Functional central limit theorems for single-stage sampling designs

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Notation

- Finite population of N individuals: $U_N = \{1, 2, \ldots, N\}$
  A sample is selected according to some sampling design
- A *sampling design* is a probability measure $p : \mathfrak{A}_N \mapsto [0, 1]$
  $(\mathcal{S}_N, \mathfrak{A}_N, p)$ is called the design probability space,
  where $\mathcal{S}_N$ is the collection of all possible subsets $s \subset U_N$; $\mathfrak{A}_N = \sigma(\mathcal{S}_N)$
- inclusion indicators $\xi_i(s) = \begin{cases} 1 & \text{if } i \in s \\ 0 & \text{otherwise} \end{cases}$
- inclusion probabilities

$$\pi_i = \mathbb{E}_d[\xi_i] = \sum_{s \in S_{(i)}} p(s),$$

  where $\mathcal{S}_{(i)}$ is the collection of all samples containing i.
- $\pi_{i_1 \ldots i_k} = \mathbb{E}_d[\xi_{i_1} \cdots \xi_{i_k}] = \displaystyle\sum_{s \in S_{(i_1)} \cap \cdots \cap S_{(i_k)}} p(s)$

Population features and estimators

- One observes a particular quantity y for each individual in the selected sample: $y_{i_1}, y_{i_2}, \ldots, y_{i_n}$

  One is interested in population features, e.g., $\overline{y} = \dfrac{1}{N}\sum_{i=1}^{N} y_i$

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- Horvitz-Thompson estimator:

$$\overline{y}_{\mathrm{HT}} = \frac{1}{N}\sum_{i\in S}\frac{y_i}{\pi_i} = \frac{1}{N}\sum_{i=1}^{N}\frac{\xi_i y_i}{\pi_i}$$

- Hajèk estimator:

$$\overline{y}_{\mathrm{HJ}} = \frac{1}{\widehat{N}}\sum_{i=1}^{N}\frac{\xi_i}{\pi_i}y_i, \qquad \widehat{N} = \sum_{i=1}^{N}\frac{\xi_i}{\pi_i}$$

Estimators as statistical functionals

- Horvitz-Thompson empirical cdf

$$\mathbb{F}_N^{\mathrm{HT}}(t) = \frac{1}{N}\sum_{i=1}^{N} \frac{\xi_i \mathbb{1}_{\{y_i \leq t\}}}{\pi_i}, \quad t \in \mathbb{R}.$$

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- Note that

$$\overline{y}_{\mathrm{HT}} = \frac{1}{N}\sum_{i=1}^{N} \frac{\xi_i y_i}{\pi_i} = \phi(\mathbb{F}_N^{\mathrm{HT}}),$$

$$\overline{y}_{\mathrm{HJ}} = \frac{1}{\widehat{N}}\sum_{i=1}^{N} \frac{\xi_i}{\pi_i} y_i = \phi(\mathbb{F}_N^{\mathrm{HT}}),$$

where $\phi(F) = \int y \, \mathrm{d}F(y)$

Limit distribution

- Of interest is the limit distribution (as $N \to \infty$) of estimators, such as

$$\sqrt{n}\left\{\frac{1}{N}\sum_{i=1}^{N}\frac{\xi_i y_i}{\pi_i} - \frac{1}{N}\sum_{i=1}^{N} y_i\right\}$$

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- This suggest to use some kind of functional delta-method, i.e., if
  - a functional CLT holds, e.g., $\sqrt{n}(\mathbb{F}_N^{\mathrm{HT}} - \mathbb{F}_N) \rightsquigarrow \mathbb{G}$
  - ϕ is differentiable in some sense

  then

$$\sqrt{n}\left\{\phi(\mathbb{F}_N^{\mathrm{HT}}) - \phi(\mathbb{F}_N)\right\} \rightsquigarrow \phi'(\mathbb{G})$$

Need for functional central limit theorems

- Functional CLT is sometimes *assumed*,

  e.g., see Dell & d'Haultfœuille (2008), Barett & Donald (2009).

- Functionals of interest are *more complex* than the mean functional, e.g.,
  - *Poverty rate*

    $$\phi(F) = F\left(\beta F^{-1}(\alpha)\right),$$

    for fixed $0 < \alpha, \beta < 1$, where $F^{-1}(\alpha) = \inf\{t : F(t) \geq \alpha\}$.

  - *Gini index*

    $$\phi(F) = \frac{2\int_0^1 \left(\int_0^q F^{-1}(t)\,\mathrm{d}t\right)\,\mathrm{d}q}{\int y\,\mathrm{d}F(y)} - 1$$

Super-population setup

Rubin-Bleuer & Schiopu Kratina (2005)

- For each individual in the population we observe $(y_i, z_i) \in \mathbb{R} \times \mathbb{R}_+^q$
  The pairs (y_i, z_i) are realizations of iid (Y_i, Z_i) on $(\Omega, \mathfrak{F}, \mathbb{P}_m)$

