Combining Conjunction with Disjunction

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Abstract. In this paper we address some central problems of combination of logics through the study of a very simple but highly informative case, the combination of the logics of disjunction and conjunction. At first it seems that it would be very easy to combine such logics, but the following problem arises: if we combine these logics in a straightforward way, distributivity holds. On the other hand, distributivity does not arise if we use the usual notion of extension between consequence relations. A detailed discussion about this phenomenon, as well as some possible solutions for it, are given.

1 Introduction

Combination of logics is still a fairly young subject. It arose from the study of some particular cases especially connected with modal logics. This was a first stage of the development of this new field of research. Then people started to put forward some general concepts, to construct a systematic theory. This was done mainly on the one hand by Dov Gabbay and his team and on the other hand by the Lisbon group at the IST leaded by Amilcar Sernadas. However this theory is still in construction: many concepts still need to be clarified, and many important problems still need to be solve. In fact, combination of logics may look at first sight as an easy subject, but after a closer examination, it looks like a hazardous field of research.

Combination of logic is related to some fundamental phenomena of logic which are still not properly understood, which are connected to what a logic is and what are the relations between different formulations of a given logic. These questions are the subject matter of universal logic. So a general theory of combination of logics must be developed within universal logic, but on the other hand it is through the study of problems such as the ones appearing in combination of logics that universal logic grows. That is one of the reasons why combination of logics is a very interesting subject.

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In this paper, we are tackling some central problems of combination of logics through the study of a very simple but highly informative case, the combination of the logics of disjunction and conjunction. As it is known in mathematics, for example with Fermat's theorem, it is not because a problem is easy to state and understand, that it is easy to solve. Mathematical problems which are very easy to formulate but very hard to solve are fascinating, because this kind of discrepancies are challenges to our minds, it is like if we were able to see the Eldorado in its full splendour but were unable to reach it.

The problem we are studying here is the combination of logic of the conjunction with the logic of disjunction. Such logics are extremely simple and have barely being studied by themselves. Considered from the outdated concept of logic taken as a set o tautologies they don't even have a sense, since there are not tautologies in such logics. At first it seems that it would be very easy to combine such logics whether it be from a proof theoretical viewpoint (Gentzen rules), semantic viewpoint (truth tables) or a purely consequence viewpoint (Consequence operators or consequence relations), but the following problem arises: if we combine these logics in a straightforward way, distributivity holds. We have introduced in a previous work [5] the suggestive terminology "copulation paradox" to describe this phenomenon, because conjunction and disjunction are interacting between each other generating distributivity.

This is a paradox and a problem, because the standard view of combination of logics is that the combined logic is the smallest one defined on the combined language, which extends the two combined logics. One could think that there is no paradox because the logic of conjunction and disjunction is necessary distributive, and as we will see this not completely false, but also someone who knows a bit of lattice theory is aware that non-distributivity is possible. One question here is to study if there is a strict parallel between logic and algebra or not. We already have pointed out the many dangers and incorrectness of a reductionist point of view, according to which logic can properly be treated by algebraic methods [3], it seems we have here again a phenomenon showing that logic does not reduce to algebra, that it is more complex.

2 The logic of conjunction and the logic of disjunction

The logic of conjunction can be defined in a very simple way using a semantical method. We just have to consider the standard following condition for bivaluations (that is, for valuations on $\{0, 1\}$):

$\beta(a \wedge b) = 1$ iff $\beta(a) = 1$ and $\beta(b) = 1$.

This condition corresponds to the truth-table of conjunction.

$\wedge$	0	1
0	0	0
1	0	1

Mathematically speaking, the semantic of the logic of conjunction is the set of homomorphisms between the absolutely free algebra generated by a binary operator called conjunction from a set of atomic formulas, and the algebra on $\{0, 1\}$ with a binary operator defined by the truth-table above. Generally people use the same symbol "$\wedge$" for the operator in the algebra of language and the operator in the algebra of truth-values.

