Bilateral $k+1$-price Auctions with Asymmetric Shares and Values

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Abstract

We study a sealed-bid auction between two bidders with asymmetric independent private values. The two bidders own asymmetric shares in a partnership. The higher bidder buys the lower bidder's shares at a per-unit price that is a convex combination of the two bids. The weight of the lower bid is denoted by $k \in [0,1]$. We partially characterize equilibrium strategies and show that they are closely related to equilibrium strategies of two well-studied mechanisms: the double auction between a buyer and a seller and the first-price auction between two buyers (or two sellers). Combining results from those two branches of the literature enables us to prove equilibrium existence. Moreover, we find that there is a continuum of equilibria if $k \in (0,1)$ whereas the equilibrium is unique if $k \in \{0,1\}$. Our approach also suggests a procedure for numerical simulations.

JEL classification: D44, D82, C78, C72

Keywords: Asymmetric auctions, Double auction, First-price auction, Partnership dissolution, Detail-free mechanism

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1 Introduction

In the $k+1$-price auction the highest bidder wins and pays a convex combination of the highest and second highest bid, where the weight on the second highest bid is denoted by $k \in [0,1]$. Unlike in a standard auction, however, the proceeds of the auction are not collected by a seller but are distributed among all bidders according to exogenously specified shares. Hence, each bidder receives a benefit in proportion to the price paid by the winner.

The $k+1$-price auction has various applications. According to Engelbrecht-Wiggans (1994) this mechanism is sometimes used in Amish estate sales or for sharing the spoils of a bidding ring among its members. Burkart (1995) as well as Bulow, Huang, and Klemperer (1999), among others, model takeover battles between bidders that own toeholds in the takeover target as $k+1$-price auctions with $k \in \{0,1\}$. Moreover, $k+1$-price auctions with symmetric revenue shares have been repeatedly proposed as a dissolution mechanism for partnerships.[1] Rather than assuming such an auction to be explicitly imposed as dissolution mechanism, we could also think of it, e.g. with $k=\frac{1}{2}$, as a simple model of bargaining between two partners about the per-unit price at which partnership shares are to be traded. In models of auctions for raising money for charity, a bidder's utility is often assumed to depend not only on the allocation of the good that is auctioned and on his payment, but also on the total revenue raised. If the utility function is linear in the auction revenue, as for example in Goeree et al. (2005) and Engers and McManus (2007), a charity auction where the winner pays a convex combination of the highest and the second highest bid is equivalent to a $k+1$-price auction.[2]

In the applications mentioned above, we may frequently encounter asymmetries, especially with respect to the shares: the size of toeholds may vary among bidders in a takeover, partners often hold different amounts of shares in a partnership, and participants in a charity event may differ in their degree of altruism. In addition, it is quite common that some bidders are known for being more interested in the object to be auctioned than other bidders, i.e., that valuations are asymmetrically distributed. Despite the importance of such asymmetries, the literature has been

[1]See, e.g., Cramton, Gibbons, and Klemperer (1987) or McAfee (1992).

[2]These are examples of auctions with linear financial externalities. Such externalities also occur when bidders have cross shareholdings as in Dasgupta and Tsui (2004) or Chillemi (2005). There, rather than depending on the auction revenue, each bidder's payoff is proportional to the individual surplus of the winner. For a general treatment of linear financial externalities see Lu (2012).

largely focusing on settings where bidders are ex ante symmetric and all obtain the same share of the auction revenue. As the main contribution of this paper, we analyze a setting with two bidders and allow for both asymmetric shares and asymmetric distributions of valuations.

Myerson and Satterthwaite (1983) study bilateral trading mechanisms where a seller and a buyer interact, both having private information. A simple mechanism in their framework is the k-double auction: both the seller and the buyer submit a sealed bid and if the buyer's bid is higher than the seller's, the good is traded at a price equal to a convex combination of the two bids (determined by the parameter k). Starting with Chatterjee and Samuelson (1983), the k-double auction has received much attention in the literature. Considering partnership dissolution Cramton, Gibbons, and Klemperer (1987) extend the analysis of Myerson and Satterthwaite (1983) to a setting with distributed ownership. The $k+1$-price auction is a straightforward extension of the k-double auction to the framework of Cramton, Gibbons, and Klemperer.[3] In contrast to the k-double auction, however, the $k+1$-price auction is not well understood except for the special case where ownership shares (and valuations) are symmetric. In this paper we take a first step towards filling this gap.

We study bilateral $k+1$-price auctions with asymmetrically distributed independent private values and asymmetric shares. We present the model in the context of partnership dissolution. Nevertheless, the main results are relevant also for other applications such as takeovers or charity. When considering partnerships we may interpret the $k+1$-price auction as determining the per-unit price of a share and the direction of trade, i.e., which partner is to sell his share to the other partner.

We restrict our attention to Bayesian Nash equilibria in continuous and strictly increasing pure strategies. In equilibrium a bidder with a low valuation overbids, whereas a bidder with a high valuation engages in bid shading. When having a low valuation, the probability of winning the auction is relatively small, such that a bidder aims at driving up the price his opponent has to pay when winning. By contrast, a high-valuation bidder expects to win and therefore shades his bid in order to lower the price he will have to pay. Owing to the asymmetry in shares and valuations, the extent of overbidding and bid shading differs between the two bidders.

Suppose, momentarily, that the two bidders have their valuations drawn from the same distribution and differ only in their shares. With one bidder owning more and the other less than 50% of the partnership, we may refer to them as the majority and

[3] Kittsteiner (2003), among others, uses the name k-double auction also for the $k+1$-price auction.

the minority owner, respectively. For the majority owner, the price he obtains for his large share when losing is relatively more important than the price at which he buys the small share from the other bidder when winning. Compared with the minority owner, the majority owner has a stronger incentive to overbid. Consequently, the majority owner bids more aggressively than the minority owner.

Now suppose, instead, that shares are symmetric but that one bidder has a reputation for being more interested in the object that the other bidder. In particular, assume there to be a “strong” and a “weak” bidder in the sense that the strong bidder’s distribution of valuations dominates the weak bidder’s in terms of the hazard and the reverse hazard rate. As in a standard first-price auction, in equilibrium the weak bidder bids more aggressively than the strong bidder.

Throughout the paper, we make a joint assumption on the asymmetry in shares and distributions that captures the two situations just discussed as special cases. We assume that bidder 1’s distribution is dominated by bidder 2’s in terms of the hazard and reverse hazard rate multiplied by the shares. This assumption is sufficient for the overall effect of asymmetric shares and asymmetric values to be that bidder 1 bids more aggressively than bidder 2.

In equilibrium, the two bidders choose their bids from the same range but, for a given type, bidder 1’s bid lies above bidder 2’s (except at the boundaries of the support). Since for bidder 1 the range of types that overbid is larger than for bidder 2, we may divide the range of bids in a given equilibrium into three parts: High bids are submitted by bid shading types of both bidders, intermediate bids come from overbidding types of bidder 1 and bid shading types of bidder 2, and when submitting low bids both bidders overbid. Equilibrium strategies make sure that no type can benefit from slightly increasing or decreasing his bid. Hence, the bid of a given type is locally determined by the behavior of types of the opponent that submit similar bids. Those types that bid in the highest part of the range of equilibrium bids are in a similar situation as in a standard first-price auction between two buyers: they find it optimal to shade their bids while competing with types of the opponent that do the same. Similarly, the mutual overbidding in the part with the lowest bids reminds of two sellers in a first-price procurement auction. For the intermediate range of equilibrium bids, the interaction resembles that in a double auction between an overbidding seller and a bid shading buyer.

As suggested by the above intuition, there is indeed a direct link to first-price and double auctions. We show that a k-double auction between a buyer and a seller can be constructed for which equilibrium strategies are equivalent to those for the intermediate range of bids in the $k+1$-price auction. In a similar manner, an asymmetric first-price auction with a reserve price can be found where equilibrium strategies are equivalent to those in the upper and lower range of bids in the $k+1$-price auction. This enables us to prove existence of an equilibrium by combining results by Satterthwaite and Williams (1989) for the k-double auction and by Lebrun (1999, 2006) for the first-price auction. Moreover, we find that there is a continuum of Bayesian Nash equilibria in continuous and strictly increasing strategies if $k \in (0,1)$. If $k \in \{0,1\}$, however, the pure-strategy Bayesian Nash equilibrium is unique.

The papers most closely related to ours, in that they also consider asymmetric $k+1$-price auctions, are Bulow, Huang, and Klemperer (1999), de Frutos (2000), and Bos (2011). They all restrict attention to $k \in \{0,1\}$. Studying takeovers, Bulow, Huang, and Klemperer (1999) consider asymmetric shares like us, but assume (symmetric) common values, which allows them to express equilibrium strategies in closed form. By contrast, our model of independent private values does not admit closed-form solutions. Bos (2011) considers asymmetric shares and symmetric uniformly distributed private values in a charity setting, focusing on comparing the revenue in the $k+1$-price auction where $k=0$ with that in the all-pay auction.

In de Frutos (2000) shares are symmetric whereas one bidder's distribution of valuations dominates the other bidder's in terms of the hazard and the reverse hazard rate. For $k \in \{0,1\}$, de Frutos proves that there is a unique equilibrium, by making use of Lebrun (1999) in a similar way as we do in this paper. We extend the uniqueness result for $k \in \{0,1\}$ by allowing for asymmetry in shares and complement de Frutos' findings by showing that if $k \in (0,1)$, there is a continuum of equilibria.

The study of symmetric $k+1$-price auctions with $k \in [0,1]$ goes at least back to Cramton, Gibbons, and Klemperer (1987) who establish it as a mechanism suitable for dissolving equal-share partnerships. They were inspired by the split-the-difference mechanism in Samuelson (1985), which corresponds to setting $k=\frac{1}{2}$. For $k \in \{0,1\}$, equilibrium bidding is further studied by Engelbrecht-Wiggans (1994) in a model of interdependent valuations and by Maasland and Onderstal (2007) when there is a reserve price. Kittsteiner (2003) examines the more general case of $k \in [0,1]$, proving uniqueness of the equilibrium.

As we show in this paper, the uniqueness result of Kittsteiner (2003) is not robust with respect to asymmetries among bidders. With $k \in (0,1)$ and private values, already a slight deviation from the case of symmetric shares and symmetric priors gives rise to a multiplicity of equilibria. Kittsteiner observes that his uniqueness result under distributed ownership is in contrast to the setting where one partner owns 100% of the partnership, i.e., a k-double auction between a seller and a buyer, which is known for having many different equilibria if $k \in (0,1)$. Our result points out that distributed ownership, per se, does not prevent equilibrium multiplicity.

When, as in our case, equilibrium strategies can in general not be expressed in closed form, additional insights may be gained from numerical simulations. Also for this purpose, our finding that relates the $k+1$-price auction to two well-studied mechanisms is very useful. It implies that when simulating the $k+1$-price auction we can make use of existing algorithms for double and first-price auctions.

In the private-values partnership framework of Cramton, Gibbons, and Klemperer (1987) a dissolution mechanism needs to satisfy ex post budget balance and interim individual rationality. Cramton, Gibbons and Klemperer find that ex post efficient dissolution is possible only if shares are not too asymmetric.[4] Chien (2007) characterizes the incentive efficient mechanism that maximizes the expected gains from trade for all asymmetric partnerships. As its main disadvantage, the incentive efficient dissolution mechanism is not detail-free: its rules depend on the distribution of valuations. In practice, such detailed information on the environment is often not available to the designer. To address this problem, one could think of a mechanism that asks players to report, in addition to their types, also their beliefs concerning the types of their partners. Of course, this would increase the complexity of the rules considerably, again posing a problem in practical applications. Hence, rules that are both simple and detail-free are required to implement the dissolution of a partnership. McAfee (1992) studies such mechanisms for equal-share partnerships.

The $k+1$-price auction is a simple and detail-free dissolution mechanism for asymmetric partnerships with private values. As equilibrium strategies differ across bidders, the resulting allocation is not guaranteed to be ex post efficient.[5] Yet numerical results for uniformly distributed valuations suggest that the gains from trade in the unique equilibrium for $k \in \{0,1\}$ are not much lower than those of the incentive efficient mechanism. For $k \in (0,1)$ there are equilibria that lead to even higher gains from trade than for $k \in \{0,1\}$. Moreover, for all $k \in [0,1]$, there are equilibria that, in

[4] Fieseler, Kittsteiner, and Moldovanu (2003) consider interdependent values and Galavotti, Muto, and Oyama (2011) study the effect of imposing ex post individual rationality.

[5] Indeed, Athanassoglou, Brams, and Sethuraman (2010) show that ex post efficiency cannot be achieved by any mechanism where the per-unit price of the partnership is determined from bids independently of the bidders' identities.

terms of expected gains from trade, outperform the buy-sell clause, another simple and detail-free dissolution mechanism which is used in practice.

The paper is organized as follows. Section 2 describes the model. In Section 3 we partially characterize equilibrium bidding. In Section 4 we show that equilibrium bidding is closely related to the equilibria of two well-known mechanisms, allowing us to establish equilibrium existence as well as multiplicity and uniqueness, respectively. Section 5 discusses the performance of the $k+1$-price auction as a mechanism for partnership dissolution, assuming asymmetric shares and uniformly distributed valuations. Section 6 concludes. Some of the proofs are relegated to the Appendix.