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  The pairs (y_i, z_i) are realizations of iid (Y_i, Z_i) on $(\Omega, \mathfrak{F}, \mathbb{P}_m)$
- The sampling design is a random measure on $(\mathcal{S}_N \times \Omega, \mathfrak{A}_N \times \mathfrak{F})$

$$p(s, Z_1(\omega), \dots, Z_N(\omega)), \quad s \in \mathcal{S}_N, \omega \in \Omega$$

$$\mathbb{P}_d(A, \omega) = \sum_{s \in A} p(s, Z_1(\omega), \dots, Z_N(\omega)), \quad A \subset \mathcal{S}_N, \omega \in \Omega$$

- Product space $(\mathcal{S}_N \times \Omega, \mathfrak{A}_N \times \mathfrak{F}, \mathbb{P}_{d,m})$ with probability measure

$$\begin{aligned} \mathbb{P}_{d,m}(\{s\} \times E) &= \int_E p(s, Z_1(\omega), \dots, Z_N(\omega)) \,\mathrm{d}\mathbb{P}_m(\omega) \\ &= \int_E \mathbb{P}_d(s, \omega) \,\mathrm{d}\mathbb{P}_m(\omega), \end{aligned} \qquad s \in \mathcal{S}_N, E \in \mathfrak{F}$$

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- $\mathbb{F}_N^{\mathrm{HT}}(t; s, \omega) = \dfrac{1}{N} \sum_{i=1}^{N} \dfrac{\xi_i(s) \mathbb{1}_{\{Y_i(\omega) \leq t\}}}{\pi_i(\omega)}$ is random element on $\mathcal{S}_N \times \Omega$

Existing results

- Horvitz-Thompson empirical processes indexed by a class of functions
  - Breslow & Wellner (2007); Saegusa & Wellner (2013); Bertail, Chautru and Clémençon (2017)
    Weighted bootstrap approach
  - Bertail & Rebecq (in preparation)
    Sampling designs with negatively associated inclusion indicators

- Empirical processes $\sqrt{n}(\mathbb{F}_N^{\mathrm{HT}} - F)$ and $\sqrt{n}(\mathbb{F}_N^{\mathrm{HJ}} - F)$ indexed by $t \in \mathbb{R}$
  - Fevrier and Ragache (2001); Wang (2012); Conti *et al* (2014, 2016)
    Incomplete and contains some (minor) gaps

- Taylor made results for specific statistical functionals
  - Bhattacharya (2007); Davidson (2009); Bhattacharya & Mazumder (2011)

Empirical processes of our interest

Recall

$$\mathbb{F}_N^{\mathrm{HT}}(t) = \frac{1}{N}\sum_{i=1}^{N}\frac{\xi_i \mathbb{1}_{\{Y_i\leq t\}}}{\pi_i}, \quad \mathbb{F}_N^{\mathrm{HJ}}(t) = \frac{1}{\widehat{N}}\sum_{i=1}^{N}\frac{\xi_i \mathbb{1}_{\{Y_i\leq t\}}}{\pi_i},$$

$$\mathbb{F}_N(t) = \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{\{Y_i\leq t\}}, \qquad F = \text{ the cdf of } Y_1,\dots,Y_N$$

$$n = \mathbb{E}_d[n_s] = \mathbb{E}_d\left[\sum_{i=1}^{N}\xi_i\right] = \sum_{i=1}^{N}\pi_i$$

AIM: establish functional CLT's with potential applications to a large class of single-stage sampling designs, for

- Horvitz-Thompson empirical processes $\begin{cases}\sqrt{n}\left(\mathbb{F}_N^{\mathrm{HT}} - \mathbb{F}_N\right)\\ \sqrt{n}\left(\mathbb{F}_N^{\mathrm{HT}} - F\right)\end{cases}$
- Hajèk empirical processes $\begin{cases}\sqrt{n}\left(\mathbb{F}_N^{\mathrm{HJ}} - \mathbb{F}_N\right)\\ \sqrt{n}\left(\mathbb{F}_N^{\mathrm{HJ}} - F\right)\end{cases}$

Weak convergence of the HT empirical process

Weak convergence of the process

$$\mathbb{X}_N(t) = \sqrt{n}\left\{\mathbb{F}_N^{\mathrm{HT}}(t) - \mathbb{F}_N(t)\right\} = \frac{\sqrt{n}}{N}\sum_{i=1}^{N}\left(\frac{\xi_i}{\pi_i} - 1\right)\mathbb{1}_{\{Y_i \leq t\}},$$

is established

Weak convergence of the HT empirical process

Weak convergence of the process

$$\mathbb{X}_N(t) = \sqrt{n}\left\{\mathbb{F}_N^{\mathrm{HT}}(t) - \mathbb{F}_N(t)\right\} = \frac{\sqrt{n}}{N}\sum_{i=1}^{N}\left(\frac{\xi_i}{\pi_i} - 1\right)\mathbb{1}_{\{Y_i \le t\}},$$

is established by proving

1. a tightness condition (see Billingsley (1968))

$$\mathbb{E}_{d,m}\left[\left(\mathbb{X}_N(t) - \mathbb{X}_N(t_1)\right)^2\left(\mathbb{X}_N(t_2) - \mathbb{X}_N(t)\right)^2\right] \le K\Big(F(t_2) - F(t_1)\Big)^2.$$

