial or naive at the first view, but it is not, and I will try to explain that partial answers to it are closely related to geometric but also topological properties of the considered Riemannian manifold.

Of course, there is a trivial way to produce arbitrarily small or large eigenvalues : take any Riemannian manifold (M, g). For any constant $c > 0$, $\lambda_k(c^2 g) = \frac{1}{c^2}\lambda_k(g)$ and an homothety produce small or large eigenvalues. So, we have to introduce some normalizations, in order to avoid the trivial deformation of the metric given by an homothety. Most of the time, these normalizations are of the type "volume is constant" or "curvature and diameter are bounded".

To investigate the Laplace equation $\Delta f = \lambda f$ is a priori a problem of analysis. To introduce some geometry on it, it is very relevant to look at the variational characterization of the spectrum. To this aim, let us introduce the Rayleigh quotient. If a function f lies in $H^1(M)$ in the closed and Neumann problems, and on $H_0^1(M)$ in the Dirichlet problem, the Rayleigh quotient of f is

$$R(f) = \frac{\int_M |df|^2 dV}{\int_M f^2 dV} = \frac{(df, df)}{(f, f)}.$$

Theorem 3. *(Variational characterization of the spectrum, [Be] p. 60-61.) Let us consider one of the 3 eigenvalues problems. We denote by $\{f_i\}$ an orthonormal system of eigenfunctions associated to the eigenvalues $\{\lambda_i\}$.*

1. *We have*

$$\lambda_k = inf\{R(u) : u \neq 0; u \perp f_0, .., f_{k-1}\}$$

where $u \in H^1(M)$ *(*$u \in H_0^1(M)$ *for the Dirichlet eigenvalue problem) and* $R(u) = \lambda_k$ *if and only if* u *is an eigenfunction for* λ_k*.*

   At view of this variational characterization, we can think we have to know the first k or $k-1$-eigenfunctions to estimate λ_k ; this is not the case :

2. *Max-Min : we have*

$$\lambda_k = \sup_{V_k} inf\{R(u) : u \neq 0,\ u \perp V_k\}$$

where V_k *runs through* k*-dimensional subspaces of* $H^1(M)$ *((*$k-1$*)-dimensional subspaces of* $H_0^1(M)$ *for the Dirichlet eigenvalue problem).*

3. *Min-Max : we have*

$$\lambda_k = \inf_{V_k} sup\{R(u) : u \neq 0,\ u \in V_k\}$$

where V_k *runs through* $k+1$*-dimensional subspaces of* $H^1(M)$ *(*k*-dimensional subspaces of* $H_0^1(M)$ *for the Dirichlet eigenvalue problem).*

Remark 4. *We can see already two advantages to this variational characterisation of the spectrum. First, we see that we have not to work with solutions of the Laplace equation, but only with "test functions", which is easier. Then, we have only to control one derivative of the test function, and not two, as in the case of the Laplace equation.*

To see this concretely, let us give a couple of simple examples.

Example 5. ***Monotonicity in the Dirichlet problem.*** *Let* $\Omega_1 \subset \Omega_2 \subset (M, g)$*, two domains of the same dimension* n *of a Riemannian manifold* (M, g)*. Let us suppose that* Ω_1 *and* Ω_2 *are both compact connected manifolds with boundary. If we consider the Dirichlet eigenvalue problem for* Ω_1 *and* Ω_2 *with the induced metric, then for each* k

$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1)$$

with equality if and only if $\Omega_1 = \Omega_2$*.*

The proof is very simple : each eigenfunction of Ω_1 may be extended by 0 on Ω_2 and may be used as a test function for the Dirichlet problem on Ω_2. So, we have already the inequality. In the equality case, the test function becomes an eigenfunction : because it is analytic, it can not be 0 on an open set, and $\Omega_1 = \Omega_2$.

Corollary 6. *If* M *is a compact manifold without boundary, and if* Ω_1*,...,*Ω_{k+1} *are domains in* M *with disjoint interiors, then*

$$\lambda_k(M, g) \leq max(\lambda_1(\Omega_1), ..., \lambda_1(\Omega_{k+1}))$$

The second example explains how to produce arbitrarily small eigenvalues for Riemannian manifold with fixed volume. This example is again very simple, but the same type of questions for the spectrum of the Laplace operator acting on p-forms is much less easy.

Example 7. ***The Cheeger's dumm-bell.*** *We explain this example for a domain in* $\mathbb{R}^n$ *but it is easily generalized as Riemannian manifold.*

The idea is to consider two balls of fixed volume V *related by a small cylinder* C *of length* $2L$ *and radius* ϵ*. The first nonzero eigenvalue of the Neumann problem converges to* 0 *as the radius of the cylinder goes to* 0*. It is even possible to estimate very precisely the asymptotic of* λ_1 *in term of* ϵ *(see [An]), but here, we just shows that it converges to* 0*.*

We choose a function f *with value* 1 *on the first ball,* -1 *on the second, and decreasing linearly, so that the norm of its gradient is* $\frac{2}{L}$*. By construction we have* $\int f dV = 0$*, so that we have* $\lambda_1 \leq R(f)$*.*

But the Rayleigh quotient is bounded above by

$$\frac{4VolC/L^2}{2V}$$

which goes to 0 *as* ϵ *does.*

A classical way to estimate the spectrum from below is to cut a manifold into small parts, where we have a reasonable knowledge of the spectrum, and try to get from this local control a control of the whole spectrum. As example, consider a compact Riemannian manifold M, and let $\Omega_1, ..., \Omega_m \subset M$ pairwise disjoint domains such that

$$M = \bar{\Omega}_1 \cup ... \cup \bar{\Omega}_m.$$

Then, consider the Neumann boundary conditions on the Ω_k and arrange all the eigenvalues of $\Omega_1, ..., \Omega_m$ in increasing order, with repetition according to multiplicity

$$0 \leq \mu_1 \leq \mu_2 \leq ...$$

Then,

Proposition 8. *With the above notations, we have*

$$\lambda_k(M) \geq \mu_{k+1}.$$

Observe that if we have m domains, then $\mu_1 = ... = \mu_m = 0$, so that we get an effective estimate only for the m-th eigenvalue λ_m.

In order to prove this proposition, we denote by $\{f_i\}$ a set or orthonormal eigenfunctions for the λ_i and by $\{\phi_i\}$ a set or orthonormal eigenfunctions for the $\{\mu_i\}$. Then consider the application

$$\psi : \mathbb{R}^{k+1} \to \mathbb{R}^k$$

given by

$$\psi(a_0, ..., a_k) = \{\int_{\Omega_{i_j}} f\phi_j\}_{j=1}^k$$

where $f = \sum_{r=0}^k a_r f_r$ and i_j is such that ϕ_j is defined on Ω_{i_j}.

The application ψ is linear, and has a non trivial kernel. There exists nonzero $(a_0, ..., a_k)$ with $f = \sum_{r=0}^k a_r f_r$ is orthogonal to $\phi_1, ..., \phi_k$, so that we have $R(f) \geq \mu_{k+1}$.

But, we have also $R(f) \leq \lambda_k$, so that we conclude that

$$\lambda_k(M) \geq \mu_{k+1}.$$

Let us finish this introduction to the Laplace operator on functions by giving two typical results which show how the geometry allows to control the first nonzero eigenvalue in the closed eigenvalue problem.

The first one is the Cheeger's inequality, which is in some sense the counter-part of the dumm-bell example. We present it in the case of a compact Riemannian manifold without boundary, but it may be generalized to compact manifolds with boundary (for both Neumann or Dirichlet boundary conditions) or to noncompact, complete, Riemannian manifolds.

Definition 9. *Let (M, g) be an n-dimensional compact Riemannian manifold without boundary. The Cheeger's isoperimetric constant $h = h(M)$ is defined as follows*

$$h(M) = \inf_C\{J(C);\ J(C) = \frac{Vol_{n-1}C}{min(Vol_n M_1, Vol_n M_2)}\},$$

where C runs through all compact codimension one submanifolds which divide M into two disjoint connected open submanifolds M_1, M_2 with common boundary $C = \partial M_1 = \partial M_2$.

Theorem 10. *Cheeger's inequality. We have the inequality*

$$\lambda_1(M, g) \geq \frac{h^2}{4}.$$

A proof may be found in Chavel's book [Ch] and developments and other statement in Buser's paper [Bu1]. This result is remarkable, because it relates an analytic quantity (λ_1) to a geometric quantity (h) without any other assumption on the geometry of the manifold. It turns out that an upper bound of λ_1 in term of the Cheeger's constant may be given, but under some geometrical assumptions : this is the Theorem of P. Buser (see [Bu2]).

Theorem 11. *Let (M,g) be an n-dimensional compact Riemannian manifold without boundary. If $Ric(M,g) \geq -(n-1)^2\delta^2 g$, then,*

$$\lambda_1(M,g) \leq c(n)(|\delta| h(M,g) + h^2(M,g)).$$

Remark 12. 1. *As far as I know, Buser's result does not extend trivially in the case of manifolds with boundary.*

2. *A difficulty with the Cheeger's inequality is precisely to estimate the Cheeger's constant. In some situations, the result has to be understand in the sense where λ_1 may be used in order to estimate h.*

3. *In the case of p-form spectrum, such an inequality is not known, and it would be a challenge to get one.*

The second one's gives an universal lower bound in term of the geometry of the manifold. Again, we state the result for a manifold without boundary, but other statements for manifolds with boundary may be found in [LY].