2 The Model

There are two risk-neutral partners who jointly own a single indivisible object. Partner $i \in \{1,2\}$ owns share $\alpha_i \in (0,1)$ in the partnership with $\alpha_1 + \alpha_2 = 1$. Partner i has valuation v_i for the entire object whereas owning share α_i generates utility $\alpha_i v_i$ for him. Valuation v_i is partner i's private information. It is common knowledge that v_i is a realization of the continuous random variable V_i that is independently distributed according to F_i with support $[0,1]$. F_i is twice continuously differentiable and has a strictly positive density f_i. Virtual valuations are increasing, i.e.,

$$\frac{d}{dv}\left(v + \frac{F_i(v)}{f_i(v)}\right) \geq 0 \qquad \text{and} \qquad \frac{d}{dv}\left(v - \frac{1-F_i(v)}{f_i(v)}\right) \geq 0 \qquad \text{for } i = 1,2. \tag{1}$$

Moreover, we assume

$$\alpha_1 \frac{F_1(v)}{f_1(v)} > \alpha_2 \frac{F_2(v)}{f_2(v)} \qquad \text{and} \qquad \alpha_2 \frac{1-F_1(v)}{f_1(v)} < \alpha_1 \frac{1-F_2(v)}{f_2(v)} \qquad \forall v \in (0,1). \tag{2}$$

This is a joint assumption on asymmetry in shares and valuations. On the one hand, if $F_1 = F_2$, (2) simplifies to $\alpha_1 > \alpha_2$. On the other hand, if $\alpha_1 = \alpha_2$, (2) requires that F_2 dominates F_1 in terms of the reverse hazard rate and the hazard rate.[6] Whereas $\alpha_1 > \alpha_2$ leads to a weaker notion of (reverse) hazard rate dominance, $\alpha_1 < \alpha_2$ corresponds to a stronger notion.[7]

The following auction is used to dissolve the partnership and assign sole owner-

[6]In this case, (2) is equivalent to assumptions (A3) and (A4) in de Frutos (2000).

[7]Suppose $\alpha_1 < \alpha_2$. An example of a pair of distributions that satisfies (2) is $F_1(v) = 1-(1-v)^a$ and $F_2(v) = v^a$ with $a > \frac{\alpha_2}{\alpha_1}$.

ship to one of the partners. Each partner i submits a sealed bid b_i. The bidder with the higher bid gets the object and pays the price $P=(1-k)\max\{b_1,b_2\}+k\min\{b_1,b_2\}$ where $k\in[0,1]$. Each bidder i obtains α_i of the revenue P generated by this $k+1$-price auction. If both bidders submit the same bid, the partnership is left intact and no payments are made.[8] Bidder i's ex post net payoff when having valuation v_i and bidding b_i while bidder j bids b_j therefore amounts to

$$u_i(v_i,b_i,b_j):=\begin{cases}(1-\alpha_i)\Big(v_i-(1-k)b_i-kb_j\Big) & \text{if } b_i>b_j,\\ 0 & \text{if } b_i=b_j,\\ \alpha_i\Big((1-k)b_j+kb_i-v_i\Big) & \text{if } b_i<b_j.\end{cases} \tag{3}$$

Throughout, we will focus on pure-strategy equilibria of the Bayesian game defined by types v_1,v_2, actions b_1,b_2 and payoffs u_1,u_2. A pure strategy for player i is a function $\beta_i\colon[0,1]\to\mathbb{R}$ such that $\beta_i(v_i)$ denotes the bid i places if he has valuation v_i. In a pure-strategy Bayesian Nash equilibrium, equilibrium strategies β_1,β_2 satisfy

$$\beta_i(v_i)\in\arg\max_{b_i} E\left[u_i(v_i,b_i,\beta_j(V_j))\right] \qquad \forall v_i\in[0,1]\,, i=1,2 \text{ and } j\neq i. \tag{4}$$

Looking at players' payoffs as defined in (3), we can make the following observation. This result will be very useful throughout the paper.

Proposition 1. *The Bayesian games induced by the following auctions are equivalent:*

1. *A $k+1$-price auction with shares α_1,α_2 and values that are realizations of V_1,V_2.*
2. *A $2-k$-price auction with shares $\tilde{\alpha}_1:=\alpha_2,\tilde{\alpha}_2:=\alpha_1$, with values that are realizations of $\tilde{V}_1:=1-V_1$ and $\tilde{V}_2:=1-V_2$, and where bidders' actions are modified as follows: a bidder who wants to place the bid b is asked to submit the number $1-b$ instead of directly announcing b.*

Proof. In a $k+1$-price auction with shares α_1,α_2 player i's ex post payoff is $u_i(v,b_i,b_j)$ as defined in (3). Similarly, let $\tilde{u}_i(v,b_i,b_j)$ denote i's payoff in a $\tilde{k}+1$-price auction with shares $\tilde{\alpha}_1,\tilde{\alpha}_2$. Observe that if $\tilde{k}=1-k$ and $\tilde{\alpha}_i=1-\alpha_i$, we have $u_i(v,b_i,b_j)=\tilde{u}_i(1-v,1-b_i,1-b_j)$. Hence, if types in the second auction are realizations of $\tilde{V}_i:=1-V_i$ for $i=1,2$ and players' actions are changed from submitting a bid to submitting

[8] This is equivalent to assuming that the partnership is also dissolved when bids are equal, with bidder i being selected as the winner with probability α_i. Note that our results will not rely on this specific tie-breaking rule. In the class of equilibria we will focus on, ties occur with probability zero.

a number that is equal to 1 minus the intended bid, the two auctions induce the same Bayesian game. □

For a specific class of distributions F_1, F_2, a direct implication of Proposition 1 is that the $k+1$- and $2-k$-price auction are equivalent in terms of the ex ante expected gains from trade, i.e., the sum of players' ex ante expected payoffs. More precisely, this holds if the densities f_1 and f_2 are mirror images to each other, implying that V_i and $1-V_j$ follow the same distribution.

Corollary 1. *Suppose valuations are drawn from distributions that satisfy $f_1(v) = f_2(1-v)\ \forall v$. Then, for every Bayesian Nash equilibrium of the $k+1$-price auction there is a Bayesian Nash equilibrium of the $2-k$-price auction that yields the same ex ante expected gains from trade.*

Moreover, Proposition 1 also has the following direct implications for pure strategies that form a Bayesian Nash equilibrium.

Corollary 2. *The $k+1$-price auction with types drawn from $F_1(v)$ and $F_2(v)$ has a pure-strategy Bayesian Nash equilibrium where players bid according to $\beta_1(v)$ and $\beta_2(v)$ if and only if the $2-k$-price auction with types drawn from $\hat{F}_1(v) := 1-F_2(1-v)$ and $\hat{F}_2(v) := 1-F_1(1-v)$ has a pure-strategy Bayesian Nash equilibrium where players bid according to $\hat{\beta}_1(v) := 1-\beta_2(1-v)$ and $\hat{\beta}_2(v) := 1-\beta_1(1-v)$.*

Note that assumption (2) is satisfied by F_1, F_2 if and only if it is satisfied by $\hat{F}_1, \hat{F}_2$. On several occasions below, Corollary 2 will allow us to apply results for a specific $k = \bar{k}$ also to the case where $k = 1-\bar{k}$.

3 Equilibrium Characterization

An important property of the equilibrium strategies is that the bids for the highest and lowest possible valuation must be the same for both bidders, that is $\beta_1(1) = \beta_2(1)$ and $\beta_1(0) = \beta_2(0)$. Suppose this was not the case, e.g., $\beta_1(1) > \beta_2(1)$. Then bidder 1 with valuation 1 would want to deviate and reduce his bid to a value just above $\beta_2(1)$ since by doing so he would still win the auction with certainty but would have to pay less. A more rigorous reasoning is given in the proof to the following lemma.[9]

[9] The proofs of Lemmata 1 and 2 make use of some arguments that are common in auction theory and can be traced back to Griesmer, Levitan, and Shubik (1967). In the context of $k+1$-price auctions related statements can also be found in Bulow, Huang, and Klemperer (1999), de Frutos (2000), and Kittsteiner (2003).

Lemma 1. *For every Bayesian Nash equilibrium in pure strategies β_1, β_2, there exist $\underline{b} < \overline{b}$ such that*

$$\beta_1(0) = \beta_2(0) = \underline{b}, \quad \beta_1(1) = \beta_2(1) = \overline{b}, \quad \text{and } \beta_i(v) \in [\underline{b}, \overline{b}]\ \forall v, i = 1,2.$$

Moreover,

$$\underline{b} \begin{cases} = 0 & \text{for} \quad k = 0, \\ > 0 & \text{for} \quad k \in (0,1] \end{cases} \qquad \text{and} \qquad \overline{b} \begin{cases} < 1 & \text{for} \quad k \in [0,1), \\ = 1 & \text{for} \quad k = 1. \end{cases}$$

Proof. See Appendix A.1. □

If either $k = 0$ or $k = 1$, equilibrium strategies can be narrowed down further.

Lemma 2. *Suppose $k \in \{0,1\}$. Then, for every pure-strategy Bayesian Nash equilibrium, equilibrium strategies β_1, β_2 are continuous and strictly increasing.*

Proof. See Appendix A.2. □

Although we do not rule out other equilibria when $k \in (0,1)$, we will exclusively look for equilibria in continuous and strictly increasing strategies. We hence assume for now that there are continuous and strictly increasing equilibrium bidding strategies $\beta_1(v), \beta_2(v)$. In the following, we will further characterize such equilibria, before proving their existence in the next section. It is useful to define the inverse bidding strategy $\phi_j(b) := \beta_j^{-1}(b)$. From bidder i's perspective, bidder j's bids are then distributed according to $F_j(\phi_j(b))$. Using (3) we obtain for i's interim expected payoff

$$\begin{aligned} E\left[u_i(v, b, \beta_j(V_j))\right] = F_j(\phi_j(b))(1-\alpha_i) &\left(v - (1-k)b - k \frac{\int_{\underline{b}}^{b} x \, dF_j(\phi_j(x))}{F_j(\phi_j(b))} \right) \\ &+ \left(1 - F_j(\phi_j(b))\right) \alpha_i \left((1-k) \frac{\int_b^{\overline{b}} x \, dF_j(\phi_j(x))}{1 - F_j(\phi_j(b))} + kb - v \right). \end{aligned}$$

As we establish in the following lemma, bidding strategies that satisfy the equilibrium condition (4) solve a system of differential equations with boundary conditions.

Lemma 3. *A pair of continuous and strictly increasing pure strategies β_1, β_2 is a Bayesian Nash equilibrium if and only if for almost all $b \in [\underline{b}, \overline{b}]$ the inverse bidding strategies $\phi_1(b) = \beta_1^{-1}(b), \phi_2(b) = \beta_2^{-1}(b)$ satisfy $\phi_i(b) \neq b$ and are solutions to the system of differential equations*

$$\phi_j'(b) = \frac{((1-\alpha_i)(1-k)+\alpha_i k)\, F_j(\phi_j(b)) - \alpha_i k}{\left(\phi_i(b) - b\right) f_j(\phi_j(b))} \qquad \textit{for } i,j = 1,2 \textit{ and } i \neq j \tag{5}$$

with boundary conditions

$$\phi_1(\underline{b}) = \phi_2(\underline{b}) = 0 \qquad \textit{and} \qquad \phi_1(\overline{b}) = \phi_2(\overline{b}) = 1. \tag{6}$$

Proof. Suppose the pair of continuous and strictly increasing strategies β_1, β_2 is a Bayesian Nash equilibrium. Since β_j is continuous and strictly increasing, and hence differentiable almost everywhere, the expected payoff $E\left[u_i(v,b,\beta_j(V_j))\right]$ is continuous and almost everywhere differentiable in b. Therefore, the first order condition

$$\tfrac{\partial E[u_i(v,b,\beta_j(V_j))]}{\partial b} = (v-b) f_j(\phi_j(b))\phi_j'(b) - ((1-\alpha_i)(1-k)+\alpha_i k)\, F_j(\phi_j(b)) + \alpha_i k = 0$$

of the maximization problem in (4) has to be satisfied at $b = \beta_i(v)$ for almost all v. Using $v = \phi_i(b)$ and rearranging yields (5). Note that $\frac{\partial E[u_i(v,b,\beta_j(V_j))]}{\partial b}\Big|_{b=v} \neq 0$ for almost all v. It follows that $b = v$ can be optimal only for a finite number of i's types, i.e., $\phi_i(b) \neq b$ for almost all b. The boundary conditions (6) are necessary by Lemma 1.

