Logic and Philosophy of Logic

Do Sentences Have Identity?

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ABSTRACT: We study here equiformity, the standard identity criterion for sentences. This notion was put forward by Lesniewski, mentioned by Tarski and defined explicitly by Presburger. At the practical level this criterion seems workable but if the notion of sentence is taken as a fundamental basis for logic and mathematics, it seems that this principle cannot be maintained without vicious circle. It seems also that equiformity has some semantical features ; maybe this is not so clear for individual signs but sentences are often considered as meaningful combinations of signs. If meaning has to play a role, we are thus maybe in no better position than when dealing with identity criterion for propositions. In formal logic, one speaks rather about well-formed formulas, but closed formulas are called sentences because they are meaningful in the sense that they can be true or false. Formulas look better like mathematical objects than material inscriptions and equiformity does not seem to apply to them. Various congruencies can be considered as identities between formulas and in particular "to have the same logical form". One can say that the objects of study of logic are rather logical forms than sentences conceived as material inscriptions.

1. What is equiformity?

Some logicians have rejected propositions in favour of sentences, arguing in particular that there is no satisfactory identity criterion for propositions (cf. Quine, 1970). But is there one for sentences? The idea that logic is about sentences rather than propositions and that sentences are nothing more that material inscriptions was already developed by Lesniewski, who also saw immediately the main difficulty of this conception and introduced the notion of *equiformity* to solve it. His attitude his well described in a footnote of one of Tarski's famous early papers:

> As already explained, sentences are here regarded as material objects (inscriptions). (...) It is not always possible to form the implication of two sentences (they may occur in widely separated places). In order to simplify matters we have (...) committed an error; this consists in *identifying equiform sentences* (as S. Lesniewski calls them). This error can be removed by interpreting *S* as the set of all types of sentences (and not of sentences) and by modifying in an analogous manner the intuitive sense of other primitve

> concepts. In this connexion by the type of a sentence *x* we understand the set of all sentences which are equiform with *x*. (Tarski, 1930a, p.31 (1983), footnote 3)(2)

Equiformity means "the same form," but what is the "form" of a sentence? What is the exact definition of "equiformity?" We don't have any definiton by Lesniewski himself but Presburger has formulated two definitions which probably express the idea of Lesniewski and of the Lvov-Warsaw's school in general. They are:

Geometrical definition: Equiformity consists only in similarity of geometrical shapes.

Rewriting definition: We say that two inscriptions *A* and *B* are *equiform* if everybody who would like to rewrite any of them might do this with the help of one and the same inscription *C*. Hence the first letter in the word "ani" is equiform with the second letter of the word "Jan."(3)

2. Challenging equiformity

2.1. The practical level

These definitions are common sense and quite simple (at least at first sight). They are easily understable and workable. But it is also possible to criticize them: equiformity so-defined is fuzzy and problems may arise. A typical difficulty is related to the bald guy paradox. Let us take a sign and *deform* it *slowly* and *continuously* in a series of steps. In each of two succesive steps, we have two equiform signs (according to the first or the second definition), but this is not necessarily the case of any two signs of the series, in particular it is not the case of the first and the last signs. Equiformity is here reflexive, symmetric but not transitive. It is not an equivalence relation and cannot be considered as an identity relation.

This kind of difficulty is not really important at the practical level. It seems rather a theoretical problem that appears only at the abstract level of a rigorous definition. And other difficulties like confusions between two signs (e.g., "1" and "l"), which have very similar shapes but are indeed different, are in general minor problems who do not seem to challenge the practicability of equiformity.(4) But can we ignore such kind of difficulties when dealing with the concept of sentences taken as a fundamental concept of logic and mathematics?

2.2. The fundamental level

The view that sentences are material objects and must be prefered to propositions is clearly related to a formalist approach to logic and mathematics, and the desire to avoid abstract problematic objects, paradoxes (especially semantical ones), as very well shown by the following declaration of Dieudonné:

> On foundations we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes we rush to hide behind formalism and say : "Mathematics is just a combinaison of meaningless symbols", and then we bring out Chapters 1 and 2 on set theory. Finally we are left in peace to go back to our mathematics and do it as we have always done, with the feeling each mathematician has that he is working with something real. This sensation is probably an illusion, but it is very convenient.That is Bourbaki's attitude towards foundations (Dieudonné, 1970, p.145).

If the notion of sentence is used to provide a fundamental secure basis for mathematics and avoid paradoxes, one can expect a precise definition of it, matching this ambitious purpose. In particular we cannot be happy with the fuzzy notion of equiformity of everyday life.