Theorem 13. *(See [LY]). Let (M,g) be a compact n-dimensional Riemannian manifold without boundary. Suppose that the Ricci curvature satisfies $Ric(M,g) \geq (n-1)K$ and that d denote the diameter of (M,g).*

Then, if $K < 0$,

$$\lambda_1(M,g) \geq \frac{exp - (1 + (1 - 4(n-1)^2 d^2 K)^{1/2})}{2(n-1)^2 d^2},$$

and if $K = 0$, then

$$\lambda_1(M,g) \geq \frac{\pi^2}{4d^2}.$$

This type of results was generalized in different directions, see for example [BBG].

1.2 The case of p-forms : the De Rham Theory

For this paragraph, we refer to the books of Goldberg [Gol].

Let M be an n-dimensional differentiable manifold.

Denote by $\wedge^p(M)$ The space of C^∞ differential forms on M with real coefficients. The exterior derivative d maps $\wedge^p(M)$ to $\wedge^{p+1}(M)$ and satisfies $d^2 = 0$. It induces a differential complex, the De Rham complex,

$$0 \to \wedge^0(M) \xrightarrow{d} \wedge^1(M) \xrightarrow{d} ... \xrightarrow{d} \wedge^{n-1}(M) \xrightarrow{d} \wedge^n(M) \xrightarrow{d} 0.$$

The cohomology of this complex is called the De Rham cohomology and the p'th cohomology group of the manifold M is given by

$$H^p_{dR}(M) = \frac{\{\omega \in \wedge^p(M) : d\omega = 0\}}{\{d\theta : \theta \in \wedge^{p-1}(M)\}}.$$

The dimension of $H^p_{dR}(M)$, when finite, is called the p'th Betti number of the manifold M and is denoted by $b_p(M)$. Since the spaces of closed and exact forms are both of infinite dimension, it is a nontrivial fact that, as M is compact, $dim\ H^p_{dR}(M) < \infty$. We can define a pairing

$$H^p_{dR}(M) \times H^{n-p}_{dR}(M) \to \mathbb{R}$$

by

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta \tag{1}$$

where α and β are closed forms representing the cohomology class $[\alpha] \in H^p_{dR}(M)$, $[\beta] \in H^{n-p}_{dR}(M)$. We have

Theorem 14. ***Poincaré duality*** *The bilinear function (1) is a nonsingular pairing and determines an isomorphism of $H^{n-p}_{dR}(M)$ with the dual of $H^p_{dR}(M)$.*

A priori, it seems that the De Rham cohomology depends on the differential structure of the manifold M. In fact, it depends only on the topology : this is precisely the meaning of the De Rham Theorem, that is the existence of an isomorphism

$$r : H^p_{dR}(M) \to H^p_{sing}(M, \mathbb{R})$$

induced by

$$\omega \mapsto (\sigma \to \int_\sigma \omega)$$

for any p-form ω and p-chain σ, where $H^p_{sing}(M, \mathbb{R})$ denote the singular cohomology group of M with real coefficients.

Theorem 15. *The cohomology of the de Rham complex of M is isomorphic to the singular cohomology of M with real coefficients.*

1.3 The case of p-forms : the Laplacian

Let (M, g) be a compact oriented Riemannian manifold of dimension n without boundary.

There exists a linear operator $*$ (the Hodge-Star operator) which assigns to each p-form $\omega \in \wedge^p(M)$ an $(n-p)$-form $*\omega \in \wedge^{n-p}(M)$ and satisfies

$$** = (-1)^{p(n-p)}.$$

The codifferential operator is defined by

$$\delta = (-1)^{n(p+1)+1} * d*$$

and the Laplace-Beltrami operator $\Delta : \wedge^p(M) \to \wedge^p(M)$ (the Laplacian acting on p-differential forms) by

$$\Delta = d\delta + \delta d.$$

Note that δ is 0 when applied to functions, so that the Laplacian acting on function reduce to $\Delta = \delta d$ on $\wedge^0(M)$.

For two smooth forms $\alpha, \beta \in \wedge^p(M)$, we define their L^2-scalar product by

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta.$$

With respect to this scalar product, δ is the adjoint of d, and Δ is a symmetric operator on $\wedge^p(M)$. Denote by $L^2(\wedge^p(M))$ the space of L^2-forms on (M, g).

We have a similar result as for the Laplacian on functions (see [Ro])

Theorem 16. *Let (M, g) be a compact Riemannian manifold without boundary. Then, $L^2(\wedge^p(M))$ has an orthonormal basis consisting of eigenforms of the Laplacian on p-forms. One can order the eigenforms so that the corresponding eigenvalues $\lambda_{k,p}$ satisfy*

$$0 < \lambda_{1,p} \le \lambda_{2,p} \le \lambda_{3,p} \le ... \to \infty$$

The eigenvalues are positive, accumulate only at infinity and have finite multiplicity.

Let $H^p(M, g) = \{\omega \in \wedge^p(M) : \Delta\omega = 0\}$ be the space of harmonic p-forms. Then $\Delta\omega = 0$ if and only if $d\omega = \delta\omega = 0$.

Theorem 17. ***(Hodge decomposition Theorem)*** *For each integer p with $0 \le p \le n$, H^p is finite dimensional and we denote dim $H^p(M)$ by $b_p(M)$, the p-th Betti number of M. Moreover, we have the following orthogonal direct sum decomposition of the space $\wedge^p(M)$:*

$$\wedge^p(M) = d(\wedge^{p-1}(M)) \oplus \delta(\wedge^{p+1}(M)) \oplus H^p(M).$$

Since every harmonic form on a compact Riemannian manifold (M, g) is closed, we have a map

$$h : Ker\Delta^p \to H^p_{dR}(M),\ \omega \mapsto H^p_{dR}(M).$$

With this, we see another aspect of the Hodge's Theorem

Theorem 18. ***(Hodge)*** *Let (M,g) be a compact, oriented manifold. Then*

$$Ker\Delta^p \tilde{=} H^p_{dR}(M).$$

So, through the multiplicity of the harmonic forms, we see that already that the topology of the underling manifold M is present. So a general question we can address in studying the first eigenvalues of the p-forms spectrum is the following : is it possible to "read" and to "measure" the importance and the influence of the topology of M in the first nonzero eigenvalues ? One of the goal of the next lectures will precisely be to give some elements of answer to this question.

As in the case of functions, we have a variational characterization of the spectrum of p-forms. To this aim, let us introduce the Rayleigh quotient of $\omega \in \wedge^p M$: it is given by

$$R(\omega) = \frac{\int_M (|d\omega|^2 + |\delta\omega|^2) dVol}{\int_M |\omega|^2 dVol} = \frac{(d\omega, d\omega) + (\delta\omega, \delta\omega)}{(\omega, \omega)}. \tag{2}$$

Theorem 19. *We have the following variational characterization for the spectrum of the Laplacian :*

$$\lambda_{k,p}(M,g) = \min_E \, max\{R(\omega) : \omega \in E\}$$

where E runs through all vector subspaces of dim $k + b_p(M)$ of $\wedge^p M$.

The differential d and the codifferential δ commute to the Laplacian. It means that if ω is a p-eigenform of Δ, then $d\omega$ is a $(p+1)$-eigenform and $\delta\omega$ is a $(p-1)$-eigenform (eventually 0). In fact, because of the Hodge decomposition, any (non harmonic) eigenform may be decomposed as a sum of exact and coexact eigenforms. If ω is an exact eigenform, $*\omega$ is a coexact eigenform.

If $\lambda > 0$, denote by $E'_p(\lambda)$ and $E''_p(\lambda)$ the eigenspaces of λ consisting respectively of exact and coexact p-forms. Then $d : E''_p(\lambda) \to E'_{p+1}(\lambda)$ and $\delta : E'_p(\lambda) \to E''_{p-1}(\lambda)$ are vector spaces isomorphisms.

In particular, there is a copy of the spectrum of function in the 1-forms spectrum.

It is useful to have a min-max characterization only for exact or coexact p-forms. Let us write

$$0 < \lambda'_{1,p} \le \lambda'_{2,p} \le \lambda'_{3,p} \le ... \to \infty$$

the spectrum of exact p-forms and

$$0 < \lambda''_{1,p} \le \lambda''_{2,p} \le \lambda''_{3,p} \le ... \to \infty$$

the spectrum of coexact p-forms.

Theorem 20. *We have the following variational characterization*

$$\lambda'_{k,p}(M,g) = \min_E \ max\{R(\omega) : \omega \in E\}$$

where E runs through all vector subspaces of dim k of $d(\wedge^{p-1}M)$;

$$\lambda''_{k,p}(M,g) = \min_E \ max\{R(\omega) : \omega \in E\}$$

where E runs through all vector subspaces of dim k of $\delta(\wedge^{p+1}M)$.

We can ask for the Laplacian acting on p-forms the same questions as for the Laplacian on functions, that is to relate the spectrum and the geometry of the Riemannian manifold. However, there is a new fact in the study of the p-forms spectrum : it is that the role of the *topology* of the manifold is much more consequent. We can see this already for the harmonic forms, which are related to the de Rham cohomology, and we will develop this in the next section.

2 Lecture 2 : Some examples and results for the p-forms spectrum

2.1 A basic example

Let us begin with a very simple example : the case of a product.

Example 21. ***The spectrum of a product.*** *Let M and N be two compact Riemannian manifolds. If $\alpha \in \wedge^p(M)$, $\beta \in \wedge^q(N)$, then, considering the Riemannian product $M \times N$, a direct calculation shows that*

$$\Delta(\alpha \wedge \beta) = \Delta\alpha \wedge \beta + \alpha \wedge \Delta\beta.$$

This implies (see [Ta] p.356), that we get spec$^p(M \times N)$ by summing s-eigenvalues of M and r-eigenvalues of N with $r + s = p$.