To show sufficiency, suppose continuous and strictly increasing ϕ_1, ϕ_2 are solutions to (5), (6) and satisfy $\phi_i(b) \neq b$ for almost all b. Hence, $\frac{\partial E[u_i(v,b,\beta_j(V_j))]}{\partial b}\Big|_{b=\beta_i(v)} = 0$ for almost all types v. From $\frac{\partial^2 E[u_i(v,b,\beta_j(V_j))]}{\partial v \partial b} = f_j(\phi_j(b))\phi_j'(b) > 0$ and $\beta_i'(v) > 0$ almost everywhere follows that $\frac{\partial E[u_i(v,b,\beta_j(V_j))]}{\partial b} > (<) 0$ for almost all $b < (>) \beta_i(v)$. Consequently, $E\left[u_i(v,b,\beta_j(V_j))\right]$ is globally maximized at $b = \beta_i(v)$ and β_1, β_2 form a Bayesian Nash equilibrium. $\square$

For the equilibrium strategies defined in Lemma 3 there is no closed-form solution. Yet, further properties of the equilibrium strategies can be obtained from studying the system of differential equations (5), as presented in the next proposition. For this, and also for later results, the parameters Λ_1, Λ_2 defined as

$$\Lambda_j := F_j^{-1}\left(\frac{\alpha_i k}{(1-\alpha_i)(1-k)+\alpha_i k}\right) \quad \text{for } i,j = 1,2 \text{ and } i \neq j \tag{7}$$

will play an important role. Note that assumption (2) implies $\Lambda_1 < \Lambda_2$ for $k \in (0,1)$, as will be shown below.

Proposition 2. *For every Bayesian Nash equilibrium in continuous and strictly increasing pure strategies, there exist* $\hat{v}_1, \hat{v}_2 \in [\underline{b}, \overline{b}]$ *such that* $\phi_i(\hat{v}_i) = \hat{v}_i$ *and* $\phi_i(\hat{v}_j) = \Lambda_i$ *for* $i, j = 1,2$ *and* $i \neq j$ *with*

$$\begin{aligned} 0 = \Lambda_1 = \hat{v}_2 = \hat{v}_1 = \Lambda_2 \quad & \textit{if } k = 0, \\ 0 < \Lambda_1 \leq \hat{v}_2 < \hat{v}_1 \leq \Lambda_2 < 1 \quad & \textit{if } k \in (0,1), \\ \Lambda_1 = \hat{v}_2 = \hat{v}_1 = \Lambda_2 = 1 \quad & \textit{if } k = 1. \end{aligned}$$

Moreover,

$$\begin{aligned} \phi_1(b) < \phi_2(b) < b \quad & \textit{for } b \in (\underline{b}, \hat{v}_2), \\ \phi_1(b) < b < \phi_2(b) \quad & \textit{for } b \in (\hat{v}_2, \hat{v}_1), \\ b < \phi_1(b) < \phi_2(b) \quad & \textit{for } b \in (\hat{v}_1, \overline{b}). \end{aligned}$$

Proof. Recall from Lemma 1 that $\phi_i(\underline{b}) = 0$ and $\phi_i(\overline{b}) = 1$ with $0 \leq \underline{b} < \overline{b} \leq 1$. Since $\phi_i(b)$ is continuous, there must be at least one fixed point x such that $\phi_i(x) = x$. According to Lemma 3 the inverse bidding strategies satisfy

$$\phi_j'(b) = \frac{((1-\alpha_i)(1-k) + \alpha_i k)\, F_j(\phi_j(b)) - \alpha_i k}{\left(\phi_i(b) - b\right) f_j(\phi_j(b))}. \tag{8}$$

Note that the numerator of the RHS of (8) is zero at exactly one point $\hat{b}$. For $b < \hat{b}$ the numerator is negative and for $b > \hat{b}$ it is positive. Since $\phi_j(b)$ is increasing, the denominator must have the same sign as the numerator wherever the RHS of (8) is defined. Therefore, $\phi_i(b) < b$ for all $b < \hat{b}$ and $\phi_i(b) > b$ for all $b > \hat{b}$. Continuity of $\phi_i(b)$ then implies $\phi_i(\hat{b}) = \hat{b}$. Moreover, from the numerator of the RHS of (8) being zero at $b = \hat{b}$ follows $\phi_j(\hat{b}) = \Lambda_j$. Let $\hat{v}_i = \hat{b}$ and the first statements in the proposition are established.

Suppose $k \in (0,1)$. We will show that in this case $\Lambda_1 < \Lambda_2$ and $\hat{v}_1 > \hat{v}_2$. Assumption (2) implies

$$\frac{\alpha_1}{\alpha_2}\frac{F_1(v)}{F_2(v)} > \frac{f_1(v)}{f_2(v)} > \frac{\alpha_2}{\alpha_1}\frac{1-F_1(v)}{1-F_2(v)} \quad \Rightarrow \quad \alpha_1^2 F_1(v)(1-F_2(v)) > \alpha_2^2(1-F_1(v))F_2(v).$$

Evaluating the last inequality at $v = \Lambda_2$ and simplifying we obtain

$$\alpha_1(1-k)F_1(\Lambda_2) > \alpha_2 k(1-F_1(\Lambda_2)) \quad \Leftrightarrow \quad \Lambda_2 > F_1^{-1}\left(\tfrac{\alpha_2 k}{\alpha_1(1-k)+\alpha_2 k}\right) = \Lambda_1.$$

Having established $\Lambda_1 < \Lambda_2$, $\hat{v}_1 = \hat{v}_2$ is impossible as this would require $\Lambda_1 = \Lambda_2$. Next, consider $\hat{v}_1 < \hat{v}_2$. From the preceding paragraph we must then have $\Lambda_1 > \phi_1(b) > b > \phi_2(b) > \Lambda_2$ for all $b \in (\hat{v}_1, \hat{v}_2)$. This clearly contradicts $\Lambda_1 < \Lambda_2$. Consequently, we must have $\Lambda_1 \leq \hat{v}_2 < \hat{v}_1 \leq \Lambda_2$ with $\phi_1(b) < b < \phi_2(b)$ for all $b \in (\hat{v}_2, \hat{v}_1)$.

Now, suppose $k \in \{0,1\}$. For $k = 0$ we have $\Lambda_1 = \Lambda_2 = 0$ such that $\hat{v}_1 = \hat{v}_2 = 0$. Similarly, for $k = 1$, $\Lambda_1 = \Lambda_2 = 1$ implies $\hat{v}_1 = \hat{v}_2 = 1$.

We are left to show that $\phi_1(b) < \phi_2(b)$ also for $b \in (\underline{b}, \hat{v}_2) \cup (\hat{v}_1, \overline{b})$. First, suppose $b \in (\underline{b}, \hat{v}_2)$ and $k \in (0,1]$. Here, we have $\phi_1(b), \phi_2(b) < b$. Suppose there exists an $x \in (\underline{b}, \hat{v}_2)$ such that $\phi_1(x) = \phi_2(x) =: r < x$. From (8) we have $\phi_1'(x) < \phi_2'(x)$ because

$$\frac{((1-\alpha_2)(1-k)+\alpha_2 k)F_1(r) - \alpha_2 k}{(r-x)f_1(r)} < \frac{((1-\alpha_1)(1-k)+\alpha_1 k)F_2(r) - \alpha_1 k}{(r-x)f_2(r)}$$

is equivalent to

$$(1-k)\alpha_1 \frac{F_1(r)}{f_1(r)} + k\alpha_1 \frac{1-F_2(r)}{f_2(r)} > (1-k)\alpha_2 \frac{F_2(r)}{f_2(r)} + k\alpha_2 \frac{1-F_1(r)}{f_1(r)}$$

which, in turn, follows from assumption (2). This implies that there can be at most one such $x \in (\underline{b}, \hat{v}_2)$, i.e., the two inverse bidding strategies intersect at most once. If they do intersect at x, then we must have $\phi_1(x) > \phi_2(x)$ for all $b \in (\underline{b}, x)$. However, (8) also implies $\phi_1'(\underline{b}) < \phi_2'(\underline{b})$ such that $\phi_1(b) < \phi_2(b)$ for b close to $\underline{b}$. Hence, the strategies cannot intersect and we have $\phi_1(b) < \phi_2(b)$ for all $b \in (\underline{b}, \hat{v}_2)$. Finally, to prove that $\phi_1(b) < \phi_2(b)$ also for $b \in (\hat{v}_1, \overline{b})$ and $k \in [0,1)$ one can use a similar line of arguments or apply Corollary 2. $\square$

The properties of the inverse bidding strategies ϕ_1, ϕ_2 identified in Proposition 2 directly translate, of course, to the bidding strategies β_1, β_2. Figure 1 depicts bidding strategies with the corresponding qualitative features. There are $\hat{v}_1, \hat{v}_2$ such that $\beta_i(v) > v$ for all $v < \hat{v}_i$ and $\beta_i(v) < v$ for all $v > \hat{v}_i$, i.e., bidders with relatively low valuations overbid whereas bidders with relatively high valuations shade their bids. If a bidder has a low valuation, he knows that he will lose the auction with a high probability. Therefore he bids more than his valuation in order to increase the expected auction revenue, even accepting a negative payoff should he win the auction

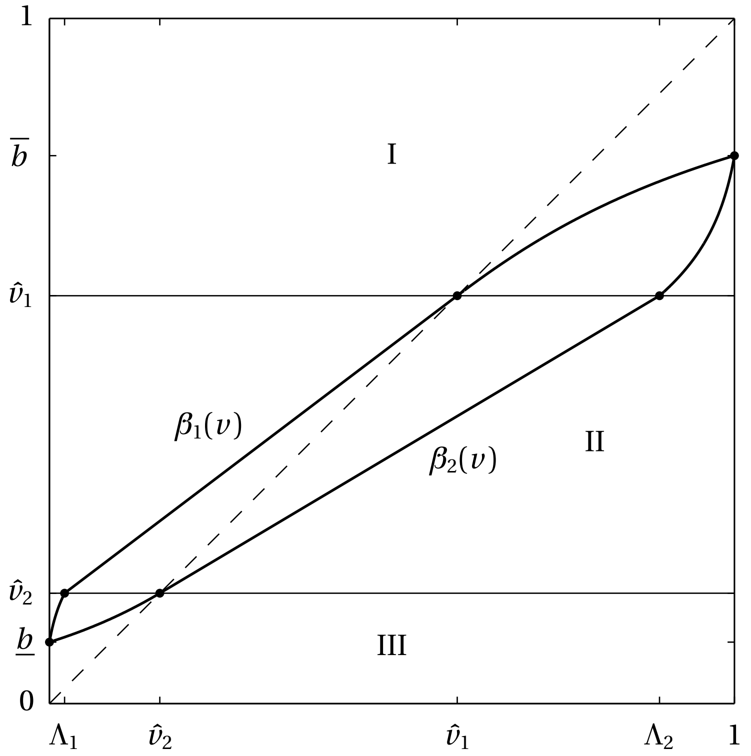

Figure 1: Equilibrium Bidding

against his expectations. Conversely, a bidder with a high valuation is likely to win the auction and aims at reducing the amount he has to pay by shading his bid.

Because of the asymmetry in shares and values, the two bidders face different trade-offs. As shown in Proposition 2, assumption (2) implies that bidder 1 bids more aggressively than bidder 2, i.e., $\beta_1(v) > \beta_2(v)$ for all $v \in (0,1)$. Asymmetry in shares means that there is a majority owner and a minority owner. Compared with the minority owner, the majority owner's payoff if he wins is scaled down while his payoff if he loses is scaled up. Since the majority owner already owns a large share of the object, there is more at stake for him when he loses the auction. Hence, his trade-off is shifted towards that case, resulting in a stronger incentive for overbidding and a weaker incentive for bid shading than for the minority owner. The effect of asymmetry in valuations on the $k+1$-price auction is similar to that on standard first-price auctions.[10] If there is a weak and a strong bidder, i.e., one bidder's distribution of valuations is dominated by the other bidder's in terms of the (reverse) hazard rate,

[10]See, e.g., Krishna (2002, Chapter 4).

the weak bidder bids more aggressively than the strong bidder. Consequently, both owning the larger share and being the weak bidder leads to higher equilibrium bids. Assumption (2) ensures that at least one of the two effects is sufficiently pronounced for bidder 1, inducing him to bid more aggressively than bidder 2.

In the special case where $k = 0$ we have $\hat{v}_1 = \hat{v}_2 = \underline{b} = 0$: all types shade their bids since one's bid has no direct effect on the payoff when losing. On the other hand, if $k = 1$, both bidders always bid more than their valuation, i.e., $\hat{v}_1 = \hat{v}_2 = \overline{b} = 1$. For $k \in (0,1)$ we know that $0 < \underline{b} < \hat{v}_2 < \hat{v}_1 < \overline{b} < 1$: for each bidder there are non-degenerate intervals both of types who overbid and of types who shade their bids.

These observations suggests that we may divide equilibrium bidding into three parts, which are labeled I, II, and III in Figure 1. For a given equilibrium, part I contains bids above $\hat{v}_1$, submitted by high-valuation types of the two bidders. In part I both bidders are shading their bids, as in a standard first-price auction between two buyers. Bids below $\hat{v}_2$ are contained in part III. Here, both bidders bid more than their valuation, similar to two sellers competing in a first-price procurement auction. The intermediate bids in part II, i.e., bids in $(\hat{v}_2, \hat{v}_1)$, come from overbidding types of bidder 1 and from bid shading types of bidder 2. Here, the situation resembles that in a double auction between a seller and a buyer.

As we show in the next section, the above intuition quite accurately describes the bidders' trade-offs: The equilibrium strategies in part II and part I/III are indeed equivalent to the equilibrium strategies in a specific double and first-price auction, respectively.