The algebra $\langle\langle 0, 1\}; \wedge\rangle$ together with the subset $\{1\}$ of distinguished values is the structure $\langle\langle\{0, 1\}; \wedge; \{1\}\rangle\rangle$, which is called a logical matrix and is used to define different logical structures. Among them, a structure where the main relation is a consequence relation, i.e. a relation between sets of formulas and formulas. When there is not ambiguity, we will call the logic of conjunction, taken as a consequence relation, just "the logic of conjunction".

The logic of conjunction can also be defined using a proof system, for example of Gentzen type. This is just the subsystem of LK including all the structural rules plus the two logical rules for conjunction. The completeness result can easily be proven in particular using the general technique presented in [6].

It is easy to check using either the matrix semantics or the Gentzen system that in the logic of conjunction, the following holds:

(1, $\wedge$) $T, a \wedge b \vdash a$
(2, $\wedge$) $T, a \wedge b \vdash b$
(3, $\wedge$) $T, a, b \vdash a \wedge b$.

It is also possible to prove that these three laws are enough to define the logic of conjunction in the sense that a model of these three laws together with the laws defining a structural consequence relation is the logic of conjunction. This is the consequence approach to logic, promoted especially by Polish logicians, not to be confused with a Gentzen type approach, although sometimes they look very similar. In the context of the consequence approach we use the terminology "law". A law should not be confused with a rule of a Gentzen system.

In order to define the logic of conjunction, instead of the three above laws, it is also possible to take the three following laws which are closer to the Gentzen rules for conjunction:

If $T, a \vdash c$ then $T, a \wedge b \vdash c$
If $T, b \vdash c$ then $T, a \wedge b \vdash c$
If $T \vdash a$ and $U \vdash b$ then $T, U \vdash a \wedge b$.

The logic of disjunction is, in a sense that can be made precise, the dual of the logic of conjunction. We will not recall here the truth-table of disjunction and the Gentzen rules for disjunction that everybody knows, but just give the following three axioms which can be used to define the logic of disjunction in the framework of consequence relations.

(1, $\vee$) If $T \vdash a$ then $T \vdash a \vee b$

(2, $\vee$) If $T \vdash b$ then $T \vdash a \vee b$
(3, $\vee$) If $T, a \vdash c$ and $U, b \vdash c$ then $T, U, a \vee b \vdash c$.

3 Combining straightforwardly the logics of conjunction and disjunction

If we put together the logic of conjunction LC with the logic of disjunction LD in a straightforward way, by putting for example the conditions defining the sets of bivaluations, we get a logic LCD in which the following laws of distributivity hold:

$a \wedge (b \vee c) \dashv\vdash (a \wedge b) \vee (a \wedge c)$
$a \vee (b \wedge c) \dashv\vdash (a \vee b) \wedge (a \vee c)$

At this point one can make the following two conjectures

- We have applied the wrong procedure to combine LC and LD: we have to find the right one, which leads to a a logic of conjunction and disjunction in which distributivity does not hold
- We have applied the right procedure: combination of LC and LD necessary leads to distributivity.

In both cases, we are in hazardous zones, for, if the first conjecture is true, we have to understand what went wrong with this combination procedure and find the proper one; and if the second conjecture is true, we have to understand how is produced the interaction between conjunction and disjunction which leads to some new feature, namely distributivity.

One may think that the first conjecture is more plausible, because non-distributivity is a real phenomenon, especially known in quantum logic and lattice theory, and also because when we examine in a closer way how we have combined the semantics of conjunction and disjunction, we see that it seems that there is confusion: we have supposed that the set of values is the same in both cases.