In particular, we have the Künneth formula

$$H^p(M \times N) = \bigoplus_{r+s=p} H^r(M) \otimes H^s(N)$$

Example 22. *Let S^1_L be the circle of length L and (N,h) an $(n-1)$-dimensional compact Riemannian manifold. We consider the product $M = S^1_L \times N$ with the product metric. We want to study the spectrum as N is fixed and as $L \to \infty$.*

Recall that the first nonzero eigenvalue of S^1_L is equal to $\frac{4\pi^2}{L^2}$ and goes to 0 as $L \to \infty$. We already deduce from this that

$$\lambda_{1,0}(S^1_L \times N) = \frac{4\pi^2}{L^2}$$

and goes to 0 as $L \to \infty$. So the same is true for $\lambda_{1,1}(S^1_L \times N)$, $\lambda_{1,n}(S^1_L \times N)$ and $\lambda_{1,n-1}(S^1_L \times N)$.

The situation may be different for $\lambda_{1,p}$, $2 \leq p \leq n-2$. To see this, it suffices to consider two examples : the first one is to take $N = S^{n-1}$, the $(n-1)$-dimensional sphere (with the canonical metric). As S^{n-1} has no harmonic p-forms if $1 \leq p \leq n-2$, then

$$\lambda_{1,p}(S^1_L \times S^{n-1}) \geq C > 0$$

independently of L if $2 \leq p \leq n-2$, where $C = min\{\lambda_{1,1}(S^{n-1}), ..., \lambda_{1,n-2}(S^{n-1})\}$.

So, the situation differs drastically as in the case of functions.

In the second example, we take $N = T^{n-1}$, where T^{n-1} is the $(n-1)$-dimensional flat torus $S^1 \times ... \times S^1$. In this case, we have for each $0 \leq p \leq n$,

$$\lambda_{1,p}(S^1_L \times T^{n-1}) = \frac{4\pi^2}{L^2}$$

as $L \to \infty$.

So, the situation differs drastically from the first example.

The explanation is that the behaviour of $\lambda_{1,p}(M)$ depends strongly in this case from the topology of the manifold N in the product, and more precisely from its cohomology. If there is an harmonic form α of degree k, then we can "lift" the spectrum of S^1_L and associate to an eigenfunction f the k-form $f \wedge \alpha$ (or the (k+1)-forms $f dt \wedge \alpha$) which is an eigenform with the same eigenvalue as f.

With these two examples, we see that two Riemannian manifolds with bounded curvature and injectivity radius, which are Hausdorff close, may have spectrum which are not comparable. This differs completely of the intuition we have in the case of functions, where two compact Riemannian manifolds with injectivity radius uniformly bounded below, Ricci curvature bounded below and which are Hausdorff close have comparable spectrum (see Mantuano [Ma] for a precise statement).

Remark 23. *Note that in the previous example, the sectional curvature satisfies $K(S^1_L \times S^{n-1}) \geq 0$. So we have a family of compact Riemannian manifolds of nonnegative sectional curvature, whose diameter goes to ∞ and $\lambda_{1,p}$ is uniformly bounded below by a strictly positive constant as $2 \leq p \leq n-2$. This contrast with a general result of Cheeger and of Cheng in the case of functions.*

In the sequel, we will generalize a lot the ideas contained in the last example. The first generalization of a product is to consider a fiber bundle. It turns out that this point of view is very powerful. In this section devoted to examples, I will look at a special case, where it is easy to do explicit calculations and which gives a very good idea of the different problems and questions we can meet.

2.2 The p-spectrum of the Berger's spheres and of the dumm-bell

Example 24. *In this example, we will study the spectrum of a family of Riemannian metrics on the sphere of odd dimension $2n+1$, but we will see it as an S^1-bundle on the complex projective space $\mathbb{C}P^n$. The details of the construction and of the calculations are in [CC1].*

We define

$$S^{2n+1} = \{(z_1, ..., z_{n+1} \in \mathbb{C}^{n+1} : |z_1|^2 + ... + |z_{n+1}|^2 = 1\}.$$

There is an isometric action of S^1 on S^{2n+1} given by

$$(e^{i\theta}, (z_1, ..., z_{n+1})) \mapsto (e^{i\theta} z_1, ..., e^{i\theta} z_{n+1}).$$

The complex projective space $\mathbb{C}P^n$ is the quotient of S^{2n+1} by this action, and this allow to see S^{2n+1} as a S^1-principal bundle

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n,$$

where the canonical projection π is a Riemannian submersion. Observe that the fiber of the bundle, which are great circles of the sphere, are geodesics and have the same length.

The connection form of the S^1-bundle is represented by $\omega = X^\sharp$, where X is the unit vector field tangent to the action of S^1. In this particular case, it is not so difficult to calculate $d\omega$. We refer to the book of A. Besse, chapter 9, for the geometry of Riemannian submersions. However, in this example, it is easy to do the calculation "with the hands".

The derivative $d\omega$ is horizontal : this comes from the classical formula

$$d\omega(U, V) = U\omega(V) - V\omega(U) - \omega([U, V])$$

for two vector fields U and V.

If $U = X$ and V is horizontal, that is orthogonal to X in each point, a property of the Lie bracket make that $[X, V] = 0$ so that

$$d\omega(X, V) = 0$$

because $\omega(X) = 1$ is a constant function and $\omega(V) = 0$.

If U, V *are horizontal, then*

$$d\omega(U,V) = -\omega([U,V])$$

that is

$$d\omega = \pi^*(\Omega)$$

where Ω *is the curvature form of the bundle, but also the Kähler form of the basis* $\mathbb{C}P^n$*.*

Now, the idea is to use the so called "canonical variation of the metric" (or to introduce the so called "Berger'sphere"). It consists in introducing a family of Riemannian metric on S^{2n+1} *obtained by deforming the canonical one's. Denote by* H_p *the horizontal space at the point p (that is the subspace of* T_pS^{2n+1} *orthogonal to* $X(p)$*). Then, define*

$$g_t(p)(U,V) = \begin{cases} t^2 g(p)(X,X) \ if \ U = V = X \\ g(p)(U,V) \ if \ U,V \in H_p \\ 0 \ if \ U = X, V \in H_p \end{cases}$$

and extend it linearly.

We can continue the calculation of $\Delta\omega$ *for the Riemannian metric* g_t*.*

Recall that $d\omega = \pi^*\Omega$*, where* Ω *is the Kähler form of* $\mathbb{C}P^n$*. So,* $*\pi^*\Omega$ *is a multiple of* $X^\sharp \wedge \pi^*(*\Omega)$*, and the multiple has to be choose such that the forms* $\pi^*\Omega$ *and* $*\pi^*\Omega$ *have the same norm with respect to* g_t*.*

As the norm of $X^\sharp$ *is now* $\frac{1}{t}$*, the multiple has to be t, so that we can write*

$$*\pi^*\Omega = tX^\sharp \wedge \pi^*(*\Omega).$$

As we apply d again, we can use exactly the same calculation as before, because

$$d\pi^*(*\Omega) = \pi^*(d * \Omega) = 0$$

because of the harmonicity of Ω*.*

At the end, we get

$$\delta d\omega = t^2\omega.$$

On the other hand, we have $*\omega = \pi^*(\Omega_0)$*, where* Ω_0 *is the volume form of* $\mathbb{C}P^n$*, so that* $d * \omega = 0$*.*

At the end, we get

$$\Delta\omega = t^2\omega.$$

Moreover, the same calculation shows the following : take any $2k$ *form* $\alpha = \Omega \wedge ... \wedge \Omega$ *of the basis* $\mathbb{C}P^n$*, then* $\pi^*(\alpha)$ *and* $\omega \wedge \pi^*(\alpha)$ *are respectively a* $2k$ *and a* $(2k+1)$*-eigenform with respect to* g_t *for the eigenvalue* t^2*.*

A classical calculation for riemannian submersion with totally geodesic fibers (see [Be] Ch. 9) shows also that as $t \to 0$*, the sectional curvature of* (S^{2n+1}, g_t) *stay bounded.*

So, we are in position to state some very interesting consequences of the two examples.

1. In the Example 22, we get (by shrinking the sphere S^{n-1}) a family of Riemannian positively curved manifolds, with diameter $\to \infty$ and $\lambda_{1,p}$ arbitrarily large for $2 \leq p \leq n-2$;
2. In Example 24, we have a family of Riemannian manifolds with bounded sectional curvature and diameter, and $\lambda_{1,p} \to 0$ for $1 \leq p \leq 2n$: this contrast with the case of functions (see Theorem 13) and will lead to a lot of questions.
3. It is also possible to understand Example 24 as follow : as $t \to 0$, the family (S^{2n+1}, g_t) converge in some sense (Gromov-Hausdorff convergence) to the basis $\mathbb{C}P^n$ of the bundle. But if S^{2n+1} has nonzero cohomology only in dimension 0 and $2n+1$, $\mathbb{C}P^n$ has non vanishing cohomology in all even degree. And the harmonic form associated to this cohomology will give a small eigenvalue for form as lifted to the sphere.

Example 25. ***The dumm-bell revisited*** *The goal of this Example is just to show that the intuition given by the dumm-bell in the case of function is no more true in the case of forms. The details of the construction and of the result I describe are rather technical and may be found in [AC2].*

Let us consider two compact Riemannian manifolds M_1 and M_2 of dimension n. We make a small hole and we join M_1 and M_2 by a thin cylinder isometric to $[O, L] \times S^{n-1}_\epsilon$ where S^{n-1}_ϵ is the round $(n-1)$-sphere of radius ϵ. Let M_ϵ denote the manifold we get with this construction. The question address in [AC2] is to study the evolution of the p-spectrum of M_ϵ as $\epsilon \to 0$.