4 Existence, Multiplicity, and Uniqueness

In order to prove existence of an equilibrium, we will show that there exist ϕ_1, ϕ_2 that solve the differential equations (5) with boundary conditions (6) identified in Lemma 3. Proposition 2 implies that for every equilibrium there are unique $\hat{v}_1, \hat{v}_2$ such that at $b = \hat{v}_i$ the RHS of (5) has a $\frac{0}{0}$ form, leaving $\phi_j'(\hat{v}_i)$ undefined. Hence, the system of differential equations (5) presents a singularity in one of the two equations at both $\hat{v}_1$ and $\hat{v}_2$. Because of the two singularities, we cannot directly apply the standard theory of ordinary differential equations.

Instead we consider the differential equations (5) separately in each of the three parts of equilibrium bidding identified in the preceding section. Note that $\hat{v}_1$ and $\hat{v}_2$ lie exactly at the boundaries of the three parts. It turns out that, with the singularities occurring at the boundaries, the system (5) has been studied in different contexts in the literature. We find that the system of equations defining equilibrium strategies in part II (parts I and III) is equivalent to the system of equations one obtains when analyzing a specific double (first-price) auction. This allows us to apply existing results to the three parts. Combining those results ultimately yields existence of a continuum of equilibria if $k \in (0,1)$ and existence of a unique equilibrium if $k \in \{0,1\}$.

4.1 Part II: A Double Auction

Let us first look at the intermediate range of bids in part II, i.e., $b \in [\hat{v}_2, \hat{v}_1]$. Of course, this part is only relevant if $\hat{v}_2 < \hat{v}_1$, so we assume $k \in (0,1)$ in the following. Define

$$H_1(x) := \left(\frac{F_1(x) - F_1(\Lambda_1)}{F_1(\Lambda_2) - F_1(\Lambda_1)}\right)^{\frac{1-k}{(1-\alpha_2)(1-k)+\alpha_2 k}} \text{ and } H_2(x) := 1 - \left(\frac{F_2(\Lambda_2) - F_2(x)}{F_2(\Lambda_2) - F_2(\Lambda_1)}\right)^{\frac{k}{(1-\alpha_1)(1-k)+\alpha_1 k}}.$$

Note that both H_1 and H_2 are continuous cumulative distribution functions with support $[\Lambda_1, \Lambda_2]$. Moreover, assumption (1) implies that virtual valuations

$$C_1(x) := x + (1-k)\frac{H_1(x)}{h_1(x)} \quad \text{and} \quad C_2(x) := x - k\frac{1 - H_2(x)}{h_2(x)}$$

are strictly increasing.[11] Using H_1, H_2 we can rewrite the system of differential equations (5) for $b \in (\hat{v}_2, \hat{v}_1)$ to

$$\phi_1'(b) = (1-k)\frac{H_1(\phi_1(b))}{h_1(\phi_1(b))}\,\frac{1}{\phi_2(b) - b} \tag{9}$$

$$\phi_2'(b) = k\frac{1 - H_2(\phi_2(b))}{h_2(\phi_2(b))}\,\frac{1}{b - \phi_1(b)}. \tag{10}$$

From Proposition 2 we add the boundary conditions

$$\phi_1(\hat{v}_2) = \Lambda_1, \quad \phi_2(\hat{v}_2) = \hat{v}_2, \tag{11}$$

$$\phi_1(\hat{v}_1) = \hat{v}_1, \quad \phi_2(\hat{v}_1) = \Lambda_2. \tag{12}$$

As it turns out, equations (9)-(12) are equivalent to the equations that characterize equilibrium strategies for the double auction of Chatterjee and Samuelson (1983).

More precisely, (9)-(12) define a Bayesian Nash equilibrium in continuous and

[11] To see this, note that $C_i(x) = \alpha_i\left(x + (1-k)\frac{F_i(x)}{f_i(x)}\right) + (1-\alpha_i)\left(x - k\frac{1-F_i(x)}{f_i(x)}\right)$ for $i = 1, 2$.

strictly increasing strategies of the $1-k$-double auction[12] where player 1 is the seller with valuation drawn from H_1 and player 2 is the buyer with valuation drawn from H_2. In such a double auction the seller and the buyer simultaneously submit a bid. If the buyer's bid exceeds the seller's, the seller's good is transferred to the buyer and the buyer pays $1-k$ times his bid plus k times the seller's bid. For each equilibrium in continuous and strictly increasing strategies there are $\hat{v}_1, \hat{v}_2$ such that the lowest type of the seller bids $\hat{v}_2$ and the highest type of the buyer bids $\hat{v}_1$. Seller types $v_1 \in [\hat{v}_1, \Lambda_2]$ and buyer types $v_2 \in [\Lambda_1, \hat{v}_2]$ trade with probability zero, bidding above $\hat{v}_1$ and below $\hat{v}_2$, respectively. Seller types $v_1 \in [\Lambda_1, \hat{v}_1]$ and buyer types $v_2 \in [\hat{v}_2, \Lambda_2]$ bid according to inverse strategies ϕ_1 and ϕ_2 that solve (9)-(12).

This type of equilibria of the double auction is very thoroughly studied in Satterthwaite and Williams (1989). In the following proposition we collect the findings that are most relevant for our purpose.

Proposition 3 (Satterthwaite and Williams, 1989)**.** *Let $k \in (0,1)$ and consider a $1-k$-double auction where the seller's valuation is drawn from H_1 and the buyer's from H_2. Continuous and strictly increasing inverse strategies ϕ_1, ϕ_2 form a Bayesian Nash equilibrium of this double auction if and only if they solve* (9)-(12)*. There is a two-parameter family of such equilibria. Moreover, for every equilibrium in this family, $\hat{v}_2 \in [\Lambda_1, C_2^{-1}(\Lambda_1)]$ and $\hat{v}_1 \in [C_1^{-1}(\Lambda_2), \Lambda_2]$.*

Proof. First note that in Satterthwaite and Williams (1989) the support of players' types is normalized to $[0,1]$. By applying the affine transformation $T: [\Lambda_1, \Lambda_2] \to [0,1]$ with $T(x) = \frac{x-\Lambda_1}{\Lambda_2-\Lambda_1}$ to valuations and bids in (9)-(12) we may obtain exactly the equations studied by Satterthwaite and Williams (1989). Their results therefore directly extend to types with support $[\Lambda_1, \Lambda_2]$. Moreover, notice that also the property of increasing virtual valuations C_1, C_2 is preserved under transformation T.

The fact that (9)-(12) are necessary and sufficient for continuous and strictly increasing ϕ_1, ϕ_2 to form a Bayesian Nash equilibrium follows from Satterthwaite and Williams (1989, Theorem 3.1). Note that Satterthwaite and Williams assume $x + \frac{H_1(x)}{h_1(x)}$ and $x - \frac{1-H_2(x)}{h_2(x)}$ to be strictly increasing whereas our assumption (1) only implies the weaker property of strictly increasing C_1, C_2. However, as is easily verified, this weaker property is enough for Satterthwaite and Williams' proof of Theorem 3.1 to hold.

[12] Note that because of the different definition of the parameter k in the double auction literature the $k+1$-price auction corresponds to a $1-k$-double auction rather than a k-double auction. In a k-double auction the buyer's bid, i.e., the higher of the two bids in case of transaction, has weight k. In contrast, in the $k+1$-price auction the lower bid has weight k.

According to Satterthwaite and Williams (1989, Theorem 3.2) Bayesian Nash equilibria in continuous and strictly increasing strategies form a two-parameter family.

Finally, from Satterthwaite and Williams (1989, Section 4) follows that at the two boundaries $\hat{v}_1, \hat{v}_2$ we must have $\phi_1'(\hat{v}_1) \geq 1$ and $\phi_2'(\hat{v}_2) \geq 1$. Consequently, evaluating (9) at $\hat{v}_1$ using (11) we obtain $\Lambda_2 \leq C_1(\hat{v}_1)$. Similarly, evaluating (10) at $\hat{v}_2$ using (12) yields $\Lambda_1 \geq C_2(\hat{v}_2)$. As C_1 and C_2 are strictly increasing and $\Lambda_1 \leq \hat{v}_2 < \hat{v}_1 \leq \Lambda_2$, we have $\hat{v}_2 \in [\Lambda_1, C_2^{-1}(\Lambda_1)]$ and $\hat{v}_1 \in [C_1^{-1}(\Lambda_2), \Lambda_2]$. $\square$

According to Proposition 3 there exists a continuum of equilibrium strategies that solve (9)-(12). Geometrically, the differential equations (9),(10) define a vector field inside the tetrahedron $\Lambda_1 \leq \phi_1 \leq b \leq \phi_2 \leq \Lambda_2$. Satterthwaite and Williams (1989) show that for any point (ϕ_1, ϕ_2, b) within the tetrahedron there is a solution curve of the differential equations passing through it. The two points where this curve exits the tetrahedron yield $\hat{v}_1$ and $\hat{v}_2$ of that specific equilibrium. Having fixed a bid $\tilde{b} \in (\Lambda_1, \Lambda_2)$ we may select a particular equilibrium by choosing the values of $\phi_1(\tilde{b}) \in (\Lambda_1, \tilde{b})$ and $\phi_2(\tilde{b}) \in (\tilde{b}, \Lambda_2)$. Hence, the equilibria constitute a two-parameter family.

From Proposition 3 we know that there are many pairs of strategies that are consistent with equilibrium bidding in part II. Yet are there also equilibrium strategies for parts I and III that fit to those strategies? The answer is yes. More precisely, as we show below, for each of the possible solutions in part II there is a unique corresponding solution in parts I and III.

4.2 Parts I and III: A First-price Auction

Let us now turn to part I of equilibrium bidding where bids are above $\hat{v}_1$ and both players are shading their bids. Assume $k \in [0,1)$ and suppose the value of $\hat{v}_1$ is given, satisfying $C_1^{-1}(\Lambda_2) \leq \hat{v}_1 \leq \Lambda_2$ (as is true for any solution in part II). Define

$$G_j(x) := \left(\frac{F_j(x) - F_j(\Lambda_j)}{1 - F_j(\Lambda_j)} \right)^{\frac{1}{(1-\alpha_i)(1-k)+\alpha_i k}} \qquad \text{for } j, i = 1,2 \text{ with } i \neq j.$$

For each j, G_j is a continuous cumulative distribution function with support $[\Lambda_j, 1]$. Using G_1, G_2, we may rewrite the system of differential equations (5) for $b \in (\hat{v}_1, \overline{b}]$ to

$$\phi_j'(b) = \frac{G_j(\phi_j(b))}{g_j(\phi_j(b))} \frac{1}{\phi_i(b) - b} \qquad \text{for } i, j = 1,2 \text{ and } i \neq j. \tag{13}$$

From Lemma 1 and Proposition 2 we add the boundary conditions

$$\phi_1(\overline{b}) = \phi_2(\overline{b}) = 1 \tag{14}$$

$$\phi_1(\hat{v}_1) = \hat{v}_1, \tag{15}$$

$$\phi_2(\hat{v}_1) = \Lambda_2. \tag{16}$$

We will argue below that equations (13)-(16) correspond exactly to the system of equations that characterizes the equilibrium of a first-price auction with a reserve price equal to $\hat{v}_1$ among two buyers with valuations drawn from G_1 and G_2. Note that, unless $k = 0$, the bidders' valuations have different supports, with the lowest value of bidder 1 lying below and the lowest value of bidder 2 lying above the reserve price $\hat{v}_1$. In equilibrium, type $\hat{v}_1$ of bidder 1 and type Λ_2 of bidder 2 both bid $\hat{v}_1$ whereas lower types of bidder 1 do not take part in the auction (or bid below the reserve price). Such equilibrium strategies have been studied, e.g., by Maskin and Riley (2000) and Lebrun (2006). Most importantly, Lebrun (2006) shows that there is a unique equilibrium. If $k = 0$, the reserve price $\hat{v}_1 = 0$ is not binding. In this case, a proof of existence and uniqueness of equilibrium can be found in Lebrun (1999).

Proposition 4 (Lebrun, 1999, 2006)**.** *Let $k \in [0,1)$. The first-price auction with reserve price $\hat{v}_1 \in [C_1^{-1}(\Lambda_2), \Lambda_2]$ and valuations drawn from G_1, G_2 has a unique Bayesian Nash equilibrium, which is in monotone pure strategies. For $k = 0$, the equilibrium strategies are strictly increasing and correspond to the unique solution to* (13)-(16)*. For $k \in (0,1)$, the equilibrium strategies are strictly increasing for bidder i's valuations in $(\phi_i(\hat{v}_1), 1]$, correspond to the unique solution to* (13)-(15)*, and satisfy $\phi_2(\hat{v}_1) \geq \Lambda_2$.*

Proof. First, suppose $k = 0$. Accordingly, both G_1 and G_2 have support $[0,1]$ and $\hat{v}_1 = 0$. This case of a first-price auction with atomless type distributions and without a binding reserve price is covered in Lebrun (1999). Observe that $G_j(x) = F_j(x)^{\frac{1}{1-\alpha_i}}$ and assumption (2) imply the stochastic dominance relation $\frac{d}{dx}\frac{G_1(x)}{G_2(x)} < 0$ for all $x \in (0,1]$. From Lebrun (1999, Corollary 4) hence follows that there is a unique Bayesian Nash equilibrium. Moreover, according to Lebrun (1999, Theorem 1), ϕ_1 and ϕ_2 form a Bayesian Nash equilibrium if and only if they correspond to strictly increasing pure strategies and solve (13)-(16).