4 Distributivity law and non-distributive lattices

A lattice can be defined as a structure of order with two binary operators verifying laws very similar to the ones of conjunction and disjunction, in fact most of the time the same symbols are used:

$a \wedge b \leq a$
$a \wedge b \leq b$
If $c \leq a$ and $c \leq b$ then $c \leq a \wedge b$
$a \leq a \vee b$
$b \leq a \vee b$
If $a \leq c$ and $b \leq c$ then $a \vee b \leq c$

The connection between lattices and the logic of conjunction and disjunction is tight: a lattice is the algebra we get if we factorize the logic of conjunction and disjunction with the usual Lindenbaum-Tarski method.

It is well-known that not every lattice is distributive, in particular there are some lattices in which the following laws do not hold:

$$a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$$
$$(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c)$$

It is easy to prove that a lattice is distributive if and only if the following law holds:

$$a \wedge (b \vee c) \leq (a \wedge b) \vee c$$

Due to the tight connection between (classical) conjunction, disjunction and lattices, one can expect that a logic of (classical) conjunction and disjunction is distributive if and only if the following law holds:

$$a \wedge (b \vee c) \vdash (a \wedge b) \vee c$$

We will call therefore this law, *the law of distributivity.*

One possibility to construct a semantics for a non-distributive logic of conjunction and disjunction would be to consider a semantics based on a non-distributive lattice. The idea behind this strategy is that in case of classical logic, its Lindenbaum-Tarksi algebra is a boolean algebra and its semantics is a matrix whose algebra is the boolean algebra on $\{0, 1\}$. Moreover, intuitively, it seems reasonable that a matrix semantics based on a non-distributive lattice could falsified the distributivity law. So the idea is to consider one of the two simplest non-distributive lattices ([8] pp. 75), knowing in particular that a lattice which is not distributive contains one of them as a sublattice. Building a matrix semantics with these non-distributive lattices means having to choose the set of distinguished elements. After toying a bit with these 5 elements lattice one sees that the task is not so easy. Either we are able to falsify distributivity but not verifying the laws for conjunction and disjunction, or we satisfy these laws but distributivity also holds. In fact it is possible to prove that there is no way out.

5 No semantics for non-distributive logics

We will now prove a very general result about the impossibility to find a semantics for a non-distributive logic of conjunction and disjunction taken as a consequence relation. To prove this result we use a very general definition of semantics. A semantic for us is any set of values divided into two subsets, of distinguished and non-distinguished values, which is then used to define a consequence relation following the standard Tarskian notion of semantical consequence. This general definition encompasses any matrix semantics finite or not and also any truth-functional semantics. Moreover, we have shown in a previous paper (see [2]) that, if we are working within the framework of Tarskian semantical consequence, any semantics can be reduced to such a concept of semantics. This means that the following result really shows that there are no semantics for non-distributive logics within the Tarskian paradigm.

Theorem 1. *The law of distributivity cannot be falsified by a bivalent semantics, truth-functional or not.*

Proof. Suppose that the we have a semantics which is complete for the LCD. This means that when the value of a is non-distinguished then, due to the law $(1, \wedge)$, the value of $a \wedge b$ should be non-distinguished. Similarly, when the value of b is non-distinguished then, due to the law $(2, \wedge)$, the value of $a \wedge b$ should be non-distinguished. Finally, if the value of a is distinguished and the value of b is distinguished then, due to the law $(3, \wedge)$, the value of $a \wedge b$ cannot be non-distinguished. Therefore the value of $a \wedge b$ is distinguished iff only the value of a is distinguished and the value of b is distinguished.

We let the reader check that if a valuation respects the law for disjunction then the value of $a \vee b$ is non-distinguished iff only the value of a is non-distinguished and the value of b is non-distinguished.

Now suppose that the law of distributivity is falsified in a semantics which verifies the law of conjunction and disjunction. There is a valuation such that the value of $a \wedge (b \vee c)$ is distinguished and the value of $(a \wedge b) \vee c$ is non-distinguished.

On the other hand, if the value of $a \wedge (b \vee c)$ is distinguished, then the value of a and the value of $b \vee c$ are distinguished and either the value of b or c is distinguished.