2. weak convergence of finite dimensional projections

$$(\mathbb{X}_N(t_1), \dots, \mathbb{X}_N(t_k)) \rightsquigarrow (\mathbb{X}(t_1), \dots, \mathbb{X}(t_k))$$

where $\mathbb{X}$ is a mean zero Gaussian process

Sufficient conditions for tightness

(C1) *There exist K_1, K_2, such that*

$$0 < K_1 \leq \frac{N\pi_i}{n} \leq K_2 < \infty, \qquad \omega\text{-a.s.}$$

- Upper bound:
  Because $N\pi_i/n \leq N/n$, the UPB is immediate if $n/N \to \lambda > 0$
- Lower bound:
  Because $N\pi_i/n \geq \pi_i$, the LWB is immediate if $\pi_i \geq \pi^* > 0$

Sufficient conditions for tightness

$$
\begin{aligned}
&\text{(C2)}\quad \limsup_{N\to\infty} \frac{N^2}{n} \max_{(i,j)\in D_{2,N}} \left|\mathbb{E}_d(\xi_i-\pi_i)(\xi_j-\pi_j)\right| < \infty \\
&\text{(C3)}\quad \limsup_{N\to\infty} \frac{N^3}{n^2} \max_{(i,j,k)\in D_{3,N}} \left|\mathbb{E}_d(\xi_i-\pi_i)(\xi_j-\pi_j)(\xi_k-\pi_k)\right| < \infty \\
&\text{(C4)}\quad \limsup_{N\to\infty} \frac{N^4}{n^2} \max_{(i,j,k,l)\in D_{4,N}} \left|\mathbb{E}_d(\xi_i-\pi_i)(\xi_j-\pi_j)(\xi_k-\pi_k)(\xi_l-\pi_l)\right| < \infty
\end{aligned}
$$

ω-a.s., where

$$
D_{\nu,N} = \Big\{(i_1,i_2,\ldots,i_\nu) \in \{1,2,\ldots,N\} : i_1,i_2,\ldots,i_\nu \text{ are all different}\Big\}
$$

Sufficient conditions for tightness

$$\text{(C2)} \quad \limsup_{N\to\infty} \frac{N^2}{n} \max_{(i,j)\in D_{2,N}} \left|\mathbb{E}_d(\xi_i - \pi_i)(\xi_j - \pi_j)\right| < \infty$$

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- Breidt & Opsomer (2000):

  Conditions similar to (C2)-(C4) hold for SRS

Sufficient conditions for tightness

$$\text{(C2)} \quad \limsup_{N\to\infty} \frac{N^2}{n} \max_{(i,j)\in D_{2,N}} \left| \mathbb{E}_d(\xi_i - \pi_i)(\xi_j - \pi_j) \right| < \infty$$

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- Boistard, L & Ruiz-Gazen (2012)

  (C2)-(C4) can be reformulated into inclusion probabilities.

  For any $k \geq 2$ and $\{i_1, \dots, i_k\} \subset \{1, \dots, N\}$:

$$\mathbb{E}_d(\xi_{i_1} - \pi_{i_1}) \cdots (\xi_{i_k} - \pi_{i_k})$$
$$= \sum_{m=2}^{k} (-1)^{k-m} \sum_{i_1 \cdots i_m \in D_{m,k}} (\pi_{i_1 \dots i_m} - \pi_{i_1} \cdots \pi_{i_m}) \pi_{i_{m+1}} \cdots \pi_{i_k}$$

Sufficient conditions for tightness under rejective sampling

Boistard, L & Ruiz-Gazen (2012)

Suppose $d_N = \sum_{i=1}^N \pi_i(1-\pi_i) \to \infty$. Then

(i) for any $A_k = \{i_1 \cdots i_k\} \subset \{1, 2, \ldots, N\}$, under a rejective sampling design,

$$\pi_{i_1 \cdots i_k} - \pi_{i_1} \cdots \pi_{i_k} = -\frac{\pi_{i_1} \cdots \pi_{i_k}}{d_N} \sum_{i,j \in A_k : i<j} (1-\pi_i)(1-\pi_j) + O(d_N^{-2})$$

uniformly in $i_1 \cdots i_k$;

(ii) for any $k \geq 3$ and any positive integers n_j, $j = 1, 2, \ldots, k$, under a rejective sampling design,

$$\mathbb{E}_d \left[\prod_{j=1}^k (\xi_{i_j} - \pi_{i_j})^{n_j} \right] = O(d_N^{-2})$$

Sufficient conditions for tightness under rejective sampling

(C1) *There exist* K_1, K_2*, such that* $0 < K_1 \leq \frac{N\pi_i}{n} \leq K_2 < \infty$

(C2) $\limsup_{N\to\infty} \frac{N^2}{n} \max_{(i,j)\in D_{2,N}} \left|\mathbb{E}_d(\xi_i - \pi_i)(\xi_j - \pi_j)\right| < \infty$

(C3) $\limsup_{N\to\infty} \frac{N^3}{n^2} \max_{(i,j,k)\in D_{3,N}} \left|\mathbb{E}_d(\xi_i - \pi_i)(\xi_j - \pi_j)(\xi_k - \pi_k)\right| < \infty$

(C4) $\limsup_{N\to\infty} \frac{N^4}{n^2} \max_{(i,j,k,l)\in D_{4,N}} \left|\mathbb{E}_d(\xi_i - \pi_i)(\xi_j - \pi_j)(\xi_k - \pi_k)(\xi_l - \pi_l)\right| < \infty$