As $1 < p < n-1$, $dim H^p(M_\epsilon) = dim H^p(M_1) + dim H^p(M_2)$, as we can see by a Mayer-Vietoris argument.

In [AC2], we prove that, if $1 < p < n-1$, the p-spectrum of M_ϵ converge to the union of p-spec(M_1) and p-spec(M_2) as $\epsilon \to 0$.

It was already known from previous works of C. Anné that, as $\epsilon \to 0$, the spectrum of functions of M_ϵ converge to the union of both 0-spectrum of M_1 and M_2, with the Dirichlet spectrum of the interval $[0, L]$. From this, we can also deduce the behaviour of the 1-spectrum.

Again, it is impossible to explain in a few lines this very technical result, but it is possible to get an intuition if we say a few word of the easier part of the problem, that is how to construct test forms on M_ϵ from eigenforms of M_1 and M_2.

As we do a small hole on a manifold N, it is straightforward (see [AC1]) to replace an eigenform α on N by a form β equal to 0 near the hole and to α outside a neighbourhood of the hole, without affecting to much the Rayleigh quotient. Then, as we do this for M_1

and M_2, we can extend the test form by 0 on the cylinder, and get a test form on M_ϵ. This give a majoration of the spectrum of M_ϵ by the union of the spectrum of M_1 and M_2.

Where a difference appears between functions and p-forms, $(1 < p < n-1)$, is that we can try to construct test forms with the cylinder. This is possible for functions (and 1-forms) by considering the Dirichlet eigenfunctions of the cylinder and extending them by 0 on M_ϵ. If we do the same for p-forms, $(1 < p < n-1)$, the Rayleigh quotient on the cylinder go to ∞ as $\epsilon \to 0$ as explained in Example 22, and in fact we can show that the eigenforms of M_ϵ tend to concentrate on both M_1 and M_2, but not on the cylinder.

2.3 Lower bound for the p-spectrum

In the case of functions, Theorem 13 shows that a control of the curvature and of the diameter of a compact Riemannian manifold of dimension n gives a lower bound for λ_1.

We have seen in Example 24 that such a control in not enough to guarantee a lower bound for $\lambda_{1,p}$, $1 \le p \le n-1$: we can ask for such a lower bound by adding some new contrainsts on the geometry of the manifolds. In [CC1], it was showed that it is enough to add a lower bound on the injectivity radius to the bound on the sectional curvature and to the diameter to garanty a lower bound on $\lambda_{1,p}$. However, the result was proved by compacity methods and did not give an explicit bound, which is very unsatisfactory.

This was done, at least partially, done in the paper of Chanillo and Trèves [CT].

We give here a statement which differs a little of the statement of the paper (which is a little more general)

Theorem 26. ***(Chanillo-Trèves)*** *Let (M, g) be an n-dimensional compact Riemannian manifold such that its sectional curvature, diameter and injectivity radius satisfies respectively $|K(M, g)| \le a$, $diam(M, g) \le d$ and $Inj(M, g) \ge r > 0$. Then, there exists a constant $c(n, a, d) >$ such that*

$$\lambda_{1,p}(M, g) \ge c(n, a, d) r^{4n^2+8n-2}.$$

The constant $c(n, a, d)$ is not given explicitly, but this may be done by a carefull reading of the proof.

Remark 27.

1. *The proof of Theorem 26 is, is some sense, based in the proof of the De Rham Theorem given in [Gol]. It shows in a nice way how the topology (particularly the cohomology) of the manifold interacts with the geometry in the estimation of the spectrum.*
2. *The end of the proof is based on a combinatorial Lemma of Trèves [Tr] (Lemma A5) which is not correct, at least as it is mentionned there, see [Ma3] for a counter-example. In Mantuano's paper, we can find a weaker version of Trèves's Lemma, which make Theorem 26 correct, with a weaker constant.*

This Theorem is at the origin of some very nice questions. The first one is very general and will be develop in the section 4 : Let us consider the set $\mathcal{M}(n, a, d)$ of n-dimensional Riemannian manifold with $|K(M, g)| \leq a$ and $diam(M, g) \leq d$. If a family $M_i \in \mathcal{M}(n, a, d)$ is such that for one p, $\lambda_{p,1}(M_i) \to 0$, then Theorem 26 implies that $Inj(M_i) \to 0$ (or on an equivalent way, $Vol(M_i) \to 0$. So, there is the converse question : If a family $M_i \in \mathcal{M}(n, a, d)$ is such that $Inj(M_i) \to 0$ (or $Vol(M_i) \to 0$), then does it gives small eigenvalues, how many, and how are they related to $Inj(M_i)$ or $Vol(M_i)$, to the topology, etc.. Now a lot is known (and also a lot is unknow) about this question (see section 4).

There are also some open questions that we can ask already

Open question 1. *Is it possible to get an analog statement as Theorem 26 with a bound only on the Ricci curvature ?*

Open question 2. *In Theorem 26, the bounds are not optimal. Is it possible to find the optimal bound in term of Inj or Vol, and, if not, at least to do much better ?*

3 Lecture 3 : The Theorem of Mc Gowan : applications and proof of the theorem

In this lecture, we will establish a result coming from the PHD thesis of J. Mc Gowan (see [MG]). It is not a good idea to give the most general formulation of this result, because it would be very complicated to read. We will give two statements in particular cases, the statement of the original paper of Mc Gowan and the statement we used to estimate specifically $\lambda_{1,p}$ (see for example [Gu], [GP]).

The idea behind this theorem is that, in order to estimate the spectrum of a manifold, it may be convenient to cut it into parts that we know, and then to estimate the whole spectrum thanks to the spectrum of the parts. This approach leads us to consider problem with boundary. So, we will first explain what are the "right" boundary conditions for the Laplacian acting on p-forms.

3.1 Boundary conditions for forms.

Dirichlet and Neumann boundary conditions on functions can be generalized to p-forms. Let M be a compact Riemannian manifold with boundary. We have a generalization of Green's formula (see [Ta] p. 361) for $\omega, \eta \in \wedge^p M$:

$$(\Delta\omega, \eta) = (d\omega, d\eta) + (\delta\omega, \delta\eta) - \int_{\partial M} (< \delta\omega, i_\nu \eta > - < i_\nu(d\omega), \eta >)$$

where i_ν denotes the inner product with ν, and ν is the outward pointing normal vector.

We choose the boundary conditions in order to have a classical Green's formula. The *absolute* boundary conditions are

$$i_\nu \omega = 0; \ i_\nu d\omega = 0 \tag{3}$$

and the *relative* boundary conditions are

$$\omega_{|\partial M} = 0; \ \delta\omega_{|\partial M} = 0 \tag{4}$$

Remark 28. *If ω satisfies the relative or absolute boundary conditions, then $\Delta\omega = 0$ implies $d\omega = \delta\omega = 0$.*

For both absolute and relative boundary conditions, the spectrum has the same properties as in the case without boundary. We can find the following result in [GLP] (Theorem 1.5.4, p. 37)

Theorem 29. *Let (M, g) be a compact Riemannian manifold with smooth boundary,and consider one of the eigenvalues problems with absolute (A) or relative (R) boundary conditions. Then, $L^2(\wedge^p(M))$ has an orthonormal basis consisting of eigenforms of the Laplacian on p-forms with (A) or (R). One can order the eigenforms so that the corresponding eigenvalues $\lambda_{k,p}$ satisfy*

$$0 < \lambda_{1,p} \leq \lambda_{2,p} \leq \lambda_{3,p} \leq ... \to \infty$$

The eigenvalues are positive, accumulate only at infinity and have finite multiplicity.

Let $H^p(M, g) = \{\omega \in \wedge^p(M) : \Delta\omega = 0\}$ be the space of harmonic p-forms. Then $\Delta\omega = 0$ if and only if $d\omega = \delta\omega = 0$.

Remark 30. *The Hodge operator $(*)$ interchanges the boundary conditions, so that there is a correspondence between the p-spectrum for absolute boundary conditions and $(n-p)$-spectrum for relative boundary conditions, where n is the dimension of M. So, in the sequel, I will focus on the case with absolute boundary conditions.*

As in the case without boundary, there is a relation between the p-harmonic forms and the topology of the underlying manifold M, and the natural injection from the harmonic forms in the De Rham cohomology gives an isomorphism, so that the dimension of the space $H^p(M, g)$ of p-harmonic forms (with absolute boundary condition) coincide with the dimension of the real De Rham cohomology of degree p of M.

We also have a Hodge decomposition (see [Ta], prop. 9.8, p.367)

Theorem 31. *If $\omega \in C^\infty(\wedge^p M)$, we have the orthogonal decomposition*

$$\omega = d\delta\alpha + \delta d\beta + \gamma \tag{5}$$

where α,β and γ are of class C^∞ and satisfies absolute boundary conditions.

Note in particular that, moreover, $d\beta$ satisfies $i_\nu(d\beta) = 0$. This implies (after calculations) a usefull property we will use in the sequel, that is $i_\nu(\delta d\beta) = 0$.