One can required a characterization of "geometrical shape" which prohibits paradoxes of the type of the bald. It can be for example based on geometry. But in this case one uses a theory which is not simpler than what one wants to found with the notion of sentences. Moreover it seems that there is a real vicious circle because geometry is part of mathematics and the notion of sentences is used to found mathematics. Even worse: as Tarski says, we must deal not with sentences, but with *sets* of equiform sentences. Therefore we need the concept of set to define the concept of sentences which is in turn used to define set theory. One can also argue that any notion of identity for sentences depends on the theory of identity based itself on the notion of sentences. Moreover if sentences are really physical bodies, we need physics in order to have an identity criterion for them.(5) Thus we need physics to found sentences and then mathematics. It seems therefore that there is no good identity criterion for sentences taken as a foundation basis, because what is needed for this criterion depends on the concept of sentences itself.

2.3. The problem of meaning

It is also possible to argue that the formalist-blind attitude rejecting meaning does not permit to provide a satisfactory identity criterion for sentences because such a criterion has necessarily some semantical features. And if meaning has to be taken in account to get a good identity criterion for sentences, we are not in a clearly better position than in the case of propositions. Claude Chevalley, the main promoter of Bourbaki's radical formalist choice, as it appears in the first Chapter of *Theory of Sets,* has rejected this choice, forty years later, saying that "a symbol cannot be "the same" if it does not have an aura of signification" (Chevalley, 1981, p.20 (1985)).(6) In fact it seems that at the practical level ambiguity of equiformity is solved by appealing to something like meaning. For example, one can have a doubt about the sign "1:" Is it equiform with the sign "l?" If this sign occurs in an American zip code, one knows that it is not a letter, but a number, and that it is therefore not equiform with the sign "l" occuring in "Berkeley."

Logicians have imported from linguistics the distinction syntax/semantics, but Saussurre himself insisted on the fact that the significant and the signified cannot be absolutely dissociated. One can even say that the notion of a meaningless sign is meaningless, because what distinguishes a sign from any object is that it is a support for meaning. In fact one of the standard definition of sign is the following: "a picture, shape, etc., that has a particular meaning" (*Longman Dictionary of Contemporary English*).

But one can claim that in the formal approach to logic and mathematics, signs are not exactly signs in the linguistics sense, but any material inscriptions. However this was not the case of Lukasiewicz's conception as described in the above quotation of Tarski's letter to Neurath. There, material inscriptions taken as ornaments, look like meaningful pictures.

Modern logicians use "symbol" rather than "sign," but it is not clear at all that these symbols are material inscriptions, for example infinite sets of symbols are often considered. In his recent book, W. Hodges writes, "The items called 'symbols' in this book need not be written down. They need not even be dreamed" (Hodges, 1997, p.2).

3. Sentences as formulas

3.1. Formulas and meaning

Usually, written sentences are made of signs and equiformity of sentences means equiformity of the signs they are built with. Equiformity of signs is something that everybody is familiar with : in some way the first letter of the latin alphabet, "a," is understood with the notion of equiformity, "a" is a set of equiform signs. One can even say that it is nothing more, rejecting arguments of section 2.3. But it is much more difficult to say the same thing for sentences, because a sentence is not any kind of combination of signs. This is true for natural languages as well as for formal languages. So the "identity" of sentences is not only based on equiformity (however we define it) but also on the way of combining signs. Let us take the ideal case of a formal language, case in which the way of combining signs is explicitly and rigorously defined by formal rules (called here *morphological rules*). Can one say that these rules are formal and that no consideration of meaning are interfering, in such a way that we can really consider a sentence as a class of equiform sets of signs and nothing else?

Tarksi, inspired by Lesniewski, identifies well-formed sentences with meaningful sentences:

> From the standpoint of metamathematics every deductive discipline is a system of sentences, which we shall also call meaningful sentences. (Instead of "meaningful sentence" we could say "well-formed sentence"..) The sentences are most conveniently regarded as inscriptions, and thus as concrete physical bodies. Naturally, not every inscription is a meaningful sentence of a given discipline : only inscriptions of a well-determined structure are regarded as meaningful. (Tarski, 1930b, p.62 (1983)).