- For rejective sampling, condition (C1) together with

$$d_N = \sum_{i=1}^{N} \pi_i(1-\pi_i) \to \infty; \qquad \frac{n}{d_N} = O(1); \qquad \frac{N^2}{nd_N} = O(1)$$

imply conditions (C2)-(C4)

Sufficient conditions for tightness under rejective sampling

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- For rejective sampling, condition (C1) together with

$$\frac{d_N}{N} = \frac{1}{N}\sum_{i=1}^{N} \pi_i(1-\pi_i) \to d > 0; \qquad \frac{n}{N} \to \lambda > 0$$

imply conditions (C2)-(C4)

Sufficient conditions for convergence of fidis

Define

$$S_N^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\pi_{ij} - \pi_i \pi_j}{\pi_i \pi_j} V_i V_j \qquad \left(= \mathbb{V}_d \left[\frac{1}{N} \sum_{i=1}^{N} \frac{\xi_i V_i}{\pi_i} \right] \right)$$

(HT1) *For any sequence of bounded i.i.d random variables* $V_1, V_2, \ldots$,

$$\frac{1}{S_N} \left(\frac{1}{N} \sum_{i=1}^{N} \frac{\xi_i V_i}{\pi_i} - \frac{1}{N} \sum_{i=1}^{N} V_i \right) \to N(0,1), \qquad \omega\text{-a.s.},$$

in distribution under $\mathbb{P}_d$.

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in distribution under $\mathbb{P}_d$.

- SRS, Poisson sampling, rejective sampling
  Hájek (1964); Víšek (1979), Thompson (1997); Fuller (2009); Prášková & Sen (2009)
- High entropy designs
  Berger (1998)

Sufficient conditions for convergence of fidis

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in distribution under $\mathbb{P}_d$.

- For high entropy designs, if (C1) holds together with

$$\frac{n}{d_N} = O(1); \qquad \frac{N}{d_N^2} \to 0; \qquad n^2 S_N^2 \to \infty$$

where $d_N = \sum_{i=1}^{n} \pi_i (1 - \pi_i)$, then (HT1) is satisfied.

Sufficient conditions for convergence of fidis

(HT2) *For all $k \in \{1,2,\ldots\}$, $i = 1,\ldots,k$ and $t_1,\ldots,t_k$, there exists a deterministic matrix $\mathbf{\Sigma}_k$, such that*

$$\lim_{N\to\infty} \frac{n}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\pi_{ij} - \pi_i\pi_j}{\pi_i\pi_j} \mathbf{Y}_{ik}\mathbf{Y}_{jk}^t = \mathbf{\Sigma}_k, \quad \omega\text{-a.s.}$$

where

$$\mathbf{Y}_{ik}^t = \left(\mathbb{1}_{\{Y_i \le t_1\}}, \ldots, \mathbb{1}_{\{Y_i \le t_k\}}\right)$$

- Similar condition is used in

  Krewski & Rao (1981); Francisco & Fuller (1991); Deville & Särndal (1992)

- When (C1)-(C2) hold then

$$\mathbf{\Sigma}_k = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}_m \left[n \frac{\pi_{ij} - \pi_i\pi_j}{\pi_i\pi_j} \mathbf{Y}_{ik}\mathbf{Y}_{jk}^t \right]$$

Horvitz-Thompson empirical process centered by $\mathbb{F}_N$

Let $D(\mathbb{R})$ be the space of càdlàg functions on $\mathbb{R}$ with the Skorohod topology.

Theorem

If conditions (C1)-(C4) and (HT1)-(HT2) hold, then

$$\sqrt{n}(\mathbb{F}_N^{\mathrm{HT}} - \mathbb{F}_N) \rightsquigarrow \mathbb{G}^{\mathrm{HT}}$$

in $D(\mathbb{R})$, where $\mathbb{G}^{\mathrm{HT}}$ is a mean zero Gaussian process with covariance kernel

$$\mathbb{E}_{d,m}\mathbb{G}^{\mathrm{HT}}(s)\mathbb{G}^{\mathrm{HT}}(t) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}_m \left[n \frac{\pi_{ij} - \pi_i \pi_j}{\pi_i \pi_j} \mathbb{1}_{\{Y_i \leq s\}} \mathbb{1}_{\{Y_j \leq t\}} \right]$$

for $s, t \in \mathbb{R}$.

Sufficient conditions for convergence of fidis

If n and π_i, π_{ij} do not depend on ω, then we can write

$$S_N^2 = \frac{1}{N^2}\sum_{i=1}^N \sum_{j=1}^N \frac{\pi_{ij} - \pi_i\pi_j}{\pi_i\pi_j} V_i V_j$$

$$= \frac{1}{N^2}\sum_{i=1}^N \left(\frac{1}{\pi_i} - 1\right) V_i^2 + \frac{1}{N^2}\sum\sum_{i\neq j} \frac{\pi_{ij} - \pi_i\pi_j}{\pi_i\pi_j} V_i V_j$$

and instead of (HT2) we require

(HT2*) *There exist constants* $\mu_{\pi 1}$, $\mu_{\pi 2} \in \mathbb{R}$ *such that*

$$\frac{n}{N^2}\sum_{i=1}^N \left(\frac{1}{\pi_i} - 1\right) \to \mu_{\pi 1}, \qquad \frac{n}{N^2}\sum\sum_{i\neq j} \frac{\pi_{ij} - \pi_i\pi_j}{\pi_i\pi_j} \to \mu_{\pi 2}.$$

As a consequence:

$$nS_N^2 \to \mu_{\pi 1}\mathbb{E}_m[V_1^2] + \mu_{\pi 2}(\mathbb{E}_m V_1)^2$$

Horvitz-Thompson empirical process centered by $\mathbb{F}_N$

Theorem

Suppose that n and π_i, π_{ij} do not depend on ω.