Remark 32. *The fact, mentionned in Theorem 31 that, for a form ω, $i_\nu\omega = 0$ implies $i_\nu(\delta\omega) = 0$ is often used without proof. We give here a short proof of this fact communicated by A. Savo.*

Let us consider an orthonormal frame field $(e_1, ..., e_{n-1}, \nu)$ near a point of the boundary, where the e_k's are tangent to ∂M. If L denotes the second fundamental form of ∂M, we have

$$\nabla_{e_k}\nu = -\sum_{j=1}^{n-1} L(e_k, e_j)e_j.$$

If $X_1, ..., X_m$ are tangent vectors, we have

$$i_\nu\delta\omega(X_1, ..., X_m) = \delta\omega(\nu, X_1, ..., X_m) =$$

$$= -\sum_{k=1}^{n-1} \nabla_{e_k}\omega(e_k, \nu, X_1, ..., X_m) - \nabla_\nu\omega(\nu, \nu, X_1, ..., X_m) =$$

$$= -\sum_{k=1}^{n-1} e_k.\omega(e_k, \nu, X_1, ..., X_m) + \sum_{k=1}^{n-1} \omega(e_k, \nabla_{e_k}\nu, X_1, ..., X_m)$$

$$= -\sum_{j,k=1}^{n-1} L(e_j, e_k)\omega(e_k, e_j, \nu, X_1, ..., X_m),$$

which is zero because L is symetric and ω skew-symmetric.

We have a variational characterisation of the spectrum in this case

Theorem 33.

$$\lambda_{k,p}(M, g) = \min_E \, max\{R(\omega) : \omega \in E\}$$

where E runs through all vector subspaces of dim $k+b_p(M)$ of $\wedge^p M$ satisfying the boundary condition $i_\nu\omega = 0$.

Again, we can restrict to coexact p-forms, and get

Theorem 34.

$$\lambda''_{k,p}(M, g) = \min_E \, max\{R(\omega) : \omega \in E\}$$

where E runs through all vector subspaces of dim k of $\delta(\wedge^p(M))$ satisfying the boundary condition $i_\nu\omega = 0$.

3.2 Statement of Mc Gowan's Theorem

We can now state a version of J. Mc Gowan's Lemma (the version of Gentile-Pagliara,[GP]).

Theorem 35. *Let M be a compact Riemannian manifold without boundary of dimension n and $\{U_i\}_{i=0}^k$ an open cover of M, such that there are no intersections of order higher than 2. Let $U_{ij} = U_i \cap U_j$. Suppose further, that $H^{p-1}(U_{ij}) = 0$ for all i, j. Denote by $\mu(U_i)$ (resp. $\mu(U_{ij})$) the smallest positive eigenvalue of the Laplacian acting on exact forms of degree p on U_i (resp. of degree $p-1$ on U_{ij}) satisfying absolute boundary conditions. Then the first nonzero eigenvalue $\lambda'_{1,p}(M)$ on exact p-forms satisfies*

$$\lambda'_{1,p}(M) \geq \frac{C}{\sum_{i=1}^{k}\left(\frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i}\left(\frac{w_{n,p}c_\rho}{\mu(U_{ij})} + 1\right)\left(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)}\right)\right)} \tag{6}$$

where m_i is the number of j, $j \neq i$, for which $U_i \cap U_j \neq \emptyset$, $w_{n,p}$ a combinatorial constant which depends on p and n, $c_\rho = (max)_i(max_{x\in U_i})|\nabla \rho_i(x)|^2$ for a fixed partition of unity $\{\rho_i\}_{i=1}^k$ subordinate to the given cover, and C a positive constant.

In order to get a lower bound on $\lambda_{1,p}(M)$, we have to control $\lambda'_{1,p}(M)$ and $\lambda'_{1,p+1}(M)$

Of course, the hypothesis of Theorem 35 seem to be very strong. We will give soon another statement of this result (in fact the original statement), but I will first explain how to use Theorem 35 to construct large eigenvalues for p-forms on any compact manifold of dimension ≥ 4.

Corollary 36. *Suppose further that we have the following uniform control :*

$$\mu(U_i) \geq \mu > 0; \;\; \mu(U_{ij}) \geq \mu > 0 \;\; m_i \leq m,$$

the latest corresponding to a bound on the number of open sets which can cut a given one's. Then we get

$$\lambda'_{1,p} \geq C(m)\mu.$$

Example 37. *There is a quit large class of examples that the Gentile-Pagliara's version of Mc Gowan's Theorem allows to control. Let us sketch an example not in the litterature as far as I know, we can deduce from the corollary : Consider a fundamental piece consisting on a compact Riemannian manifold with boundary (M, g_0). Moreover, suppose that the boundary ∂M consists 4 disjoint geodesic hypersurfaces isometric to the same sphere S^{n-1}. Around the boundary, we can also suppose that (M, g_0) is isometric to a product $S^{n-1} \times I$.*

Then, it is possible to glue different pieces together, following, as example, a regular graph of degree 4 (see [CM] for detailled construction of this type). Then, an application of the above corollary is that, for $2 \leq p \leq n-2$, the p-spectrum of such a compact manifold *is controled from below by the spectrum of (M, g_0) with absolute boundary condition and by the spectrum of the product of the connected component of $(\partial M, g_0)$ with an interval. This is in particular true in the case of a Riemannian covering.*

3.3 Construction of large eigenvalues for p-forms

Theorem 38. *Every compact, connected manifold M^n of dimension $n \geq 4$ admits metrics g of volume one with arbitrarily large $\lambda_{1,p}(g)$ for all $2 \leq p \leq n-2$.*

Moreover, if a Riemannian metric g_1 on M is given, we can choose g in the conformal class of g_1.

For motivations and history about this question, see the lecture of A. El Soufi, or the reference [Co].

Proof of Theorem 38 (For the details, see [GP]) We begin with a metric g_1 on M, and we deform this metric around a point, in order to "add a long cylinder" to the manifold M. Note that does not change the topology $of M$. It is like if we have a manifold (M_1, g_1) gluing with a cylinder $Z = I \times S^{n-1}$ closed by half a sphere H_1.

More precisely, we put a family g_t of Riemannian metric equal to g_1 on M_1, to $[0,t] \times S^{n-1}$ on Z and to the canonical metric of the half-sphere on H_1. Moreover, we identify ∂M_1 with ∂Z at $\{t\} \times S^n$ and ∂H_1 to $\{0\} \times S^{n-1}$.

We set $Z_1 = [0,1[\times S^{n-1}$, $Z_2 =]0,t[$ and $Z_3 =]t-1,t] \times S^{n-1}$ and

$$U_1 = H_1 \cup Z_1;\ U_2 = Z_3 \cup M_1\ U_3 = Z_2.$$

So, U_1, U_2, U_3 is an open covering of M, and satisfies exactly the hypothesis of Theorem 35, because

$$U_1 \cap U_2 = \emptyset;\ U_1 \cap U_3 =]0,1[\times S^n;\ U_2 \cap U_3 =]t-1,t[\times S^{n-1},$$

and $U_1 \cap U_3$, $U_2 \cap U_3$ have the cohomology of the sphere S^{n-1}, that is 0 in degree $2, ..., n-2$.

Theorem 35 will allow us to control the first nonzero eigenvalue of p-exact forms with the first nonzero eigenvalue of the absolute boundary problem for exact p-forms on U_1, U_2, U_3 and exact $(p-1)$-forms on $U_1 \cap U_3$ and $U_2 \cap U_3$.

The fundamental observation is now that as t varies, U_1, U_3, $U_1 \cap U_3$ and $U_2 \cap U_3$ are fixed and U_2 is depending on t, but in a very simple way, because $U_2 =]0,t[\times S^{n-1}$.

Exactly as in Example 22, we have

$$\mu_{1,p}(U_3) \geq C > 0$$

if $2 \leq p \leq n-1$, where C is independent of t. This allow to control the p-exact spectrum of M from below by a positive constant not depending on t, for $2 \leq p \leq n-1$, and as a consequence, $\lambda_{1,p}(M)$ for $2 \leq p \leq n-2$.

But, as $t \to \infty$, $Vol(M, g_t) \to \infty$. So, after renormalization to a volume 1 metric, $\lambda_{1,p} \to \infty$ with t.

It turn out that our construction of the cylinder is a conformal deformation of a euclidean metric. Now, if we begin with any metric g', we first replace it by a metric g'' flat around a point, and the, we do our construction. We show then that the same conformal deformation of the initial metric g' gives also large eigenvalue. The reason is that firstly, g' is quasi-isometric to g'', with a controled ratio, close to 1, and this does not affect the spectrum to much, as explained in Theorem 45 du to Dodziuk. Secondly, the ratio of quasi-isometry between two metric is not affected by a conformal deformation : we have $\frac{g'(p)(v,v)}{g''(p)(v,v)} = \frac{f^2(p)g'(p)(v,v)}{f^2(p)g''(p)(v,v)}$. The details are in [CE1] and [CE2].

Note that the construction of large eigenvalues for function is possible, but different, see [CD] and the lecture of A. El Soufi. So, the case of 1-forms is very special, and we have the following

Open question 3. *Let M be a given compact manifold of dimension $n \geq 3$. Is it possible to construct on M a family of volume 1 Riemannian metric with arbitrarily large $\lambda_{1,1}$?*

In a joint work in progress with El Soufi and Takahashi, we have some results in the direction of a positive answer in some cases, but, till now, our results are not complete.

A problem in Theorem 35 is of course that, in general, there is no reason for $U_i \cap U_j = \emptyset$. It turns out that in the initial statement of J. Mc Gowan, this was not supposed. But there is a price to pay : it is not possible to estimate the *first* nonzero eigenvalue, but only the N-th. The lemma was also stated for 1-forms, which do that it is not to difficult to read.