One could argue that meaning is present in the aim of the construction of well-formed sentences, but not in the resulting construction itself. But if the meaning is nothing else that the way the sentences are constructed, then one must admit that the concept of sentence depends on meaning. If one thinks that meaning is not an essential feature of the construction, then one must speak only of well-formed sentences, ruling out the expression "meaningful sentences." In fact, in first-order logic, there is a tendency to speak only about "formulas;" as K.Schütte says: "in mathematical logic sentences are represented by formulas" (Schütte, 1960, p.3). However the vocable "sentence" is not totally banished from first-order logic, it is used to denote *closed formulas*. A closed formula is true or false, this is no the case of an open formulas (unless we identify it to its universal closure, transforming it into a sentence). One can suppose that closed formulas are called sentences just for this reason. Thus "sentences" applies here to formulas which are meaningful in the sense that they can be truth or false.

According to Hodges (Hodges, 1985-86), the reason why it was so difficult for Tarski to establish the notion of truth for first-order logic is that even a closed formula, as we understand it today, can look like something that is not a sentence, i.e., something to which truth and falsity cannot be applied : in the formula $\forall x \exists y Rxy$, the binary predicate must be interpreted until we can speak of the truth of the formula. According to Hodges this formula would not have been taken as a sentence (*Satz*) by Frege. He summarizes the difficulty as following:

> The issue is simply this. If a sentence contains symbols without a fixed interpretation, then the sentence is meaningless and doesn't express a determinate thought. But then we can't properly call it true or false. (Hodges, 1985-86, pp.147-148). All this shows the importance of meaning in the conception of formulas. Thus, once again, one is here not clearly in a better position than dealing with propositions.

However one could admit meaning as an essential feature of the notion of sentence but argue that equiformity does not depend on this feature. This raises the difficult question about relations between an identity criterion of some entities and the "identity" (i.e., nature) of these entities. Anyway, one can doubt of the validity of an identity criterion which depends only on some accidental features.

3.2. Formulas as abstract mathematical objects

Formulas constructed in a formal language can be considered as a kind of mathematical idealizations of sentences taken as material inscriptions (one could say that the relation between a formula and a physical sequence of signs is the same as the relation between a geometrical circle and a circle drawn on the sand). But they are indeed mathematical objects with some mathematical properties. For example the set of formulas is an infinite inductive set, allowing to carry out proofs by induction on the complexity of formulas. The metatheory necessary to construct such simple set as the set of sentential formulas is in fact strictly stronger than first-order logic (the informal statement in the definition saying "nothing else is a formula" is equivalent to a second-order principle of induction, at the first-order level, like in arithmetics, non-standard objects cannot be eliminated; for more details, see (Béziau, 199?)).

The notion of equiformity does not seem a good identity criterion for formulas because the "identity" of the formula $F \wedge G$ does not lie in its apparent shape, but in the way it is built, which can be visualized by its decomposition tree. We can represent this formula in a different manner, for example in the Polish style: $\wedge FG$. There are many different ways of writing this *same* formula. This formula is not a set of equiform material inscriptions, it is an abstract object in a mathematical structure.(7) In the better case, a material sequence of signs is just one possible representation of the formula, and the physical shape of this sequence is not essentialy relevant.

3.3. Congruency

The two formulas $\forall xFx$ and $\forall yFy$ are, according the standard construction of the set of first-order formulas, two different formulas. Kleene says that these are *congruent* formulas (Kleene, 1952, p.153). Bourbaki in his *Theory of Sets* uses a different method of construction (with the help of the Bourbakian square), and according to it, there is only one formula corresponding to $\forall xFx$ and $\forall yFy$. In fact the Bourbakian construction can be seen as the result of factorizing the standard construction by Kleene's congruency. This identification can be justified by the fact that Kleene's congruency is really a congruency: two formulas which are congruent in the sense of Kleene are logically equivalent and logical equivalence is a congruency due to the replacement theorem.

Zermelo's axiom of choice (AC) is logically equivalent to Zorn's lemma (ZL), modulo the axioms of ZF. These two formulas are the *same* under this notion of equivalence. Can we say they are the *same* like when saying that $\forall xFx$ and $\forall yFy$ are the same or that the difference between ACx and ACy (two formulations of AC differing only by the name of one bounded variable) is similar to the difference between AC and ZL?

One can say that the difference between $\forall xFx$ and $\forall yFy$ is rather morphological and that they can indeed be identified at the morphological level, as shown by Bourbaki. But there are many other cases where the difference between two formulas seems morphological rather than logical and no morphological reduction is operated, for example the difference between a formula and its prenex form. Is the difference between AC and its prenex form of the same nature as the difference between AC and ZL?