If conditions (C1)-(C4) and (HT1)-(HT2) hold, then*

$$\sqrt{n}(\mathbb{F}_N^{\mathrm{HT}} - \mathbb{F}_N) \rightsquigarrow \mathbb{G}^{\mathrm{HT}}$$

in $D(\mathbb{R})$, where $\mathbb{G}^{\mathrm{HT}}$ is a mean zero Gaussian process with covariance kernel

$$\mathbb{E}_{d,m}\mathbb{G}^{\mathrm{HT}}(s)\mathbb{G}^{\mathrm{HT}}(t) = \mu_{\pi 1}F(s \wedge t) + \mu_{\pi 2}F(s)F(t), \quad s, t \in \mathbb{R};$$

Horvitz-Thompson empirical process centered by F

Weak convergence of the fidis of $\sqrt{n}(\mathbb{F}_N^{\mathrm{HT}} - F)$ requires a CLT for

$$\sqrt{n}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{\xi_i V_i}{\pi_i} - \mu_V\right), \qquad \text{where } \mu_V = \mathbb{E}_m(V_i)$$

Decompose as follows

$$\sqrt{n}S_N \cdot \underbrace{\frac{1}{S_N}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{\xi_i V_i}{\pi_i} - \frac{1}{N}\sum_{i=1}^{N} V_i\right)}_{\to N(0,1),\ \omega\text{-a.s., by (HT1)}} + \frac{\sqrt{n}}{\sqrt{N}} \cdot \underbrace{\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} V_i - \mu_V\right)}_{\to N(0,\sigma_V^2) \text{ by the traditional CLT}}$$

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Combine limits using a result from Rubin-Bleuer & Schiopu Kratina (2005). This requires extra conditions

(HT3) $n/N \to \lambda \in [0,1]$, *$\omega$-a.s.*

(HT4) $\boldsymbol{\Sigma}_k$ *is positive definite*

Horvitz-Thompson empirical process centered by F

Theorem

If conditions (C1)-(C4) and (HT1)-(HT4) hold, then

$$\sqrt{n}(\mathbb{F}_N^{\mathrm{HT}} - F) \rightsquigarrow \mathbb{G}_F^{\mathrm{HT}}$$

in $D(\mathbb{R})$, where $\mathbb{G}_F^{\mathrm{HT}}$ is a mean zero Gaussian process with covariance kernel

$$\mathbb{E}_{d,m}\mathbb{G}_F^{\mathrm{HT}}(s)\mathbb{G}_F^{\mathrm{HT}}(t) = \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}_m \left[n \frac{\pi_{ij} - \pi_i \pi_j}{\pi_i \pi_j} \mathbb{1}_{\{Y_i \leq s\}} \mathbb{1}_{\{Y_j \leq t\}} \right] + \lambda \Big\{ F(s \wedge t) - F(s)F(t) \Big\}$$

for $s, t \in \mathbb{R}$.

Horvitz-Thompson empirical process centered by F

When n and π_i, π_{ij} do not depend on ω, instead of (HT4) we assume

(HT4*) $$\lim_{N\to\infty} \frac{n}{N^2} \sum_{i=1}^{N} \left(\frac{1}{\pi_i} - 1 \right) = \mu_{\pi 1} > 0$$

Theorem

Suppose that n and π_i, π_{ij} do not depend on ω.

If conditions (C1)-(C4) and (HT1), (HT3), (HT2), and (HT4*) hold, then*

$$\sqrt{n}(\mathbb{F}_N^{\mathrm{HT}} - F) \rightsquigarrow \mathbb{G}_F^{\mathrm{HT}}$$

in $D(\mathbb{R})$, where $\mathbb{G}_F^{\mathrm{HT}}$ is a mean zero Gaussian process with covariance kernel

$$\mathbb{E}_{d,m} \mathbb{G}_F^{\mathrm{HT}}(s) \mathbb{G}_F^{\mathrm{HT}}(t) = (\mu_{\pi 1} + \lambda) F(s \wedge t) + (\mu_{\pi 2} - \lambda) F(s) F(t), \quad s, t \in \mathbb{R};$$

Hajèk empirical processes

- Similar results hold for the Hajèk empirical processes
- Useful relationships

$$\sqrt{n}\left(\mathbb{F}_N^{\mathrm{HJ}}(t) - \mathbb{F}_N(t)\right) = \mathbb{Y}_N(t) + \left(\frac{N}{\widehat{N}} - 1\right)\mathbb{G}_N^{\pi}(t)$$

$$\sqrt{n}\left(\mathbb{F}_N^{\mathrm{HJ}}(t) - F(t)\right) = \frac{N}{\widehat{N}}\mathbb{G}_N^{\pi}(t)$$

where

$$\mathbb{Y}_N(t) = \frac{\sqrt{n}}{N}\sum_{i=1}^{N}\left(\frac{\xi_i}{\pi_i} - 1\right)\left(\mathbb{1}_{\{Y_i \leq t\}} - F(t)\right)$$

$$\mathbb{G}_N^{\pi}(t) = \frac{\sqrt{n}}{N}\sum_{i=1}^{N}\frac{\xi_i}{\pi_i}\left(\mathbb{1}_{\{Y_i \leq t\}} - F(t)\right)$$