Theorem 39. *Let M be a compact Riemannian manifold without boundary of dimension n and $\{U_i\}_{i=0}^k$ an open cover of M.Let $U_{ij} = U_i \cap U_j$. Denote by $\mu(U_i)$ (resp. $\mu(U_{ij})$) the smallest positive eigenvalue of the Laplacian acting on exact forms of degree p on U_i (resp. of degree $p-1$ on U_{ij}) satisfying absolute boundary conditions. Let $N_1 = \sum_{i,j} dimH^1(U_{i,j})$, $N_2 = \sum_{i,j,l} dimH^0(U_i \cap U_j \cap U_l)$ and set $N = 1 + N_1 + N_2$.*

Then the N-th eigenvalue $\lambda'_{N,2}(M)$ on exact 2-forms satisfies

$$\lambda'_{N,2}(M) \geq \frac{1}{\sum_{i=1}^k (\frac{1}{\mu(U_i)} + \sum_{j=1}^{m_i} (\frac{w_{n,p} c_\rho}{\mu(U_{ij})} + 1)(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)}))} \quad (7)$$

where m_i is the number of j, $j \neq i$, for which $U_i \cap U_j \neq \emptyset$, $w_{n,p}$ a combinatorial constant which depends on p and n, $c_\rho = (max)_i(max_{x\in U_i})|\nabla\rho_i(x)|^2$ for a fixed partition of unity $\{\rho_i\}_{i=1}^k$ subordinate to the given cover.

Remark 40. 1. *If one want to estimate eigenvalues for forms of higher degree, we have to take account the cohomology of higher degree of the intersection $U_{i_1} \cap ... \cap U_{i_r}$.*

2. *If we try to apply this Theorem on function (that is on 1-exact forms), we need to take account of the cohomology of degree 0 of the intersection. This cohomology is never 0, so that we never have informations on $\lambda_{1,0}$.*

3.4 Proof of Mc Gowan's Theorem

The goal is to prove the Gentile-Pagliara version of Mc Gowan's Theorem

The difficulty is that the restriction of an eigenform to an open subset does not, in general, satisfies the absolute boundary conditions. However, there is an approach of J. Dodziuk which allow to turn more or less this problem :

Theorem 41. *Let (M, g) be a compact Riemannian manifold with boundary. The spectrum of the Laplacian $0 < \lambda'_{1,p} \leq \lambda'_{2,p} \leq ...$, acting on exact p-forms which satisfies absolute boundary conditions can be computed by*

$$\lambda'_{k,p} = \inf_{V_k} \sup_{V_k-\{0\}} \{\frac{(\phi, \phi)}{(\eta, \eta)} : d\eta = \phi\} \tag{8}$$

where V_k range over all dimension k subspaces of $\wedge^p(M) \cap L^2(\wedge^p(M))$ exact p-forms, and $\eta \in \wedge^{p-1}(M) \cap L^2(\wedge^{p-1}M)$.

Démonstration. First, for each $\phi \in V_k$, we choose η to maximize the quotient $\frac{(\phi,\phi)}{(\eta,\eta)}$.

If $d\eta = \phi$, we use the Hodge decomposition of $\eta = d\alpha + \delta\beta + \gamma$ and choose $\eta_0 = \delta\beta$. Recall that we have $i_\nu\eta_0 = 0$, as said in Theorem 31.

It follows then that

$$\inf_{V_k} \sup_{V_k-\{0\}} \{\frac{(\phi, \phi)}{(\eta, \eta)} : d\eta = \phi\} = \inf_{V_k} \sup_{V_k-\{0\}} \{\frac{(\phi, \phi)}{(\eta_0, \eta_0)}\},$$

and this is equal to

$$\inf_{W_k} \sup_{\eta_0\in W_k-\{0\}} \{\frac{(d\eta_0, d\eta_0)}{(\eta_0, \eta_0)}\}$$

where W_k ranges over subspaces of dimension k of p-forms satisfying the first boundary condition $i_\nu\eta_0 = 0$.

But this is exactly the variational characterization of the spectrum of coexact (p-1)-forms with absolute boundary condition, which allows to conclude.

□

It is of fundamental importance to note that ϕ or η are *not* suppose to satisfy absolute (or relative) boundary conditions. Theorem 41 has some very useful consequences.

1. If ϕ is an exact p-form (and in the sequel, it will be the restriction to a domain of an exact p-form), then in order to estimate $\lambda'_{1,p}$ we can choose for V the vector space generated by ϕ, and we get

$$\lambda'_{1,p} \leq \underset{\eta}{sup}\{\frac{(\phi,\phi)}{(\eta,\eta)} : d\eta = \phi\} \tag{9}$$

2. Moreover, the supremum is achieve if η is coexact. It follows that, if η is coexact and $d\eta = \phi$, we get

$$\frac{(\phi,\phi)}{(\eta,\eta)} \geq \lambda'_{1,p} \tag{10}$$

3. If now ϕ_1 is an exact eigenform for $\lambda'_{1,p}$ (with absolute boundary conditions) and if η_1 is coexact, satisfies the absolute boundary conditions and is such that $d\eta_1 = \phi_1$, then

$$\lambda'_{1,p} = \frac{(\phi_1,\phi_1)}{(\eta_1,\eta_1)} \geq \frac{(\phi_1,\phi_1)}{(\eta,\eta)} \tag{11}$$

for each η such that $d\eta = \phi_1$.

Proof of Theorem 35 Let α be an eigenform for $\lambda'_{1,p}(M)$ and β with $d\beta = \alpha$. We have

$$\lambda'_{1,p} \geq \frac{(\alpha,\alpha)}{(\beta,\beta)}.$$

The goal will be to find a "good" β for this equation, where "good" mean that we can find β with (β,β) bounded from above thanks to the informations we have from the U_i and from the U_{ij}.

Let denote by α_i the restriction from α to U_i. It follow from (10) that there exists β_i coexact on U_i with $d\beta_i = \alpha_i$. and

$$\mu(U_i) \leq \frac{(\alpha_i,\alpha_i)}{(\beta_i,\beta_i)} \leq \frac{(\alpha,\alpha)}{(\beta_i,\beta_i)} \tag{12}$$

so that

$$(\beta_i,\beta_i) \leq \frac{1}{\mu(U_i)}(\alpha,\alpha) \tag{13}$$

Now, suppose (what is in general not correct!) that there exist β on M such β_i is the restriction of β to U_i. It would become easy to have a minoration of $\lambda_{1,p}$, because

$$(\beta,\beta) \leq \sum_{i=1}^{k}(\beta_i,\beta_i) \leq (\alpha,\alpha)\sum_{i=1}^{k}\frac{1}{\mu(U_i)},$$

so

$$\lambda'_{1,p}(M) \geq \frac{1}{\sum_{i=1}^{k}\frac{1}{\mu(U_i)}}.$$

As this is not true, the idea is to correct the situation to make this in some sense possible.

The difficulty comes from the fact that, in general, the restriction of β_i and β_j to U_{ij} do not coincide. We will try to replace β_i by $\bar{\beta}_i$ with this property. We are looking at solutions of type

$$\bar{\beta}_i = \beta_i + d\tau_i \tag{14}$$

where τ_i is defined over U_i and together with a control form above of $\|d\tau_i\|$.

To do this, we will take advantage of the fact that, on U_{ij}, $d\beta_i = \alpha = d\beta_j$, so that $d(\beta_j - \beta_i) = 0$. We can use the hypothesis that there is no cohomology of degree $(p-1)$ in U_{ij}, which implies the exactness of $\beta_j - \beta_i$. There exists γ_{ij} defined on U_{ij}, with

$$d\gamma_{ij} = \beta_j - \beta_i,$$

and γ_{ij} coexact.

The goal will be to write

$$\gamma_{ij} = \tau_i - \tau_j$$

with τ_i defined on U_i and then to write

$$\bar{\beta}_i = \beta_i + d\tau_i,\ i = 1,...,k.$$

We introduce a partition of unity $\{\rho_i\}_{i=1}^{k}$ subordinated to the covering (U_i) and write

$$\tau_i = \sum_{l=1}^{k}\rho_l\gamma_{il}.$$

Observe that τ_i is well defined on U_i by extension by 0 of $\rho_l\gamma_{il}$ on U_i.

First, we have to show that $\bar{\beta}_i$ and $\bar{\beta}_j$ coincide on U_{ij}.

We have

$$\bar{\beta}_i - \bar{\beta}_j = (\beta_i - \beta_j) + d(\tau_i - \tau_j) = -d\gamma_{ij} + d(\tau_i - \tau_j).$$

So, we have to show that $\tau_i - \tau_j = \gamma_{ij}$ on U_{ij}.

Let $x \in U_{ij}$. Because the partition of unity $\{\rho_i\}_{i=1}^k$ is subordinated to the covering (U_i), and because of the hypothesis (no intersection of order 3), only $\rho_i(x)$ and $\rho_j(x)$ may differ from 0. We get

$$\tau_i(x) - \tau_j(x) = \rho_j(x)\gamma_{ij}(x) - \rho_i(x)\gamma_{ji}(x) = (\rho_i(x) + \rho_j(x))\gamma_{ij}(x) = \gamma_{ij}(x).$$

To have the right estimate, we have to control the norm $\|\bar{\beta}_i\|$. We have

$$\|\beta_i + d\tau_i\|^2 \leq 2(\|\beta_i\|^2 + \|d\tau_i\|^2) \tag{15}$$

and we already know how to control the term $\|\beta_i\|$.

We also have

$$\|d\tau_i\| = \|\sum_{l=1}^{k}(d\rho_l \wedge \gamma_{il} + \rho_l d\gamma_{il})\| \tag{16}$$

This is the reason we need to control the L^∞-norm of the partition of unity. With this a priori estimate, it is enough to control $\|\gamma_{ij}\|$ and $\|d\gamma_{ij}\|$.