P. Halmos (Halmos, 1956) suggests to call *proposition* a class of logically equivalent *sentences*. From this point of view Boolean algebra deals with propositions and classical sentential logic with sentences. (8) There are a lot of intermediate solutions, like Kleene's congruency, and other partial factorizations under logical equivalence. Using Halmos's terminology, the problem here is to know where drawing the line between propositions and sentences. One can consider ACx and ACy as two different sentences but as the same proposition, or as just one sentence. But can we consider that AC and its prenex form are the same sentence or that AC and ZL are the same sentence? P.Suppes in his "Congruency theory of propositions" (Suppes, 1986) defines a proposition as any class of congruent token utterances (staying at the level of written language, we can say here: any class of congruent physical sequence of signs) under a given congruency, and for him "two propositions are identical just when the utterances from which they are "abstracted" are congruent" (Suppes, 1986, p.279). Note that from this viewpoint sentences, taken as sets of equiform physical inscriptions, are propositions: a sentence is a class of congruent physical objects under equiformity. Of course this is a question of terminology and one could say that this kind of congruency defines sentences, but then the problem is to know what are the level of abstraction and the kind of congruency which distinguish sentences from propositions.

It seems that in natural language, as well as in formal language, the congruency which leads to what is called sentence, is a mixture of syntax and semantics, more complex than equiformity. And maybe this kind of congruency is not of an essentialy different nature than the congruency which leads to "propositions" or "facts." (9)

4. A possible solution for identity of sentences in logic

In logic, sentences are considered from the point of view of truth, thus the related congruency should be connected with this notion. A schematic formula is the class of all substitutional instances of a given formula, a class of formulas which have the same *logical form*. As it is known, the fundamental feature of formal logic is that logical truths are invariable under substitution. This means that "to have the same logical form" is a congruency relatively to the notion of logical truth.

Logical truths are rather schematic formulas than formulas, in the sense that the logician wants to characterize the logical forms which are logical truths. This notion of logical form has nothing to do with geometrical form of material inscriptions and maybe the right identity criterion for sentences in logic (formulas) is logical equiformity and not material equiformity.

The notion of logical form has been precisely characterized in mathematical terms in (Los and Suszko, 1958) and is closely connected with the structure which defines formulas : substitutions are endomorphims of the absolute free algebra of formulas. Moreover the identity criterion "to have the same logical form" and the "identity" of logical forms coincide.

Identity is here perfectly well-defined in mathematical terms. This definition is more rigorous than Presburger's informal definitions of material equiformity. Of course it presupposes mathematics, but to define rigorously material equiformity one has to use not only mathematics (geometry, sets, etc.) but also physics, and maybe alchemy.

Notes

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(2) Let us quote here some parts of a letter of Tarski to Neurath which gives some interesting informations about the development of the concept of sentences as material objects within the Polish school and that will be of interest for our discussion later on: "You write in your letter that the thesis, according to which sentences, punctuation marks, etc., are physical pictures, was debated in the Vienna Circle still before my coming to Vienna and met in part a positive judgment. I certainly have no right to contest this. I only want to observe, that this thesis has been prevailing in Warsaw for years before the years 1928/29. Also it appears to me that the characterization of pictures of sentences as "ornaments" is not an original Viennese formulation: among us, we spoke of figures of speech as "arabesques"; Lukasiewicz gave a lecture (around 1925), in which he endeavored to show just this, that one can look upon pictures as sentences simply as "ornaments": he brought along to his lecture a large number of differently colored squares, trapezoids, etc., and developed a system of sentential calculus, in which he employed these "ornaments" as logical variables and constants." (Tarski, 1936)

(3) These definitions of Presburger appear respectively in a book by Lukasiewicz and in a book by Ajdukiewicz (cf. Zygmunt, 1991, p.216) ; names for these defintions are ours.

(4) We can also wonder if "Snow is white" and "*Snow is white*" are equiform (cf. Curry, 1963, p.169).

(5) And in quantum physics the notion of identity criterion is problematic, as stressed e.g. by Schrödinger.

(6) Chevalley even claims : "The idea of a symbol which is "the same", althought written in different places and at different time, is not at all an idea that stands by itself. Not only can this idea not possible be realized, but its content is absurd." (*Ibid*)

(7) At the level of sentential logic, this structure is an absolute free algebra, as it is known for years in Poland.

(8) Wittgenstein calls "Satz" what Halmos calls "proposition" (Cf. *Tractacus* 5.141), although apparently he is dealing rather with what we now call sentential logic than Boolean algebra.

(9) In French grammar the word "proposition" is currently used and not the word "énoncé" and in fact there is no distinction equivalent to the English distinction sentence/proposition. Quite the same can be said of German language, for example "Satz" has a wider meaning than the English "sentence", as shows the expression "Der Satz vom Grund".

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