Hajèk empirical processes

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$$\sqrt{n}\left(\mathbb{F}_N^{\mathrm{HJ}}(t) - \mathbb{F}_N(t)\right) = \mathbb{Y}_N(t) + \left(\frac{N}{\widehat{N}} - 1\right)\mathbb{G}_N^{\pi}(t) \approx \mathbb{Y}_N(t)$$

$$\sqrt{n}\left(\mathbb{F}_N^{\mathrm{HJ}}(t) - F(t)\right) = \frac{N}{\widehat{N}}\mathbb{G}_N^{\pi}(t) \approx \mathbb{G}_N^{\pi}(t)$$

where

$$\mathbb{Y}_N(t) = \frac{\sqrt{n}}{N}\sum_{i=1}^{N}\left(\frac{\xi_i}{\pi_i} - 1\right)\left(\mathbb{1}_{\{Y_i \leq t\}} - F(t)\right)$$

$$\mathbb{G}_N^{\pi}(t) = \frac{\sqrt{n}}{N}\sum_{i=1}^{N}\frac{\xi_i}{\pi_i}\left(\mathbb{1}_{\{Y_i \leq t\}} - F(t)\right)$$

High entropy designs

Based on results by Berger (1998a, 1998b, 2011)

The entropy of a sampling design P is defined as

$$H(P) = \sum_{s \in \mathcal{S}_N} P(s) \log P(s)$$

- Given inclusion probabilities $\pi_1, \ldots, \pi_N$, the rejective sampling design R maximizes the entropy among all sampling designs with inclusion probabilities $\pi_1, \ldots, \pi_N$.
- Divergence of a sampling design P from the rejective design R is measured by

$$D(P||R) = \sum_{s \in \mathcal{S}_N} P(s) \log \left(\frac{P(s)}{R(s)} \right)$$

- P is called a high entropy design, if $D(P||R) \to 0$, as $N \to \infty$.

High entropy designs

Theorem

Let P be a high entropy design.
Suppose that, instead of (C2)-(C4) and (HT1), P satisfies

$$\frac{1}{N}\sum_{i=1}^{N}\pi_i(1-\pi_i)\to d>0; \qquad \frac{n}{N}\to\lambda>0; \qquad n^2S_N^2\to\infty$$

as well as all other former conditions.

Then all the previous functional CLT's are valid.

Hadamard differentiability and the functional δ-method

Let $\mathbb{D}$ and $\mathbb{E}$ be metrizable topological vector spaces

Definition

A map $\phi : \mathbb{D}_\phi \subset \mathbb{D} \mapsto \mathbb{E}$ is called *Hadamard differentiable* at $\theta \in \mathbb{D}_\phi$ tangentially to a set $\mathbb{D}_0 \subset \mathbb{D}$, if there exists a continuous linear map $\phi'_\theta : \mathbb{D} \mapsto \mathbb{E}$ such that

$$\frac{\phi(\theta + t_n h_n) - \phi(\theta)}{t_n} \to \phi'_\theta(h), \quad \text{as } n \to \infty$$

for all $t_n \to 0$ and $h_n \to h \in \mathbb{D}_0$ such that $\theta + t_n h_n \in \mathbb{D}_\phi$ for every n.

Hadamard differentiability and the functional δ-method

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for all $t_n \to 0$ and $h_n \to h \in \mathbb{D}_0$ such that $\theta + t_n h_n \in \mathbb{D}_\phi$ for every n.

Theorem (functional δ-method)

Suppose $\phi : \mathbb{D}_\phi \subset \mathbb{D} \mapsto \mathbb{E}$ is Hadamard differentiable at θ tangentially to $\mathbb{D}_0$. Let $r_n(X_n - \theta) \rightsquigarrow X$ as $r_n \to \infty$, where $X \in \mathbb{D}_0$ is separable. Then $r_n\big(\phi(X_n) - \phi(\theta)\big) \rightsquigarrow \phi'_\theta(X)$

see, e.g., van der Vaart and Wellner (1996); van der Vaart (2000)

Example: the poverty rate

- $\mathbb{D}_\phi$ consists of $F \in D(\mathbb{R})$ that are non-decreasing, with $f(F^{-1}(\alpha)) > 0$.
  The poverty rate is defined as

$$\phi(F) = F\left(\beta F^{-1}(\alpha)\right)$$

  for fixed $0 < \alpha, \beta < 1$, where $F^{-1}(\alpha) = \inf\{t : F(t) \geq \alpha\}$.

Example: the poverty rate

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for fixed $0 < \alpha, \beta < 1$, where $F^{-1}(\alpha) = \inf\{t : F(t) \geq \alpha\}$.