As $\|d\gamma_{ij}\| = \|\beta_j - \beta_i\|$, the above mentioned estimate of $\|\beta_i\|$ allows to estimate $\|d\gamma_{ij}\|$.

To estimate $\|\gamma_{ij}\|$, we use again (10). As γ_{ij} is coexact, we have

$$\|\gamma_{ij}\|^2 \leq \frac{1}{\mu(U_{ij})}\|d\gamma_{ij}\|^2 \tag{17}$$

which allows to get the desired estimate of $\lambda'_{1,p}(M)$.

The details of the calculation go as follow :

$$(\beta, \beta) \leq \sum_{i=1}^{k}(\bar{\beta}_i, \bar{\beta}_i) \leq \sum_{i=1}^{k} 2((\beta_i, \beta_i) + \|d\tau_i\|^2) \leq 2\sum_{i=1}^{k}(\frac{(\alpha, \alpha)}{\mu(U_i)} + \|d\tau_i\|^2).$$

$$\|d\tau_i\|^2 \leq 2\sum_{j=1}^{m_i}(c_\rho\|\gamma_{ij}\|^2 + \|d\gamma_{ij}\|^2) \leq 2\sum_{j=1}^{m_i}(\frac{c_\rho}{\mu(U_{ij})} + 1)\|d\gamma_{ij}\|^2 \leq$$

$$\leq 2(\alpha, \alpha)\sum_{j=1}^{m_i}(\frac{c_\rho}{\mu(U_{ij})} + 1)(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)}),$$

so that we can conclude

$$(\beta, \beta) \leq 2(\alpha, \alpha)\sum_{i=1}^{k}(\frac{(\alpha, \alpha)}{\mu(U_i)} + \sum_{j=1}^{m_i}(\frac{c_\rho}{\mu(U_{ij})} + 1)(\frac{1}{\mu(U_i)} + \frac{1}{\mu(U_j)})).$$

Proof of the corollary

We can modify as follow the proof of the theorem : we have $(\beta_i, \beta_i) \leq \frac{1}{\mu}(\alpha_i, \alpha_i)$, and we do not use directly the fact that $(\alpha_i, \alpha_i) \leq (\alpha, \alpha)$.

Then we follow the proof of Theorem 35 :

$$(\beta, \beta) \leq \sum_{i=1}^{k} (\bar{\beta}_i, \bar{\beta}_i) \leq \sum_{i=1}^{k} 2((\beta_i, \beta_i) + \|d\tau_i\|^2) \leq 2 \sum_{i=1}^{k} (\frac{(\alpha_i, \alpha_i)}{\mu} + \|d\tau_i\|^2).$$

$$\|d\tau_i\|^2 \leq 2 \sum_{j=1}^{m_i} (c_\rho \|\gamma_{ij}\|^2 + \|d\gamma_{ij}\|^2) \leq 2 \sum_{j=1}^{m_i} (\frac{c_\rho}{\mu} + 1) \|d\gamma_{ij}\|^2 \leq$$

$$\leq 2 \sum_{j=1}^{m_i} (\frac{c_\rho}{\mu} + 1)(\frac{1}{\mu}((\alpha_i, \alpha_i) + (\alpha_j, \alpha_j)).$$

Now, we observe that a given point x is at most in m different U_i, and for each indice i, there is at most m indices j related to i. This implies that

$$(\beta, \beta) \leq \frac{C(m)}{\mu} (\alpha, \alpha).$$

4 Lecture 4 : Small eigenvalues under collapsing

In this lecture, we will investigate the question :

"Does a family of compact Riemannian manifolds with bounded diameter and curvature and volume (or injectivity radius) converging to 0 has nonzero eigenvalues for p-forms converging to 0 ?"

This question would justify a series of lectures for itself, because of a lot a recent and interesting developments (see [CC1], [CC2], [Ja1], [Ja2],[Ja3], [Lo1], [Lo2], [Lo3]). The goal of the section is just to give an overview of the problem. Note that a related question, we will not study at all, is about the adiabatic limits (see [Fo]).

Recall that Example 24 shows that we may have small eigenvalue as the injectivity radius (or the volume) goes to 0, but that the example of a product $M \times S^1$ shows that it may have no small eigenvalue.

4.1 A few words about collapsing

We begin by some geometrical considerations :

Definition 42. *We say that a compact manifold M admits a collapsing if there exists two positive constant a and d and a family $\{g_i\}_{i=1}^{\infty}$ of Riemannian metrics such that $|K(g_i)| \leq a$, $diam(g_i) \leq d$ and $inj(g_i) \to 0$ as $i \to \infty$ (or equivalently, $vol(g_i) \to 0$ as $i \to \infty$).*

Notation In the sequel, we will call *a (a,d)-metric* a Riemannian metric with sectional curvature $|K| \leq a$ and diameter $diam \leq d$.

It is in general not easy to decide if a manifold admit or not a collapsing, but this depends clearly of its topology. As example, if a manifold admits a collapsing, its minimal volume is 0 and all its characteristic numbers have to vanish. The metric description of a collapsing manifold was investigate of lot during the 80', and a quit complete, but very complicated, description is given in the paper of Cheeger-Fukaya-Gromov [CFG]. However, for the purpose of these notes, it is easier to give a partial result of Fukaya [Fu1]

Theorem 43. *Let (M_i, g_i) a sequence of compact Riemannian manifolds of dimension n with sectional curvature and diameter uniformly bounded, and (N, h) a compact manifold of dimension $m < n$. If (M_i, g_i) converge to (N, h) for the Gromov-Hausdorff distance, then, for i large enough, there is a fiber bundle structure $\pi : M_i \to N$ whose fiber is an infranilmanifold.*

The prototype of an infranilmanifold is a torus (or a quotient of a torus) and the prototype of such a bundle is a torus bundle; (note however that not all torus bundle may collapse). A simple example of collapsing manifold is the product of a fixed Riemannian manifold N with a flat torus whose injectivity radius goes to 0 (which does not affect its curvature!), and roughly speaking, we may think to a collapsing manifold as a manifold which is *locally* a product of a manifold with a flat torus (or a nilmanifold). The simplest non trivial examples we can imagine are the following :

- At first, a S^1-bundle on a manifold N;
- Then a torus bundle on a "simple" manifold N like a circle.

And these examples were the first one's to be deeply investigated from the point of view of the spectral theory ([CC1], [CC2] for the S^1-bundle and [Ja1] for torus bundle over a circle).

The easiest result to present is the case of S^1-bundle, and it allows to ask a lot of open questions.

4.2 The case of S^1-bundle

The goal is to give the main steps of the paper [CC2]. We will state a precise result at the end of the first step.

Intuitively, in order to study the spectrum of the S^1-bundle, we try to be close to a situation where we have a good intuition and where we can do explicit calculations : this situation is precisely Example 24 of lecture 2, the case of the Berger's spheres. **Step 1** The goal of this fisrt step is to show that "the general situation" of an S^1-bundle, with bounded curvature and diameter, is not so different of a very particular situation where the Riemannian metric is easy to understand. By close, we mean "quasi-isometric with a controled ratio". Indeed, there is an easy but very useful fact we can use when we study the spectrum from a qualitative viewpoint (and to decide if there are small eigenvalues and even to have informations about their asymptotic behaviour is a prototype of qualitative question). It is the property that two quasi-isometric Riemannian manifolds have comparable spectrum. This is a result du to J. Dodziuk (see [Do])

Definition 44. *Let M be a manifold and $\tau > 1$. Two Riemannian metrics g_1 and g_2 on M are said τ-quasi-isometric if, for each p in M and $v \in T_pM$, then*

$$\frac{1}{\tau} \leq \frac{g_1(p)(v,v)}{g_2(p)(v,v)} \leq \tau.$$

The number τ is the ratio *of the quasi-isometry.*

Theorem 45. *Let M be a compact manifold of n dimension, and g_1, g_2 two Riemannian metric τ-quasi-isometric on it. Then, for each $0 \leq p \leq n$ and $k > 0$, we have*

$$\frac{1}{\tau^{n+2p+1}} \lambda_{k,p}(M, g_1) \leq \lambda_{k,p}(M, g_2) \leq \tau^{n+2p+1} \lambda_{k,p}(M, g_2) \tag{18}$$

If $S^1 \hookrightarrow (M, g') \to (N, h')$ is a S^1 principal bundle with a (a', d')-metric on M, we show in the first step that we can replace this by $S^1 \hookrightarrow (M, g) \to (N, h)$, where g is a (a, d)-metric (a, d depending only on a', d') and g, h are τ-quasi-isometric to g' and h' respectively, τ depending only on a', d', such that the bundle

$$S^1 \hookrightarrow (M, g) \xrightarrow{\pi} (N, h)$$

has very nice properties :

1. The principal bundle $S^1 \hookrightarrow (M, g) \xrightarrow{\pi} (N, h)$ is a Riemannian submersion ;
2. The fibers are geodesics all of the same length ϵ ;
3. The action of S^1 is isometric ;
4. If ω is the vertical 1-form of norm 1 associated to the action of S^1, then $d\omega = \epsilon \pi^*(e(M))$, where $e(M)$ is the harmonic representant of the Euler class of the bundle.

Definition 46. *Such a couple (g, h) of Riemannian metric will be called a (a, d)-adapted metric.*

InExample 24, the Riemannian metric is precisely of this type. The Euler class of the Hopf-bundle is the Kähler form of the complex projective space.

In the context of (a, d)-adapted metric, we can state our result, which will be also true in general because of the quasi-isometry property. In the sequel, we set $m_p = b_p(N) + b_{p-1}(N) - b_p(M)$, which measure in some sense the defect of M to be a product.