Now, let $k \in (0,1)$. In this case, $0 < \Lambda_1 < \hat{v}_1 \leq \Lambda_2$, i.e., the lower bound of the support of valuations of bidder 1 is strictly below that of bidder 2. Existence of a unique Bayesian Nash equilibrium of the first-price auction with a reserve price $\hat{v}_1$ and valuations drawn from G_1, G_2 is directly implied by Lebrun (2006, Theorem 1).

Moreover, according to Lebrun (2006, Section 5.1), equilibrium bidding strategies have the following properties. Bidder 2 with valuation Λ_2 bids the amount $\underline{v}$. Equilibrium strategies are strictly increasing whenever bids are strictly larger than $\underline{v}$. There is a $\overline{b}$ such that for $b \in (\underline{v}, \overline{b}]$ the inverse bidding strategies ϕ_1, ϕ_2 solve the system of differential equations (13) with the upper boundary condition (14) and the lower boundary condition $\phi_1(\underline{v}) = \underline{v}$. For bidder 2 whose valuation has support $[\Lambda_2, 1]$, we must have $\phi_2(\underline{v}) \geq \Lambda_2$ at the lower boundary. For valuations $v_2 \in [\Lambda_2, \phi_2(\underline{v})]$ bidder 2 bids $\underline{v}$. Bidder 1 bids strictly less than $\underline{v}$ for valuations $v_1 < \underline{v}$. According to Lebrun (2006, Definition 1), $\underline{v}$ is defined as

$$\underline{v} := \max \arg \max_{b \in [\hat{v}_1, \Lambda_2]} (\Lambda_2 - b) G_1(b).$$

Observe that because $\hat{v}_1 \in [C_1^{-1}(\Lambda_2), \Lambda_2]$ the first derivative of the objective function satisfies

$$\left(\Lambda_2 - b - \frac{G_1(b)}{g_1(b)} \right) g_1(b) < 0 \quad \text{for all } b \in (\hat{v}_1, \Lambda_2].$$

To see this, note that $b + \frac{G_1(b)}{g_1(b)} = C_1(b)$ where $C_1(b)$ is strictly increasing because of assumption (1). Therefore, we obtain $\underline{v} = \hat{v}_1$ and the lower boundary condition $\phi_1(\underline{v}) = \underline{v}$ becomes (15). Moreover, we have $\phi_2(\hat{v}_1) \geq \Lambda_2$.

Finally, note that Lebrun (2006) proves his uniqueness result by showing that there is a unique solution to the differential equations with boundary conditions that characterize the equilibrium, i.e., (13)-(15). □

In his general analysis of asymmetric first-price auctions, Lebrun (2006) does not rule out the possibility that there is a non-degenerate interval $[\Lambda_2, \phi_2(\hat{v}_1)]$ of types of bidder 2 that in equilibrium all submit a bid equal to the reserve price $\hat{v}_1$. That is why, for $k \in (0,1)$, Proposition 4 only states that there is a unique solution to (13)-(15) but not that this solution also satisfies (16). In the Appendix we prove the following lemma, showing that in our setting also (16) must hold for the solution to (13)-(15). We do so by studying the geometry of the solution in part I in a similar way as Satterthwaite and Williams (1989) do for the double auction in part II.

Lemma 4. *Suppose* $k \in (0,1)$. *For all* $\hat{v}_1 \in [C_1^{-1}(\Lambda_2), \Lambda_2]$, *the unique solution to* (13)-(15) *also satisfies* (16).

Proof. See Appendix A.3. □

Given equilibrium strategies for the intermediate range of bids in part II, Proposition 4 and Lemma 4 show that there are unique strictly increasing equilibrium strategies for the bids above $\hat{v}_1$ contained in part I. This result also has direct implications for the equilibrium strategies in part III where bids are below $\hat{v}_2$. Parts I and III are closely related: Using Corollary 2 we may obtain strategies for part III from strategies for part I by a simple transformation. Recall that if β_1, β_2 form an equilibrium of the $k+1$-price auction, then the strategies $\tilde{\beta}_i(v) := 1-\beta_j(1-v)$ form an equilibrium of the $2-k$-price auction with values drawn from $\tilde{F}_i(v) = 1-F_j(1-v)$. Moreover, $\tilde{\Lambda}_i = 1-\Lambda_j$ and $\tilde{\hat{v}}_i = 1-\hat{v}_j$ such that part III strategies of the $k+1$-price auction are transformed part I strategies of the $2-k$-price auction. Accordingly, Proposition 4 and Lemma 4 easily carry over to part III. Indeed, in part III bidders behave like the sellers in a first-price procurement auction. In such auctions equilibrium strategies can typically be obtained by transforming equilibrium strategies of the standard first-price auction.

4.3 Main Results and Discussion

Combining the results for the three parts allows us to obtain the main results of this section. A Bayesian Nash equilibrium in continuous and strictly increasing pure strategies generally exists. The number of equilibria, however, depends on the parameter k. Let us first consider $k \in (0,1)$ where equilibria consist of all three parts.

Theorem 1. *For the $k+1$-price auction with $k \in (0,1)$, there is a two-parameter family of Bayesian Nash equilibria in continuous and strictly increasing pure strategies.*

Proof. From Proposition 3 there is a two-parameter family of solutions for the equilibrium strategies in part II. Each solution corresponds to a pair of continuous and strictly increasing inverse bidding strategies $\phi_1(b), \phi_2(b)$ for $b \in (\hat{v}_2, \hat{v}_1)$ with $\hat{v}_2 \in [\Lambda_1, C_2^{-1}(\Lambda_1)]$ and $\hat{v}_1 \in [C_1^{-1}(\Lambda_2), \Lambda_2]$. Moreover, $\lim_{b\downarrow\hat{v}_2} \phi_1(b) = \Lambda_1$, $\lim_{b\downarrow\hat{v}_2} \phi_2(b) = \hat{v}_2$, $\lim_{b\uparrow\hat{v}_1} \phi_1(b) = \hat{v}_1$, and $\lim_{b\uparrow\hat{v}_1} \phi_2(b) = \Lambda_2$. For each $\hat{v}_1 \in [C_1^{-1}(\Lambda_2), \Lambda_2]$, Proposition 4 and Lemma 4 imply that there is a unique solution for part I, corresponding to continuous and strictly increasing $\phi_1(b), \phi_2(b)$ for $b \in (\hat{v}_1, \overline{b}]$ for some $\overline{b}$. Moreover, $\lim_{b\downarrow\hat{v}_1} \phi_1(b) = \hat{v}_1$ and $\lim_{b\downarrow\hat{v}_1} \phi_2(b) = \Lambda_2$. By Corollary 2, this result also applies to part III, implying existence of unique continuous and strictly increasing $\phi_1(b), \phi_2(b)$ for $b \in [\underline{b}, \hat{v}_2)$, with $\lim_{b\uparrow\hat{v}_2} \phi_1(b) = \Lambda_1$ and $\lim_{b\uparrow\hat{v}_2} \phi_2(b) = \hat{v}_2$. Connecting the solutions for the three parts, we obtain a two-parameter family of continuous and strictly increasing $\phi_1(b), \phi_2(b)$ that solve (5) and (6) and satisfy $\phi_i(b) \neq b$ for almost all for $b \in [\underline{b}, \overline{b}]$ and hence, by Lemma 3, constitute a Bayesian Nash equilibrium. $\square$

Now, suppose $k \in \{0,1\}$. According to Lemma 2, any pure-strategy Bayesian Nash equilibrium must be in continuous and strictly increasing strategies. If $k = 0$, equilibrium bidding only consists of part I. Similarly, if $k = 1$, there is only part III.

Theorem 2. *For the $k+1$-price auction with $k \in \{0,1\}$, there is a unique pure-strategy Bayesian Nash equilibrium.*

Proof. Let $k = 0$. According to Lemma 2 any pure-strategy Bayesian Nash equilibrium is in continuous and strictly increasing strategies. By Proposition 2, $\Lambda_1 = \hat{v}_2 = \hat{v}_1 = \Lambda_2 = 0$ such that equilibrium strategies are entirely contained in part I. Consequently, Proposition 4 implies that there is a unique solution to (5) and (6) which, using Lemma 3, yields existence of a unique pure-strategy Bayesian Nash equilibrium. Thanks to Corollary 2 the result carries over to $k = 1$. □

When equilibrium strategies cannot be expressed in closed form, additional insights can usually be obtain from numerically approximating the equilibrium, especially if equilibrium existence has been established. At first glance, the singularities of the system of differential equations (5) at $b = \hat{v}_1$ and $b = \hat{v}_2$ represent a major obstacle for numerical computations. The findings of this section, culminating in Theorems 1 and 2, suggest a procedure to overcome this problem. We may again consider the three parts of equilibrium bidding separately and numerically compute equilibrium strategies for each part. First, we fix some bid $b^* \in (\Lambda_1, \Lambda_2)$ and choose two starting values $\phi_1(b^*) \in (\Lambda_1, b^*)$ and $\phi_1(b^*) \in (b^*, \Lambda_2)$. From this we obtain equilibrium strategies for part II by extrapolating the differential equations (5) forward and backward, respectively, until we hit the boundaries of part II where $\phi_1(b) \to b$ and $\phi_2(b) \to b$, respectively. Thereby we obtain $\hat{v}_1$ and $\hat{v}_2$. Now, we can compute numerical solutions for parts I and III by applying methods developed for numerically solving first-price auctions. As described in the pioneering contribution by Marshall et al. (1994), for part I we may use a backward-shooting method (in order to find $\overline{b}$), as we cannot use $\hat{v}_1$ as a starting point because $\phi_2'(\hat{v}_1)$ is undefined.[13] A similar method, of course, works for part III.

Assuming $\alpha_1 = 0.8$, $k = 0.3$, and that valuations of both bidders are uniformly distributed, Figure 2 shows four different pairs of equilibrium strategies. We obtained them by varying the starting points in part II. The strategies drawn in Figure 1 also correspond to actual equilibrium strategies. Except for $\alpha_1 = 0.95$, the assumptions there are as for Figure 2.

[13]See Fibich and Gavish (2011) for an alternative to the backward-shooting method.

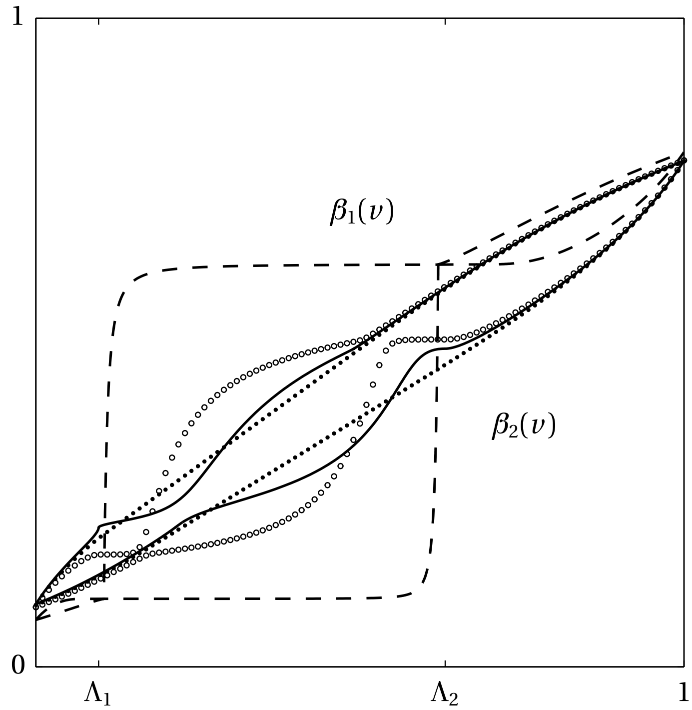

Figure 2: Examples of equilibrium strategies for $\alpha_1 = 0.8$, $k = 0.3$, and uniform $F_1 = F_2$.

Suppose assumption (2) does not hold and bidders are symmetric, i.e., $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $F_1 = F_2 = F$. In this case, part II of equilibrium bidding disappears and $\Lambda_1 = \hat{v}_2 = \hat{v}_1 = \Lambda_2 = F^{-1}(k)$. As first shown by Cramton, Gibbons, and Klemperer (1987), there is a symmetric closed-form solution for the equilibrium strategies defined in Lemma 3:

$$\beta_1(v) = \beta_2(v) = \beta(v) = v + \int_v^{F^{-1}(k)} \frac{(F(x)-k)^2}{(F(v)-k)^2} dx.$$

Kittsteiner (2003) proves that β corresponds to the unique Bayesian Nash equilibrium of the symmetric $k+1$-price auction. Note that for valuations $v \geq F^{-1}(k)$ where bids belong to part I, $\beta(v)$ is equivalent to the equilibrium strategy of a symmetric first-price auction where buyers' valuations are drawn from $G(v) = \left(\frac{F(v)-k}{1-k}\right)^2$ with support $[F^{-1}(k), 1]$. For valuations $v \leq F^{-1}(k)$, bids are in part III and there is a corresponding first-price procurement auction with symmetric sellers.