On the other hand if the value of $(a \wedge b) \vee c$ is non-distinguished, the value of $a \wedge b$ and c are both non-distinguished. The value of b must therefore be distinguished. But since the value of a is distinguished, the value of $a \wedge b$ must be distinguished, which is absurd. □

6 The logic of conjunction and disjunction is necessarily distributive

If one allows systems of sequents, as the standard ones, with many formulas on the left side of the sequent, and whatever is the number of formulas on the right (be it reduced to one as in the case of intuitionistic sequent system or not), it is possible to prove distributivity in such sequent systems with the standard rules for conjunction and disjunction and the usual structural rules. We will not present this proof here, but a analogous version of it, which corresponds to a meta-theorem in the theory of consequence relations showing that the logic of conjunction, taken as a consequence relation, combined with the logic of disjunction, taken also as a consequence relation, is necessary distributive.

Theorem 2. *The logic of conjunction and disjunction, taken as a consequence relation, is distributive.*

Proof.
(1) $a \vdash a$ (identity)
(2) $b \vdash b$ (identity)
(3) $a, b \vdash (a \wedge b)$ from (1) and (2) using (3, $\wedge$)
(4) $a \wedge (b \vee c)), b \vdash (a \wedge b)$ from (3) using (1, $\wedge$)
(5) $a \wedge (b \vee c)), b \vdash (a \wedge b) \vee c$ from (4) using (1, $\vee$)
(6) $c \vdash c$ (identity)
(7) $c \vdash (a \wedge b) \vee c$ from (6) using (2, $\vee$)
(8) $a \wedge (b \vee c)), (b \vee c) \vdash (a \wedge b) \vee c$ from (5) and (7) using (3, $\vee$)
(9) $(a \wedge (b \vee c)) \vdash (a \wedge b) \vee c$ from (8) using (1, $\wedge$) □

7 The combination of the logic of conjunction and disjunction is not necessarily distributive

In the preceding section, we have considered the logic of conjunction and disjunction to be the logic verifying the laws (1, $\wedge$), (2, $\wedge$), (3, $\wedge$), (1, $\vee$), (2, $\vee$) and (3, $\vee$) of section 2. One may argue that this logic is not the combination of the logic of conjunction and the logic of disjunction. One may in fact define the combination of these two logics as the smallest logic on the combined language which is an extension of both. If we take here extension in the sense of inclusion of consequence relation, one may find a logic which is an extension of both where (3, $\vee$) does not hold and where also distributivity does not hold.

In order to prove this, we will consider now structural Tarskian consequence relations, that is, satisfying:

(EXT) If $a \in T$ then $T \vdash a$
(CUT) If $T \vdash a$ and $U \vdash b$ for every $b \in T$ then $U \vdash a$
(FIN) If $T \vdash a$ then $U \vdash a$ for some finite $U \subseteq T$
(STR) If $T \vdash a$ then $\sigma(T) \vdash \sigma(a)$ for every substitution σ

It is worth noting that, from (EXT) and (CUT), any consequence relation satisfies the usual property of monotonicity:

(MON) If $T \vdash a$ and $T \subseteq U$ then $U \vdash a$

The set of formulas For of any logic consider here is the absolutely free algebra generated by their connectives from the set of atomic formulas. Then a substitution is just an endomorphism $\sigma : \mathsf{For} \to \mathsf{For}$.

Consider now the set For of formulas generated by conjunction $\wedge$ and disjunction $\vee$. Let $\vdash_{\wedge\vee}$ be the relation defined as follows: for any $T \cup \{a\} \subseteq \mathsf{For}$, $T \vdash_{\wedge\vee} a$ iff there exists a nonempty finite set $U \subseteq T$ such that $(\bigwedge_{b \in U} v(b)) \leq v(a)$ holds good in L, for every lattice L and every homomorphism $v : \mathsf{For} \to L$.

The reader can check that the relation $\vdash_{\wedge\vee}$ is in fact a structural Tarskian consequence relation over the set For of formulas.