- ϕ is Hadamard-differentiable at F with derivative

$$\phi'_F(h) = -\beta \frac{f(\beta F^{-1}(\alpha))}{f(F^{-1}(\alpha))} h(F^{-1}(\alpha)) + h(\beta F^{-1}(\alpha))$$

where h is continuous at $F^{-1}(\alpha)$

HT estimator for the poverty rate $\phi(F)$

- According to the δ-method

$$\sqrt{n}(\phi(\mathbb{F}_N^{\mathrm{HT}}) - \phi(F)) \rightsquigarrow \phi'_F\left(\mathbb{G}_F^{\mathrm{HT}}\right)$$
$$= -\beta \frac{f(\beta F^{-1}(\alpha))}{f(F^{-1}(\alpha))} \mathbb{G}_F^{\mathrm{HT}}(F^{-1}(\alpha)) + \mathbb{G}_F^{\mathrm{HT}}(\beta F^{-1}(\alpha)),$$

where $\mathbb{G}_F^{\mathrm{HT}}$ is a mean zero Gaussian process with covariance kernel

$$\mathbb{E}_{d,m} \mathbb{G}_F^{\mathrm{HT}}(s) \mathbb{G}_F^{\mathrm{HT}}(t) = (\mu_{\pi 1} + \lambda) F(s \wedge t) + (\mu_{\pi 2} - \lambda) F(s) F(t)$$

- $\phi'_F\left(\mathbb{G}_F^{\mathrm{HT}}\right)$ is a mean zero normal random variable with variance

$$\mathbb{E}\left[\phi'_F\left(\mathbb{G}_F^{\mathrm{HT}}\right)^2\right] = \cdots$$

that can be derived from the covariance kernel of $\mathbb{G}_F^{\mathrm{HT}}$

Simulation study

- Populations are generated from a standard exponential distribution
  - Population size $N = 10\,000$ and 1000
- Sampling designs
  - sample size $n = 500$, 100, and 50
  - Simple random sampling without replacement (SI)
  - Bernoulli sampling (BE) with parameter n/N
  - Poisson sampling (PO) with $\pi_i = 0.4n/N$ for half of the population and $\pi_i = 1.6n/N$ for the other half
- Replicating $N_R = 1000$ populations.

  For each population, $n_R = 1000$ samples are drawn.
- Poverty rate
  - $\phi(F) = F\left(\beta F^{-1}(\alpha)\right) = 1 - \exp(\beta \log(1-\alpha))$
  - Estimators $\widehat{\phi}_{\mathrm{HT}} = \phi(\mathbb{F}_N^{\mathrm{HT}})$ and $\widehat{\phi}_{\mathrm{HJ}} = \phi(\mathbb{F}_N^{\mathrm{HJ}})$
  - Population and model parameters $\phi(\mathbb{F}_N)$ and $\phi(F)$

Relative bias

Define the relative bias w.r.t. $\phi(F)$ in percentages by

$$\mathrm{RB}_F(\widehat{\phi}) = \frac{100}{N_R\, n_R} \sum_{i=1}^{N_R} \sum_{j=1}^{n_R} \frac{\widehat{\phi}_{ij} - \phi(F)}{\phi(F)}$$

Here, $\widehat{\phi}_{ij}$ is either $\widehat{\phi}_{\mathrm{HT}}$ or $\widehat{\phi}_{\mathrm{HJ}}$ for the ith generated population and jth drawn sample.

Define the relative bias w.r.t. $\phi(\mathbb{F}_N)$ in percentages by

$$\mathrm{RB}_N(\widehat{\phi}) = \frac{100}{N_R\, n_R} \sum_{i=1}^{N_R} \sum_{j=1}^{n_R} \frac{\widehat{\phi}_{ij} - \phi(\mathbb{F}_{N_i})}{\phi(\mathbb{F}_{N_i})}$$

Here, $\phi(\mathbb{F}_{N_i})$ is the population parameter for the ith population, $i = 1, \ldots, N_R$

Relative bias

Table: RB (in %) of the HT and the HJ estimators for the finite population $\phi(\mathbb{F}_N)$ and the super-population $\phi(F)$ poverty rate parameter

			$N = 10\,000$			$N = 1000$		
			$n = 500$	$n = 100$	$n = 50$	$n = 500$	$n = 100$	$n = 50$
SI	HT-HJ	$\phi(\mathbb{F}_N)$	-0.17	-0.89	-1.82	-0.05	-0.84	-1.62
		$\phi(F)$	-0.20	-0.91	-1.86	-0.18	-0.72	-1.85
BE	HT	$\phi(\mathbb{F}_N)$	-0.12	-0.66	-1.29	0.01	-0.65	-1.12
		$\phi(F)$	-0.15	-0.68	-1.34	-0.12	-0.54	-1.36
	HJ	$\phi(\mathbb{F}_N)$	-0.17	-0.92	-1.87	-0.04	-0.88	-1.68
		$\phi(F)$	-0.20	-0.93	-1.92	-0.17	-0.76	-1.91
PO	HT	$\phi(\mathbb{F}_N)$	-0.05	-1.05	-2.06	-0.06	-0.30	-0.37
		$\phi(F)$	-0.08	-1.07	-2.11	-0.19	-0.19	-0.63
	HJ	$\phi(\mathbb{F}_N)$	-0.20	-1.27	-2.95	-0.04	-1.08	-1.99
		$\phi(F)$	-0.23	-1.28	-3.00	-0.17	-0.97	-2.23