Theorem 47. *If ϵ is small enough, there exists $C_i = C_i(n, a, d) > 0$, $i = 1, 2$, such that*

$$0 < \lambda_{k,p}(M, g) \leq C_1(\epsilon\|e\|_2)^2, \ 1 \leq k \leq m_p; \tag{19}$$

$$\lambda_{m_p+1,p}(M, g) \geq C_2. \tag{20}$$

Theorem 48. *If ϵ is small enough, there exists $C_i = C_i(n, a, d) > 0$, $i = 1, 2$, such that, if $e(M) \neq 0$, then*

$$C_1(\epsilon\|e\|_2)^2 < \lambda_{1,1}(M, g) \leq C_2(\epsilon\|e\|_2)^2. \tag{21}$$

Theorem 49. *If ϵ is small enough, there exists $C_i = C_i(n, a, d) > 0$, $i = 1, 2$, such that, if $e(M) \neq 0$ and if $dim H^2(N, \mathbb{R}) = 1$, then*

$$C_1(\epsilon\|e\|_2)^2 < \lambda_{k,p}(M, g) \leq C_2(\epsilon\|e\|_2)^2, \ 1 \leq k \leq m_p. \tag{22}$$

Some comments

1. The first clear point is that it cannot be more small eigenvalues for p-forms as the maximal number of p-harmonic forms (the case of the product), and that the number of small nonzero p-eigenvalues measure precisely the defect to be product ;
2. A second question is to estimate asymptotically the small eigenvalues. It is clear that it is not enough to consider ϵ (which correspond to the injectivity radius), but that we have to take account of $\|e\|$ (see Example 50 to understand why) ;
3. Then, Theorem 49 says that, if the 2 nd cohomology of the basis is not complicated, it is indeed true that $(\epsilon\|e\|)^2$ give a good estimate of the small eigenvalues, Theorem 48 says that this is always true for $\lambda_{1,1}$, and Theorem 47 says that in general it gives only an upper bound. An (non easy) example given in [CC2] explains why we cannot hope to get a lower bound in term of $(\epsilon\|e\|)^2$.

Example 50. *There is an isometric action of the finite group $\mathbb{Z}/q\mathbb{Z}$ on S^{2n+1} given by Example 24 if we see $\mathbb{Z}/q\mathbb{Z}$ as the q-root of unity.*

Let denote by L_q the quotient of S^{2n+1} by this action, with the induced metric. We have a S^1-bundle

$$S^1 \hookrightarrow L_q \to \mathbb{C}P^n$$

with totally geodesic fiber and injectivity radius π/q going to 0 as $q \to \infty$. However, there are no small eigenvalues on L_q, because each eigenform may be lift to an eigenform of the *round sphere. This comes from the fact that the Euler class e_q of L_q has a norm going to ∞ as $q \to \infty$, so that the product ($inj(L_q)\|e_q\|_2$) stay bounded away from* 0.

Step 2 In the context of an (a, d)-adapted metric, this step is very natural : it consists to show that only the S^1-invariant p-forms may produce small eigenvalues. This is well explain in the paper of P. Jammes [Ja2].

Step 3 In this step, we have to prove Theorem 47, 48 and 49 for S^1-invariant forms. We just explain here the beginning of the calculations.

Let ψ be a S^1-invariant p-form on M. We can write

$$\psi = \pi^*\alpha + \pi^* a \wedge \omega \tag{23}$$

where we recall that ω is the vertical 1-form of norm 1 associated to the action of S^1. Since we have $d\omega = \epsilon e$, a standard calculation shows that

$$\|\psi\|_2^2 = \epsilon(\|\alpha\|_2^2 + \|a\|_2^2) \tag{24}$$

$$\|d\psi\|_2^2 = \epsilon(\|d\alpha + (-1)^{p-1}\epsilon(a \wedge e)\|_2^2 + \|da\|_2^2) \tag{25}$$

$$\|\delta\psi\|_2^2 = \epsilon(\|\delta\alpha\|_2^2 + \|\delta a + (-1)^{np}\epsilon * (*\alpha \wedge e)\|_2^2) \tag{26}$$

Now the goal is to control the apparition of small eigenvalues as $\epsilon \to 0$. It is intuitively clear that and it is not difficult to show that we have only to look at the lift of harmonic forms of the basis manifold N.

So we lift $b_p(N) + b_{p-1}(N)$ harmonic forms of N to M. But $b_p(M)$ of them will correspond to harmonic forms. We will see that the other ones gives m_p small eigenvalues.

If α and a are harmonic forms, we get the expression

$$\|\psi\|_2^2 = \epsilon(\|\alpha\|_2^2 + \|a\|_2^2) \tag{27}$$

$$\|d\psi\|_2^2 = \epsilon^3\|a \wedge e\|_2^2 \tag{28}$$

$$\|\delta\psi\|_2^2 = \epsilon^3\| * \alpha \wedge e\|_2^2 \tag{29}$$

If we estimate the Rayleigh quotient $R(\psi)$, we note that as a and α are harmonic of L^2-norm ≤ 1, their L^∞ norm is controled by the geometry of the manifold by Sobolev inequality (see [Li]), so that we get

$$R(\psi) \leq C\epsilon^2\|e\|_2^2, \tag{30}$$

where C is a positive constant depending only on the geometric bounds of the problem.

It is much more technical to understand to what extend this gives the good asymptotic, and this is the purpose of the paper [CC2] to do it.

4.3 Other developments from J. Lott and P. Jammes

The work of P. Jammes In a series of very interesting papers ([Ja1], [Ja2], [Ja3]), Pierre Jammes has investigated the following question : there is only one way to collapse a circle, but as soon as we collapse a bundle whose fiber has dimension > 1, there are a lot of different way to collapse. Does the way of collapsing affect the apparition of small eigenvalues ?

P. Jammes investigated mainly the torus bundles, and the answer is clear : the way of collapsing affects a lot the apparition of small eigenvalues. The things are so tricky that the works of P. Jammes leads to a lot of new open questions. One can find a small survey in [Ja4]. Let us just mention two in my opinion very spectacular results of P. Jammes (see [Ja2])

Theorem 51. *Let $k \geq 1$. For each Riemannian manifold (N, h) such that $b_2(N) \geq k$, there exists a principal T^k- torus bundle on N, a family of Riemannian metric $(g_\epsilon)_{0<\epsilon<1}$ on M, and two positive constants $C = C(k, h)$, $\epsilon_0 = \epsilon_0(k, h)$ such that :*

$$|K(M, g_\epsilon)| \leq a; diam(M, g_\epsilon) \leq d \text{ and } vol(M, g_\epsilon) = \epsilon \tag{31}$$

and, for $p = 1, 2$ and ϵ small enough,

$$\lambda_{1,p}(M, g_\epsilon) \leq C(inj(M, g_\epsilon)^{2k}. \tag{32}$$

This result shows that (at least for the small degrees) the dependance of λ_1, p with the injectivity depends on the dimension, what istotally unclear when we study the S^1-bundle. This lead also to the question (see [Ja4])

Open question 4. *For a compact Riemannian manifold M of dimension n with bounded sectional curvature by k and diameter bounded from above by d, do we have an estimation of the type*

$$\lambda_{1,p} \geq C(n, d, k)(inj(M))^{an+b}$$

with a, b universal constants.

Theorem 51 leads to try to estimate the spectrum with respect to the volume. P. Jammes has also a result in this direction

Theorem 52. *Let $T^k \hookrightarrow (M, g) \xrightarrow{\pi} (N, h)$ a T^k-principal bundle so that $|K(g)| \leq a$ and $diam(g) \geq d$. If (M, g) is ϵ-Hausdorff close to (N, h), then and ϵ small enough, then*

$$\lambda_{1,1}(M, g) \geq C(n, a, d)Vol^2(M, g).$$

This lead to the two natural questions (see [Ja4])

Open question 5. *In the context of Theorem 52, what about the other degrees ?*

Open question 6. *In general (diameter and sectional curvature bounded), could we hope to have a minoration of* $\lambda_{1,p}$ *with respect to* Vol^2 *?*

Two last general questions in these directions are the following

Open question 7. *Does each compact manifold of dimension* ≥ 3 *admit a family of metric with* $Ric(g)diam(g)^2 \geq C$ *and arbitrarily small* $\lambda_{1,p}$*. Recall that the hypothesis* $Ric(g)diam(g)^2 \geq C$ *implies that* $\lambda_{1,0}$ *is bounded away from* 0 *(Theorem 13).*

The work of J. Lott The approach of J. Lott ([Lo1], [Lo2]) is a generalization to the p-forms spectrum of the work of Fukaya for functions [Fu2]. If a family of Riemannian manifolds (M_i, g_i) collapses to a limit X, is it possible to construct a limit operator on X so that the spectrum of X gives the limit of the spectrum of (M_i, g_i) ? In fact, in the case of form, the idea is to construct a vector bundle on the limit, and then an operator on this limit.

The results of Lott are related to the question of the apparition of small eigenvalues as follow. The number of small eigenvalues for p-forms is the small eigenvalues which are not 0. But all the small eigenvalues for p-forms (which are 0 or not) converge to 0 at the limit and give a contribution to the kernel of the limit operator. So, if we are able to investigate the dimension of the kernel of this operator, we get a qualitative information about the number of small eigenvalues, and this is well illustrated in Lott's papers. This is also quit difficult in general.

Open question 8. *The paper [Lo2] is not published, perhaps because it contains some gaps in some proofs, but it is in any case very interesting. It would be interesting to study this paper and to illustrate it with examples.*

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