Now, consider the combination CD of the logics LC and LD obtained as the smallest logic defined by a structural Tarskian consequence relation over the combined language For which is an extension of both logics. In more precise terms, let $CONS_{\wedge\vee}$ be the set of structural Tarskian consequence relations defined over For. The set $CONS_{\wedge\vee}$ ordered by inclusion is a complete lattice (cf. [14]). Then, CD is the logic defined over For by the consequence relation

$$\vdash_{CD} = \bigwedge \{ \vdash \in CONS_{\wedge\vee} \ : \ \vdash_{LC} \subseteq \vdash \ \text{ and } \ \vdash_{LD} \subseteq \vdash \}.$$

Of course the infimum is taken in the complete lattice $CONS_{\wedge\vee}$. Here $\vdash_{LC}$ and $\vdash_{LD}$ denote the consequence relations of LC and LD, respectively. Obviously $\vdash_{CD}$ is the supremum in $CONS_{\wedge\vee}$ of $\vdash_{LC}$ and $\vdash_{LD}$. Then we obtain the following:

Theorem 3. *The logic* CD *obtained by the combination of the logic of conjunction and the logic of disjunction is non-distributive.*

Proof. Let $\mathcal{R} = \{ \vdash \in CONS_{\wedge\vee} \ : \ \vdash_{LC} \subseteq \vdash \ \text{ and } \ \vdash_{LD} \subseteq \vdash \}$. It is clear that $\vdash_{\wedge\vee}$ defined above belongs to $\mathcal{R}$ and so $\vdash_{CD} \subseteq \vdash_{\wedge\vee}$. But $\vdash_{\wedge\vee}$ is non-distributive (since there exist non-distributive lattices), therefore CD is non-distributive. □

The trick is that if we consider the logic of disjunction just as a consequence relation, the law (3, $\vee$), which appears as a kind of meta-property, does not necessarily hold in a logic which extend LD. The question here is to know whether a logic in which (3, $\vee$) does not hold can properly be called a logic of disjunction. If we think this is not reasonable, then combination of logics should not be defined as the "smallest logic on the combined language which is an extension of both in the sense of inclusion of consequence relation". But then how to proceed? It is not possible to proceed in general as we have done here by putting together some laws that define the two logics, because we want to have a definition of combination of logics even when the logics are not given by some laws. One solution is to consider that a logical structure is not completely given by the consequence relation but must be further be specified by some meta-properties, but then how to define the set of relevant meta-properties?

At this point, it is worth noting that the operation of categorial fibring (see [13]) is defined as a coproduct in the category of logic systems under consideration. In the case of the category of logics defined by structural Tarskian consequence relations and translations between logics as morphisms,[1] the coproduct coincides with the supremum of the consequence relations. Thus, the logic CD is the coproduct (categorial fibring) of LC and LD in this category. However, it is possible to substitute the usual notion of translation between a logic L_1 and a logic L_2 by a stronger one, viz. a mapping f : $\mathsf{For}_1 \to \mathsf{For}_2$ satisfying the following: if the logic L_1 (defined over the set For_1 of formulas) satisfies a

[1] Recall that a translation between a logic L_1 (defined over the set For_1 of formulas) and a logic L_2 (defined over the set For_2 of formulas) is a mapping $f : \mathsf{For}_1 \to \mathsf{For}_2$ such that $T \vdash_{L_1} a$ implies $f(T) \vdash_{L_2} f(a)$.

meta-property of the form

$$\texttt{If } T_1 \vdash_{L_1} a_1 \texttt{ and } \ldots \texttt{ and } T_n \vdash_{L_1} a_n \texttt{ then } T \vdash_{L_1} a$$

then the logic L_2 (defined over the set For_2 of formulas) must satisfy the meta-property

$$\texttt{If } f(T_1) \vdash_{L_2} f(a_1) \texttt{ and } \ldots \texttt{ and } f(T_n) \vdash_{L_2} f(a_n) \texttt{ then } f(T) \vdash_{L_2} f(a).$$

If this kind of meta-translations are considered, then the coproduct of LC and LD in this category (that is, the smallest logic given by a structural Tarskian consequence relation over the set For which extend both LC and LD by means of meta-translations) is exactly the logic LCD obtained in Theorem 2, which is distributive. In [10] was defined a formal framework for combining consequence relations by means of meta-translations. The importance of meta-properties (such as those considered above) in the analysis of logics was already studied in [4].