Theorem 1 shows that if $k \in (0,1)$, the uniqueness result for symmetric $k+1$-price auctions is not robust. Already a slight asymmetry among bidders that is consistent with assumption (2) results in a continuum of equilibria. In contrast, for $k \in \{0,1\}$

the equilibrium remains unique also for asymmetric $k+1$-price auctions. For asymmetric valuations, this is shown in de Frutos (2000). Theorem 2 generalizes de Frutos' result to a setting that also accommodates asymmetry in shares.

In the limit case of extremely asymmetric shares with $\alpha_1 = 1$, the $k+1$-price is equivalent to the $1-k$-double auction of Chatterjee and Samuelson (1983) where partner 1 is the seller and partner 2 the buyer. Here, only part II of equilibrium bidding is relevant. As stated in Proposition 3, Satterthwaite and Williams (1989) show that for $k \in (0,1)$ there is a continuum of equilibria in continuous and strictly increasing strategies. If $k = 0$ ($k = 1$), the seller (buyer) has a dominant strategy to bid his valuation and the equilibrium is unique, as proved by Williams (1987).

5 Dissolution Performance: The Uniform Case

In the following, we make a first step towards evaluating the suitability of $k+1$-price auctions for dissolving partnerships in the framework of Cramton, Gibbons, and Klemperer (1987). As in that framework, we assume $F_1 = F_2 = F$ so that partners differ only in the size of their shares. Accordingly, assumption (2) simplifies to $\alpha_1 > \alpha_2$. Making use of numerical simulations, we will devote special attention to the specific example where valuations are uniformly distributed, i.e., $F(v) = v$ for $v \in [0,1]$. We start by reviewing some results from the literature.

5.1 Mechanisms for Dissolving a Partnership

Following Cramton, Gibbons, and Klemperer (1987), let us consider a direct mechanism $\langle s,t \rangle$ where partners report their types and surrender their shares. Depending on the reported v_1, v_2, partner i obtains the object with probability $s_i(v_i, v_j) \in [0,1]$ as well as a transfer $t_i(v_i, v_j) \in \mathbb{R}$. The allocation rule s satisfies $s_1(v_1,v_2)+s_2(v_2,v_1)=1$. In addition, $\langle s,t \rangle$ fulfills ex post budget balance, i.e., $t_1(v_1,v_2)+t_2(v_2,v_1)=0$. Recall that partner i derives utility $\alpha_i v_i$ from an intact partnership, which he loses when surrendering his shares.[14] If players i and j both report truthfully, the interim ex-

[14]Hence, we assume i's outside option to be worth $\alpha_i v_i$ to him. An alternative interpretation of this is that, if no other mechanism can be applied, the partnership is dissolved through a lottery where partner i receives the object with probability α_i, i.e., $s_i(v_i,v_j)=\alpha_i$ and $t_i(v_i,v_j)=0$.

pected net payoff of player i is

$$U_i(v_i) := v_i \int_0^1 \big(s_i(v_i, v_j) - \alpha_i\big)\, dF(v_j) + \int_0^1 t_i(v_i, v_j)\, dF(v_j).$$

The mechanism $\langle s, t\rangle$ is incentive compatible if, for $i = 1, 2$ and $j \neq i$,

$$U_i(v_i) \geq v_i \int_0^1 \Big(s_i(u, v_j) - \alpha_i\Big)\, dF(v_j) + \int_0^1 t_i(u, v_j)\, dF(v_j) \quad \forall v_i, u \in [0,1]. \tag{17}$$

For $\langle s, t\rangle$ to be interim individually rational it has to satisfy

$$U_i(v_i) \geq 0 \quad \forall v_i \in [0,1], i = 1, 2. \tag{18}$$

For all mechanisms we will consider, the allocation rule s can be represented by an increasing function $\psi : [0,1] \to [0,1]$ such that

$$s_1(v_1, v_2) = \begin{cases} 1 & \text{for} \quad \psi(v_1) > v_2, \\ 0 & \text{for} \quad \psi(v_1) < v_2. \end{cases}$$

When evaluating a dissolution mechanism we are particularly interested in the ex ante expected gains from trade

$$W := \int_0^1 U_1(v_1)\, dF(v_1) + \int_0^1 U_2(v_2)\, dF(v_2) = \int_0^1 \int_0^{\psi(v_1)} (v_1 - v_2)\, dF(v_2)\, dF(v_1).$$

A dissolution mechanism is ex post efficient if and only if it ensures that the object is allocated to the partner with the highest valuation, i.e., $\psi(v_1) = v_1$. Cramton, Gibbons, and Klemperer (1987) show that an ex post efficient and budget-balanced dissolution mechanism satisfying constraints (17) and (18) exists if and only if ownership shares α_1, α_2 are not too asymmetric. For uniformly distributed valuations this is the case if and only if $\alpha_1 \leq \frac{1}{2} + \frac{1}{\sqrt{12}}$. While the incentive compatibility constraint (17) is the same for both partners (i.e., the effect of a change in u is independent of α_i), the individual rationality constraint (18) differs among partners. A mechanism that treats the two partners symmetrically dispossesses player 1 to some extent. Above a certain degree of asymmetry in shares a transfer from player 2 to player 1 is therefore needed to ensure individual rationality for player 1. Since the ex post efficient allocation rule treats both partners the same, this transfer (as well as incentive compatibility) must be achieved through the payment rule t alone. Yet this is possible only as long as α_1 is not too large. For larger asymmetries, we must depart from the efficient allocation rule in order for the mechanism to meet (17) and (18).

Chien (2007) characterizes the incentive efficient mechanism that maximizes the ex ante expected gains from trade W. When the asymmetry in ownership shares is large, the incentive efficient allocation rule departs from the ex post efficient allocation rule in a way that favors partner 1 and thus helps ensure individual rationality for partner 1. For uniformly distributed values, the incentive efficient allocation is[15]

$$\psi_{IE}(v_1) := \begin{cases} v_1 & \text{for} \quad v_1 \in [0, 1-\alpha_1] \cup [\alpha_1, 1], \\ v_1 + \bar{x} & \text{for} \quad v_1 \in (1-\alpha_1, \alpha_1 - \bar{x}), \\ \alpha_1 & \text{for} \quad v_1 \in [\alpha_1 - \bar{x}, \alpha_1), \end{cases} \tag{19}$$

$$\bar{x} = \arg\min_{x \geq 0} x \quad \text{s.t. } \tfrac{1}{3}x^3 - (\alpha_1 - \tfrac{1}{4})x^2 + (\alpha_1 - \tfrac{1}{2})x + \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_1^2 - \tfrac{1}{12} \geq 0.$$

With its rules depending on F, the incentive efficient mechanism is not detail free.

A simple and detail-free dissolution mechanism that is used in practice is the buy-sell clause:[16] One partner, denoted by p, proposes a price r for the object whereupon the other partner, denoted by c, chooses whether to sell his share for $\alpha_c r$ or buy the other's share at the price $\alpha_p r$. Player c, the chooser, will decide to sell as long as $r \geq v_c$ and will buy otherwise. Player p, the proposer, therefore solves $\max_r F(r)(v_p - \alpha_c r) + (1 - F(r))\alpha_p r$. Under assumption (1), the first order condition of this maximization problem uniquely defines the increasing equilibrium strategy $r(v_p)$. In the case of uniformly distributed valuations, we obtain $r(v_p) = \frac{1}{2}(v_p + \alpha_p)$. To some extent, bidding behavior is similar to that in the $k+1$-price auction: if the proposer has a relatively low valuation, he will overbid. If his valuation and therefore also the probability of ending up buying the other's share are high, he will shade his bid. The allocation rule induced by the buy-sell clause is

$$s_p(v_p, v_r) = \begin{cases} 1 & \text{for} \quad r(v_p) > v_c, \\ 0 & \text{for} \quad r(v_p) < v_c. \end{cases} \tag{20}$$

[15] Note that for $\alpha_1 \leq \frac{1}{2} + \frac{1}{\sqrt{12}}$ we have $\bar{x} = 0$ and therefore $\psi_{IE}(v_1) = v_1$ for all v_1.

[16] This mechanism is known under various names, including "cake-cutting mechanism", "Texas shootout", or "shotgun clause". For symmetric shares McAfee (1992) analyzes equilibrium strategies, whereas de Frutos and Kittsteiner (2008) show that there results an ex post efficient dissolution if the buy-sell clause is preceded by an ascending auction where partners bid for the right to choose.

For distributions F that satisfy $f(v) = f(1-v)$ for all v, it can be shown that the resulting gains from trade W are independent of whether $p=1$ or $p=2$.

5.2 The $k+1$-price Auction and Uniform Valuations

Also the $k+1$-price auction is a simple and detail-free dissolution mechanism. By the revelation principle, the $k+1$-price auction is equivalent to an incentive compatible direct mechanism with $U_i(v) = E[u_i(v,\beta_i(v),\beta_j(V))]$ and allocation rule $\psi(v) = \phi_2(\beta_1(v))$. Since Proposition 2 implies $\psi(v) > v\ \forall v \in (0,1)$ the allocation rule is not ex post efficient. By design, the $k+1$-price auction has an ex post balanced budget. Moreover, interim individual rationality is satisfied for every Bayesian Nash equilibrium of the $k+1$-price auction. Regardless of j's strategy, player i can, by placing a bid equal to v_i, guarantee himself the ex post payoff $u_i(v_i, v_i, b_j) \geq 0$, implying a non-negative interim expected payoff. If i chooses not to bid truthfully in equilibrium, it must be because he obtains a higher interim payoff by doing so.

Now, assume that valuations are uniformly distributed and let $k \in (0,1)$. For this special case, there is a closed-form solution for equilibrium bidding strategies in part II, which turn out to be linear.

Proposition 5. *Suppose F is uniform and $k \in (0,1)$. The $k+1$-price auction has a Bayesian Nash equilibrium in continuous and strictly increasing strategies where, for $i,j=1,2$ and $i \neq j$,*

$$\hat{v}_i = \frac{k+\Lambda_j}{2} \quad \textit{and} \quad \phi_i(b) = b + ((1-\alpha_i)(1-k)+\alpha_i k)(b-\hat{v}_i) \quad \forall b \in [\hat{v}_2, \hat{v}_1].$$

The k-double auction is known to have an equilibrium in linear strategies under specific distributional assumptions. In particular, for $k=\frac{1}{2}$ Leininger, Linhart, and Radner (1989) determine such linear strategies for a one-parameter family of distributions. When F is uniform, H_1 and H_2 are members of this family.[17]

If $k=0$, the above result, of course, does not apply as there is no part II. Yet also for this case an additional analytical result is available under the uniform assumption. There is a closed-form expression for the upper bound of bids $\overline{b}$, which is due to Marshall et al. (1994).[18] Using Corollary 2, we may also obtain an expression for $\underline{b}$ if

[17]For $\alpha_1 = 1$, the strategies provided in Proposition 5 are equivalent to the linear equilibrium strategies studied by Chatterjee and Samuelson (1983) in their Example 1.

[18]They study first-price auctions where valuations are drawn from two different power distribu-

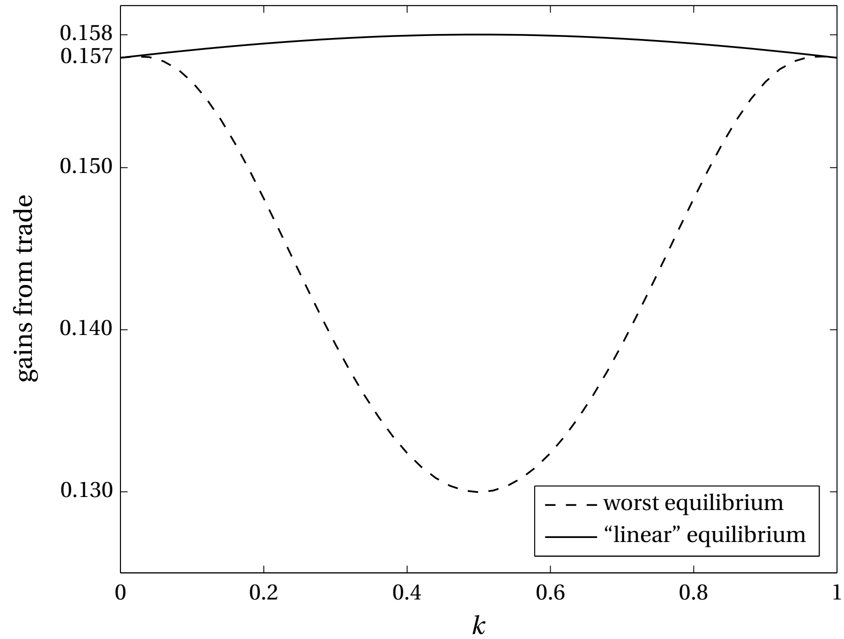

Figure 3: Gains from trade for $\alpha_1 = 0.8$, uniform F and varying k.

$k = 1$. Knowing the value of $\overline{b}$ or $\underline{b}$, respectively, simplifies the numerical simulations for $k \in \{0, 1\}$ considerably.