Relative bias of variance estimator

Recall $\sqrt{n}(\widehat{\phi}_{\mathrm{HT}} - \phi(F)) \rightsquigarrow N(0, \mathrm{AV}(\widehat{\phi}_{\mathrm{HT}}))$, where

$$\begin{aligned}\mathrm{AV}(\widehat{\phi}_{\mathrm{HT}}) = \beta^2 \frac{f(\beta F^{-1}(\alpha))^2}{f(F^{-1}(\alpha))^2} \left(\gamma_{\pi 1}\alpha + \gamma_{\pi 2}\alpha^2\right) \\ + \gamma_{\pi 1}\phi(F) + \gamma_{\pi 2}\phi(F)^2 - 2\beta \frac{f(\beta F^{-1}(\alpha))}{f(F^{-1}(\alpha))}\phi(F)\big(\gamma_{\pi 1} + \gamma_{\pi 2}\alpha\big),\end{aligned}$$

where $\gamma_{\pi 1} = \mu_{\pi 1} + \lambda$ and $\gamma_{\pi 2} = \mu_{\pi 2} - \lambda$.

Relative bias of variance estimator

Recall $\sqrt{n}(\widehat{\phi}_{\mathrm{HT}} - \phi(F)) \rightsquigarrow N(0, \mathrm{AV}(\widehat{\phi}_{\mathrm{HT}}))$, where

$$\mathrm{AV}(\widehat{\phi}_{\mathrm{HT}}) = \beta^2 \frac{f(\beta F^{-1}(\alpha))^2}{f(F^{-1}(\alpha))^2} \left(\gamma_{\pi 1}\alpha + \gamma_{\pi 2}\alpha^2\right) + \gamma_{\pi 1}\phi(F) + \gamma_{\pi 2}\phi(F)^2 - 2\beta \frac{f(\beta F^{-1}(\alpha))}{f(F^{-1}(\alpha))}\phi(F)\big(\gamma_{\pi 1} + \gamma_{\pi 2}\alpha\big),$$

where $\gamma_{\pi 1} = \mu_{\pi 1} + \lambda$ and $\gamma_{\pi 2} = \mu_{\pi 2} - \lambda$.

Define the relative bias of the variance estimator

$$\mathrm{RB}(\widehat{\mathrm{AV}}(\widehat{\phi})) = \frac{100}{N_R\, n_R} \sum_{i=1}^{N_R} \sum_{j=1}^{n_R} \frac{\widehat{\mathrm{AV}}(\widehat{\phi}_{ij}) - \mathrm{AV}(\widehat{\phi})}{\mathrm{AV}(\widehat{\phi})}, \quad \widehat{\phi} = \widehat{\phi}_{\mathrm{HT}} \text{ or } \widehat{\phi}_{\mathrm{HJ}}$$

where $\widehat{\mathrm{AV}}(\widehat{\phi}_{ij})$ denotes the variance (plug-in) estimate for the ith generated population and the jth drawn sample.

Relative bias of variance estimator

Table: RB (in %) for the variance estimator of the HT and the HJ estimators for the poverty rate parameter

		$N = 10\,000$			$N = 1000$		
		$n = 500$	$n = 100$	$n = 50$	$n = 500$	$n = 100$	$n = 50$
SI	HT-HJ	−2.21	−3.08	−2.97	−2.25	−3.26	−3.00
BE	HT	−4.15	−5.11	−4.21	−3.31	−5.11	−4.19
	HJ	−2.22	−3.06	−3.03	−2.26	−3.24	−3.03
PO	HT	−4.43	−4.96	−3.45	−3.74	−5.72	−4.59
	HJ	−2.36	−3.43	−3.36	−2.44	−3.75	−4.13

Confidence intervals

Table: Coverage probabilities (in %) for 95% confidence intervals of the HT and the HJ estimators for the finite population $\phi(\mathbb{F}_N)$ and the super-population $\phi(F)$ poverty rate parameter

			$N = 10\,000$			$N = 1000$		
			$n = 500$	$n = 100$	$n = 50$	$n = 500$	$n = 100$	$n = 50$
SI	HT-HJ	$\phi(\mathbb{F}_N)$	95.2	94.4	93.5	98.8	95.1	94.6
		$\phi(F)$	94.6	93.2	92.2	94.7	93.2	92.0
BE	HT	$\phi(\mathbb{F}_N)$	94.9	94.3	94.6	98.4	94.8	94.6
		$\phi(F)$	94.4	93.7	94.9	94.6	93.6	94.7
	HJ	$\phi(\mathbb{F}_N)$	95.1	94.3	93.9	98.7	94.9	94.2
		$\phi(F)$	94.7	94.2	93.9	94.7	94.2	93.9
PO	HT	$\phi(\mathbb{F}_N)$	94.5	94.2	94.3	96.8	94.0	93.6
		$\phi(F)$	94.5	94.0	94.3	94.6	93.6	93.5
	HJ	$\phi(\mathbb{F}_N)$	94.8	93.9	93.6	97.2	94.2	93.3
		$\phi(F)$	94.6	93.9	93.6	94.6	93.9	93.2

Thank you for your attention

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