There is still another solution in the case of the logic of disjunction, which is to consider logics as multiple-conclusion consequence relations. In this case we can state the law (3, $\vee$) without going at the meta-level. In this case we can also reformulate the result of Section 6 and show that the combination of the logics of disjunction and conjunction is necessarily distributive.

8 Are there non-distributive logics? and what is the right method to combine logics?

Following the result of Section 6, one may want to conclude that there are no non-distributive logics, that the combination of the logic of disjunction and disjunction is the standard fragment of classical propositional logic, whose semantics is given by the usual truth-tables for conjunction and disjunction and whose proof theory is provided by the usual Gentzen rules.

From this point of view, there are here no problems related to combination of logics, and one may conjecture that from a semantical viewpoint, to combine bivalent semantics, truth-functional or not, we just have to put together the set of conditions in the same way as what we combine proof systems by putting the rules together.

However it is not that simple. We can argue that there really are non-distributive logics in a strong sense, not only degenerated ones as given in Section 7 where (3, $\vee$) does not hold. The idea of non-distributivity makes sense in lattice theory and in algebra in general. If we cannot express it in the standard framework of consequence relations, it it perhaps because this framework is not good enough, because it is not the right framework to develop logic. On the other hand, by comparing Theorems 2 and 3, it seems that the underlying notion of morphisms between logics adopted for the process of combining logics is crucial.

The framework of consequence relations in fact has been over the last decades challenged in different ways, for example a alternative to it is multiple-conclusion logics. However this option it is not a solution for our problem, since our theorem still hold for this framework, as we have seen. A possible solution is going in the opposite direction, by restricting to the cardinality one, the set of formulas on both side of the consequence relation. If we do that we find our way to a non-distributive logic of conjunction and disjunction which can be generated by a sequent systems having a analogous feature. It is easy to prove that in a sequent system with only one formula on both side of the sequent, distributivity cannot be derived. The combination of the logic of conjunction and disjunction from a proof-theoretical viewpoint follows the standard definition which consists in putting the rules together. However we have here a problem for defining the combination of semantics, since the standard semantics for conjunction is a semantics for the logic of conjunction considered as a monomonoconsequence relation and respectively the same for the logic of disjunction, but their standard combination is not an adequate semantics.

At this point it seems reasonable to conjecture that Lisbon Group's theorem about combination of Hilbert proof systems, stating that it is enough to put the rules together to get the combined logic (cf. [13]), can be extended to Gentzen's systems with restriction or not on the cardinality of the set of formulas on the left or/and on the left provided analogous restrictions are set up on the corresponding logics. This approach was adopted in [10]. However from a semantical point of view, we don't know how to proceed.

One may say that the problem we face with non-distributivity is connected to substructurality. We prove distributivity in the standard sequent system for conjunction and disjunction by using contraction. As we have seen, one way to block this phenomenon is to reduce the cardinality of the sets of the formulas to one, which already leads to sequent systems that can be called substructural. A monomonoconsequence relation could by analogy also be called a substructural logic. Another possibility in order to block distributivity is to throw away or to restrict contraction. But then several problems arise: we have to see how we will define logics, i.e. consequence relations, without contraction rules. This problem is in general not faced by people working in linear logic, who rather considered logic as set of tautologies. A solution to this problem is to consider multiset or alternatively further operators between formulas which properly form a substructure. Then we have to study the techniques to combine such substructural logics and related proof systems and semantics for such logics.

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