Let us look at the effect of the parameter k on the expected gains from trade W under uniformly distributed valuations. For $k \in (0, 1)$ we have a continuum of equilibria, each implying different gains from trade. Extensive numerical calculations[19] suggest that the partially "linear" equilibrium identified in Proposition 5 leads to the highest gains from trade. The equilibrium strategies with the lowest W look similar to those that are represented by the dashed lines in Figure 2: they approximate a pair of discontinuous strategies where $\hat{v}_2 = \Lambda_1$, $\hat{v}_1 = \Lambda_2$ as well as $\beta_1(v) = \Lambda_2$ and $\beta_2(v) = \Lambda_1$ for all $v \in (\Lambda_1, \Lambda_2)$.[20] Assuming $\alpha_1 = 0.8$, Figure 3 displays the gains from trade for the best and the worst equilibrium as a function of k. Whereas $k = \frac{1}{2}$ yields the highest gains from trade in the linear equilibrium, it also leads to the biggest range of possible equilibrium W. When approaching $k = 0$ or $k = 1$ this range becomes smaller,

tions. Our distributions G_1 and G_2 are power distributions when assuming F to be uniform. For a derivation of $\overline{b}$ see Marshall et al. (1994, Appendix A) where $t_* = \overline{b}$ and $k_i = \frac{1}{\alpha_i}$ for $i = 1, 2$.

[19]If F is uniform, $C_2^{-1}(\Lambda_1) < C_1^{-1}(\Lambda_2)$. Hence, the bid $b^* := \frac{1}{2}(C_2^{-1}(\Lambda_1) + C_1^{-1}(\Lambda_2))$ belongs to part II of any possible equilibrium. Computing equilibria and the implied W for various combinations of $\phi_1(b^*) \in (\Lambda_1, b^*)$ and $\phi_2(b^*) \in (b^*, \Lambda_2)$ we may obtain a good estimate of the range of possible W.

[20]This is equivalent to a no-trade equilibrium of the corresponding $1 - k$-double auction in part II.

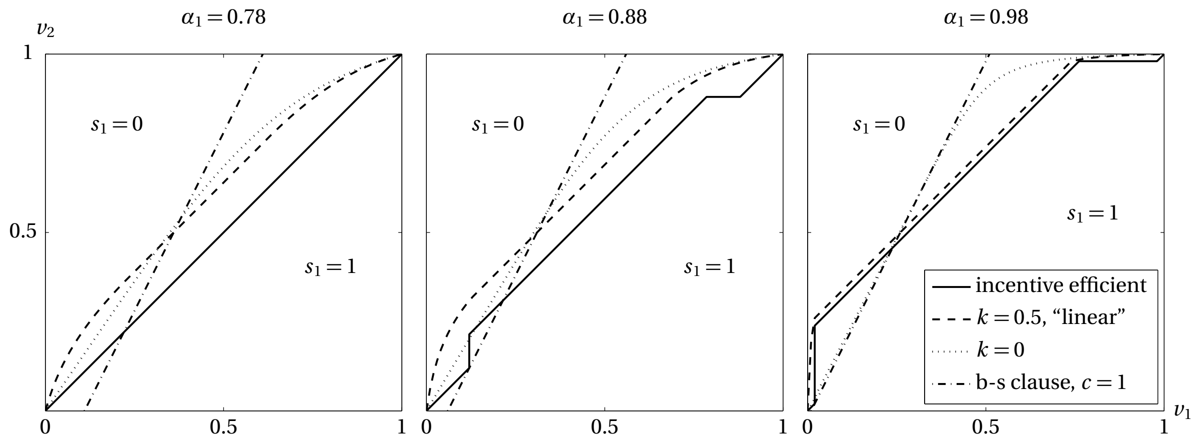

Figure 4: Comparison of allocation rules.

as the size of part II of equilibrium bidding decreases. In the unique equilibrium for $k = 0$ and $k = 1$, gains from trade are lower than in the linear equilibrium for other k. As predicted by Corollary 1, for every equilibrium of the $k + 1$-price auction there is an equilibrium of the $2 - k$-price auction that realizes the same W. This also holds for non-uniform distributions as longs as the density is symmetric about the mean.

5.3 Comparision

Using numerical results for the $k + 1$-price auction together with (20) and (19), we compare in Figure 4 the allocation rules of the auction with $k = 0$, of the linear equilibrium for $k = \frac{1}{2}$, and of the buy-sell clause with the incentive efficient allocation rule under uniformly distributed valuations. Note that $\alpha_1 = 0.78$, as set in the first panel of the figure, is slightly below the highest α_1 for which the incentive efficient mechanism still exhibits the ex post efficient allocation rule $\psi(v) = v$. When we increase α_1 further to 0.88 and 0.98, the incentive efficient mechanism deviates from the efficient allocation rule in a way similar to $k+1$-price auctions: in some situations where v_2 is not much higher than v_1 the object is allocated to partner 1. Note that the allocation rule that most closely resembles the incentive efficient mechanism is the linear equilibrium for $k = \frac{1}{2}$.

Figure 5 shows the result of numerically computing the expected gains from trade W for $\alpha_1 \in \{0.50, 0.51, 0.52, \ldots, 1.00\}$. In terms of W, the buy-sell clause is generally outperformed by the auctions with $k \in \{0, 1\}$. The linear equilibrium for $k = \frac{1}{2}$ realizes even larger gains from trade, being incentive efficient in the two limit cases $\alpha_1 =$

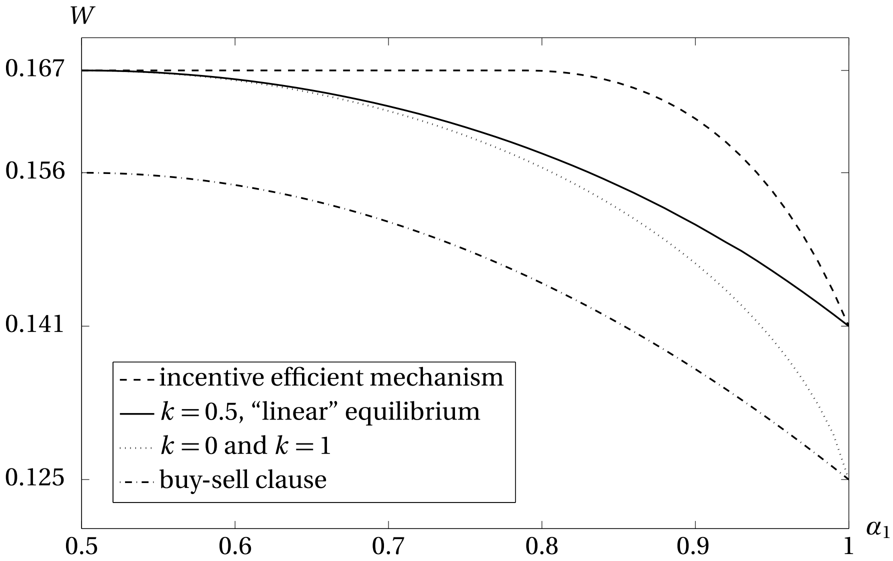

Figure 5: Expected gains from trade.

0.50 and $\alpha_1 = 1.00$.[21] We find that for $\alpha_1 = 0.89$ the linear equilibrium for $k = \frac{1}{2}$ realizes 93.3% of the gains from trade of the incentive efficient mechanism whereas this relative performance measure is higher for all other values we considered for α_1. For the auction with $k \in \{0, 1\}$, the expected gains from trade relative to the incentive efficient mechanism are decreasing in α_1, dropping from 100% at $\alpha_1 = 0.50$ to 88.9% at $\alpha_1 = 1.00$. The highest relative performance of the buy-sell clause is 93.8% at $\alpha_1 = 0.50$ and the lowest is 84.2% at $\alpha_1 = 0.91$.

6 Conclusion

We have analyzed asymmetric bilateral $k+1$-price auctions, allowing both the shares bidders receive in the auction revenue and the distribution of valuations to differ across bidders. Characterizing equilibrium bidding, we found that bidder 1 bids more aggressively than bidder 2 if bidder 2's distribution of valuations dominates bidder 1's in terms of the share-weighted hazard and reverse hazard rate. Both bidders shade their bids if they have a high valuation and they overbid if they have a low valuation for the auctioned object. The system of differential equations defin-

[21] For $\alpha_1 = 1$, the auction with $k = \frac{1}{2}$ is equivalent to the $\frac{1}{2}$-double auction, the linear equilibrium of which is shown by Myerson and Satterthwaite (1983) to be incentive efficient in the uniform case.

ing equilibrium strategies can be transformed so as to reveal a strong link between the $k+1$-price auction and both the k-double auction as well as the standard first-price auction. Exploiting this relationship, we established equilibrium existence and found there to be a continuum of equilibria if $k \in (0,1)$. For $k \in \{0,1\}$ there is a unique equilibrium. An interesting task for future research is to determine to what extent our approach can also be applied when weakening the assumption of strictly private and independently distributed valuations.

As a dissolution mechanism for asymmetric partnerships, the advantage of the $k+1$-price auction over the incentive efficient mechanism is that its rules are both simple and detail free. As a first step of evaluating the suitability of the $k+1$-price auction for dissolving bilateral partnerships, we presented numerical results for uniformly distributed valuations and asymmetric shares. Regarding the choice of k there is a trade-off: On the one hand, $k \in \{0,1\}$ implies that the equilibrium is unique, avoiding any coordination problems for the bidders. On the other hand, for $k \in (0,1)$ there are equilibria that lead to higher expected gains from trade. Provided that the partially linear equilibrium is played, the loss in expected gains from trade when using the auction with $k = \frac{1}{2}$ instead of the incentive efficient mechanism is less than 7%. In addition, the $k+1$-price auction outperforms the buy-sell clause where this relative loss in gains from trade can be as large as 15.8%. To what extent our findings for uniformly distributed valuations generalize is an issue left for future research.

A Appendix

A.1 Proof of Lemma 1

Before proving the lemma, we will first derive two auxiliary results. Suppose β_1, β_2 are pure strategies that form a Bayesian Nash equilibrium of the $k+1$-price auction.

Claim 1. *If $\beta_i(v)$ is locally decreasing, i.e., $\beta_i(v^*) > \beta_i(v^{**})$ for some $v^* < v^{**}$, then*

$$\Pr[\beta_i(v^{**}) \le \beta_j(V_j) \le \beta_i(v^*)] = 0 \quad \textit{and} \quad \Pr[\beta_j(V_j) < \beta_i(v^{**})] = \frac{\alpha_i k}{\alpha_i k + (1-\alpha_i)(1-k)}.$$

Proof. Consider two valuations v^* and v^{**} where $v^* < v^{**}$. Since β_i maximizes i's interim expected utility, we have $E\Big[u_i(v^*, \beta_i(v^*), \beta_j(V_j))\Big] \ge E\Big[u_i(v^*, \beta_i(v^{**}), \beta_j(V_j))\Big]$

and $E\Big[u_i(v^{**},\beta_i(v^{**}),\beta_j(V_j))\Big] \geq E\Big[u_i(v^{**},\beta_i(v^{*}),\beta_j(V_j))\Big]$ such that

$$E\Big[u_i(v^{*},\beta_i(v^{*}),\beta_j(V_j)) - u_i(v^{**},\beta_i(v^{*}),\beta_j(V_j))\Big] \\ \geq E\Big[u_i(v^{*},\beta_i(v^{**}),\beta_j(V_j)) - u_i(v^{**},\beta_i(v^{**}),\beta_j(V_j))\Big]. \quad (21)$$

Suppose β_i is locally decreasing, i.e., $\beta_i(v^*) > \beta_i(v^{**})$. Using (3), (21) then implies

$$(v^* - v^{**})\Big((1-\alpha_i)\Pr[\beta_i(v^{**}) \leq \beta_j(V_j) < \beta_i(v^*)] + \alpha_i \Pr[\beta_i(v^{**}) < \beta_j(V_j) \leq \beta_i(v^*)]\Big) \geq 0.$$

Hence, i's strategy can be decreasing only if $\Pr[\beta_i(v^{**}) \leq \beta_j(V_j) \leq \beta_i(v^*)] = 0$, i.e., only if there is a gap $\big[\beta_i(v^{**}),\beta_i(v^*)\big]$ in β_j. Therefore, using the fact that $\Pr[\beta_j(V_j) > \beta_i(v^*)] = 1 - \Pr[\beta_j(V_j) < \beta_i(v^{**})]$, bidder i's interim expected utility when bidding an amount $b \in \big[\beta_i(v^{**}),\beta_i(v^*)\big]$ is

$$E\Big[u_i(v_i,b,\beta_j(V_j))\Big] = b\,\Big(\alpha_i k - \Pr[\beta_j(V_j) < \beta_i(v^{**})]\,(\alpha_i k + (1-\alpha_i)(1-k))\Big) + Q$$

where Q summarizes all the terms that are independent of b. Consequently, i's strategy can be decreasing only if $\Pr[\beta_j(V_j) < \beta_i(v^{**})] = \frac{\alpha_i k}{\alpha_i k + (1-\alpha_i)(1-k)}$. Otherwise it would be optimal for i to bid either $\beta_i(v^*)$ or $\beta_i(v^{**})$ for all $v_i \in [v^*, v^{**}]$. □

Claim 2. *The equilibrium strategies β_1 and β_2 cannot both have an atom at the same bid, i.e., for all b*

$$\Pr[\beta_i(V_i) = b] > 0 \Rightarrow \Pr[\beta_j(V_j) = b] = 0 \qquad \textit{for } i,j = 1,2 \textit{ and } i \neq j.$$

Proof. Suppose $\Pr[\beta_i(V_i) = b] > 0$. Then almost all types of bidder j are better off by bidding slightly more or slightly less than b instead of b, which hardly changes payments but significantly changes the winning probability. □

We are now ready to prove Lemma 1. Let us start with the lower bound. First, consider $k \in (0,1]$. Without loss of generality assume $\inf_v \beta_1(v) > \inf_v \beta_2(v)$. This cannot be an equilibrium since a type of bidder 2 who bids below $\inf_v \beta_1(v)$ could increase his bid by some amount so that he still loses with certainty while benefiting from an increased auction revenue. Thus, in equilibrium we must have $\inf_v \beta_1(v) = \inf_v \beta_2(v)$. Claim 1 then implies that $\beta_i(v_i)$ is weakly increasing for v_i near $\arg\inf_v \beta_i(v)$ and therefore $\beta_i(0) = \inf_v \beta_i(v)$ so that $\beta_1(0) = \beta_2(0) =: \underline{b}$.

We will next show that $\underline{b} > 0$. From Claim 2 we know that at most one bidder can have an atom at $\beta_i(0)$. Assume without loss of generality that bidder j does not have an atom at $\beta_j(0)$. Consider bidder i with type $v_i = 0$ who deviates from the equilibrium by bidding $\underline{b} + \varepsilon$ instead of $\underline{b}$. His gain in ex post payoff is

$$u_i(0, \underline{b} + \varepsilon, b_j) - u_i(0, \underline{b}, b_j) \begin{cases} > \alpha_i k \varepsilon - (\underline{b} + \varepsilon) & \text{if } \underline{b} + \varepsilon > b_j > \underline{b}, \\ = \alpha_i k \varepsilon - \alpha_i(\underline{b} + \varepsilon) & \text{if } \underline{b} + \varepsilon = b_j, \\ = \alpha_i k \varepsilon & \text{if } \underline{b} + \varepsilon < b_j. \end{cases}$$

Suppose we had $\underline{b} \leq 0$. Then, i's gain in interim expected payoff is bounded below by

$$E\Big[u_i(0, \underline{b} + \varepsilon, \beta_j(V_j)) - u_i(0, \underline{b}, \beta_j(V_j))\Big] \geq \alpha_i k \varepsilon - \Pr[\beta_j(V_j) \leq \underline{b} + \varepsilon]\varepsilon$$

Note that we can always find a small enough ε such that $\alpha_i k > \Pr[\beta_j(V_j) \leq \underline{b} + \varepsilon]$, implying that i strictly prefers to deviate. Thus, we must have $\underline{b} > 0$ to prevent such a profitable deviation from the equilibrium strategy.

Now, consider $k = 0$. Without loss of generality, assume $\inf_v \beta_1(v) \leq \inf_v \beta_2(v)$. Claim 1 implies that $\beta_2(v)$ is weakly increasing and therefore $\beta_2(0) = \inf_v \beta_2(v)$. Then, for some $\tilde{v}_1$, all types $v_1 \leq \tilde{v}_1$ bid at most $\beta_2(0)$ while all types $v_1 > \tilde{v}_1$ bid more than $\beta_2(0)$ and $\beta_1(v)$ is weakly increasing in that range. Note that we must have $\beta_2(0) \leq 0$ in order for the type $v_2 = 0$ of bidder 2 to accept winning the auction with a positive probability. Now, consider type $\tilde{v}_1$ of bidder 1. If bidder 2 does not have an atom at $\beta_2(0)$, type $\tilde{v}_1$ loses the auction with probability 1, and therefore we must have $\beta_2(0) \geq \tilde{v}_1$. If bidder 2 has an atom at $\beta_2(0)$ and if $\beta_2(0) < \tilde{v}_1$, then type $\tilde{v}_1$ would want to bid just above $\beta_2(0)$. Therefore, we must have $\beta_2(0) \geq \tilde{v}_1$ in this case as well. From $\beta_2(0) \leq 0$ and $\beta_2(0) \geq \tilde{v}_1$ follows that $\tilde{v}_1 = 0$ and $\beta_1(0) = \beta_2(0) = 0$.

Making use of Corollary 2, the results for the upper bound directly follow from our findings for the lower bound. Finally, $\underline{b} < \overline{b}$ follows from Claims 1 and 2. □

A.2 Proof of Lemma 2

From Claim 1 in conjunction with Lemma 1 follows, for $k = 1$ and $k = 0$, that both $\beta_1(v)$ and $\beta_2(v)$ are weakly increasing in v.

Let $k = 0$. We begin by showing that the bidding strategies have no gaps, i.e., there is no interval $[b', b'']$ with $b' < b''$ so that $\Pr\big[\beta_i(V_i) \in [b', b'']\big] = 0$. Suppose without loss of generality that $\beta_1(v)$ has a gap $[b', b'']$. Bidder 2 then prefers bidding b' to bidding any other amount $b \in (b', b'']$ since bidding b' leads to a higher payoff when winning without changing the probability of winning. Therefore, $\beta_2(v)$ must also have a gap that ends at b''. In addition, if bidder 1 does not have an atom at b'', bidder 2 would prefer bidding b' to bidding just above b'', so that the gap in $\beta_2(v)$ would have to end even above b''. Of course, this would in turn lead the gap in $\beta_1(v)$ to end above b'', etc. Since we know from Claim 2 that the two bidders cannot both have an atom at b'', we conclude that there can be no gaps in $\beta_1(v)$ and $\beta_2(v)$, i.e., the equilibrium bidding strategies are continuous functions.

From Lemma 1 we know that $\beta_i(0)=0$. Observe that $\beta_i(v)$ cannot have an atom at 0, because no type $v_i > 0$ would want to lose the auction with certainty at a price of 0. Now, suppose $\beta_1(v)$ has an atom at $\tilde{b} > 0$. Note that a necessary condition for bidder 2 to be willing to bid $\tilde{b}$ is that he is indifferent between winning and losing the auction at that price. This is only the case if $v_2 = \tilde{b}$. Types $v_2 < \tilde{b}$ would prefer to bid just below $\tilde{b}$ and types $v_2 > \tilde{b}$ would prefer to bid just above $\tilde{b}$ instead of bidding $\tilde{b}$. However, it can be shown that type $v_2 = \tilde{b}$ is strictly better off when decreasing his bid $\tilde{b}$ by a finite amount ε. To see this, note that

$$u_2(\tilde{b},\tilde{b}-\varepsilon)-u_2(\tilde{b},\tilde{b}) > \Pr[\beta_1(V_1)<\tilde{b}-\varepsilon]\varepsilon - \Pr[\beta_1(V_1)<\tilde{b}]\alpha_2\varepsilon + \frac{1}{2}\Pr[\beta_1(V_1)=\tilde{b}-\varepsilon]\varepsilon$$

so that bidder 2 strictly prefers bidding $\tilde{b}-\varepsilon$ if $\alpha_2 < \frac{\Pr[\beta_1(V_1)<\tilde{b}-\varepsilon]}{\Pr[\beta_1(V_1)<\tilde{b}]}$. Such an ε always exists. Therefore, the atom in $\beta_1(v)$ creates a gap in $\beta_2(v)$. Since this contradicts the continuity property we have proved above, equilibrium strategies must be atomless.

Thanks to Corollary 2 our proof for $k=0$ also applies to $k=1$. Thus we have shown for $k \in \{0,1\}$ that, in addition to being continuous, β_1 and β_2 are weakly increasing and have no atoms, which implies that they are strictly increasing. $\square$

A.3 Proof of Lemma 4

In order to show that the unique solution to (13)-(15) also satisfies (16), we analyze the geometry of solutions in part I as Satterthwaite and Williams (1989) do for part II. Using the notation $\dot{\phi}_i := \frac{d\phi_i}{db}$ and combining the differential equations (13) with the tautology $\dot{b}=1$, we obtain

$$\dot{\phi}_1 = \frac{G_1(\phi_1)}{g_1(\phi_1)}\frac{1}{\phi_2 - b}, \qquad \dot{\phi}_2 = \frac{G_2(\phi_2)}{g_2(\phi_2)}\frac{1}{\phi_1 - b}, \qquad \dot{b}=1. \tag{22}$$

We know that $\phi_i \in [\Lambda_i, 1]$ and $b < \phi_1 < \phi_2$. The last inequality is due to assumption (2), as shown in the proof of Proposition 2. Solutions for part I are hence con-

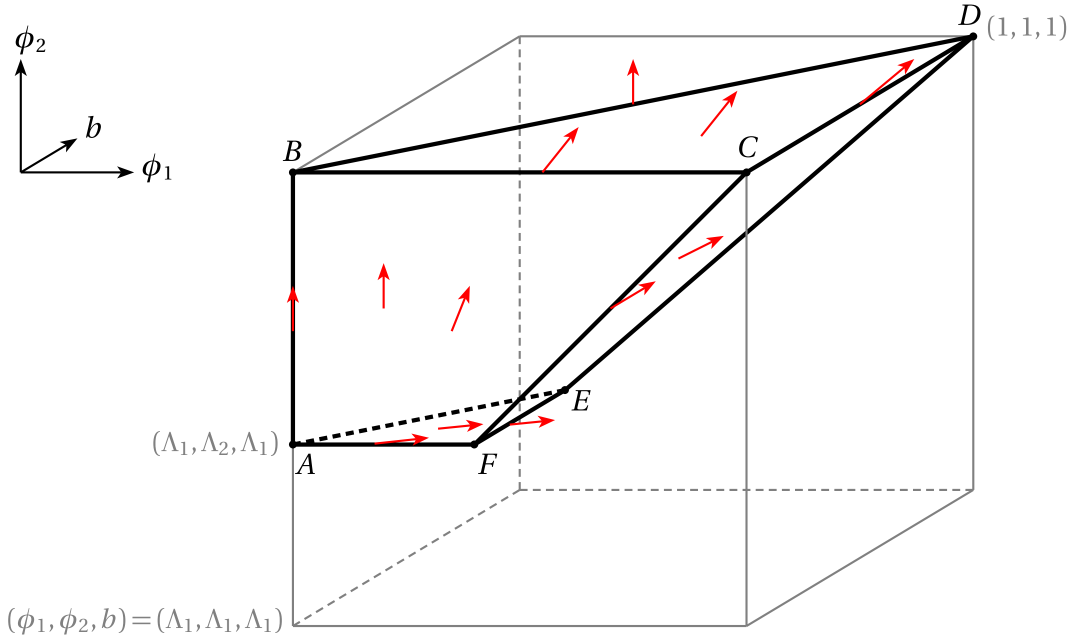

Figure 6: The frustum of a tetrahedron that contains solutions for part I. The red arrows show the limit of the normalized vector field on the faces and edges of the frustum.

tained in that portion of the tetrahedron $\Lambda_1 \le b \le \phi_1 \le \phi_2 \le 1$ which also satisfies $\Lambda_2 \le \phi_2$. Equations (22) define a vector field at each point (ϕ_1, ϕ_2, b) in this frustum of a tetrahedron within $[\Lambda_1, 1]^3$, which is shown in Figure 6.[22] Following Satterthwaite and Williams (1989, Section 4), we examine the limiting values of the vector field $(\dot{\phi}_1, \dot{\phi}_2, \dot{b})$ on the faces and edges of the frustum. We do so by looking at the normalization of the vector field since $\dot{\phi}_1$ or $\dot{\phi}_2$ may equal infinity on some faces and edges. The limits of the normalized field are reported in Table 1 and are indicated by red arrows in Figure 6.

According to Proposition 4, Lebrun (2006) shows that given $\hat{v}_1$ there is a unique solution to (13)-(15). Because of boundary condition (14), the corresponding solution curve exits the frustum on the edge CD, whereas, in accordance with boundary condition (15), it must enter the frustum somewhere on the face $ABDE$ or on one of its edges AB, BD, DE, or AE. Note that the normalized vector field on the face $ABDE$ lies within that face, and the same is true for the edge AB. On the edge BD the vector field points outside the frustum. A solution curve can therefore not enter through

[22]Note that the edge AE of the frustum in Figure 6 corresponds to the edge BD of the tetrahedron for part II in Satterthwaite and Williams (1989, Fig. 3.1).

	$\dot{\phi}_1$	$\dot{\phi}_2$	$\dot{b}$
face $ABDE$ and edges AB, BD	0	1	0
faces $ABCF$, BCD, $CDEF$ and edges BC, CD, CF	>0	>0	>0
face AEF and edges AF, EF	>0	0	>0
edges AE, DE		undefined	

Table 1: Direction of the normalized vector field on the faces and edges of the frustum.

$ABDE$, AB, or BD. Moreover, we can also rule out points on DE as potential entry points since there we have $\phi_1 = \phi_2 = b > \Lambda_2$ which contradicts $\hat{v}_1 \in [C_1^{-1}(\Lambda_2), \Lambda_2]$. We conclude that the solution curve must necessarily enter the frustum on the edge AE where the direction of the vector field is undefined. On AE we have $b = \phi_1 < \phi_2 = \Lambda_2$, which completes the proof. □

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