The Fields Medalists 2014*

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The 4 Fields medals of 2014 have been awarded on August 13, 2014 during the Opening Ceremony of ICM 2014 in Seoul, by Mrs Park Geun-hye, the President of the Republic of Korea.

1 Ladies first: Maryam MIRZAKHANI

Citation: *For her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces.*

An iranian citizen, Maryam Mirzakhani was born in Tehran in 1977. She was one of the two first female students in the iranian team at International Mathematical Olympiads, with marks of 41/42 in 1994 and 42/42 in 1995. She appears in the center of the picture below, taken at IMO 1995.

After studying mathematics at Sharif University in Tehran (1999), she got her thesis in 2004 from Harvard, under the supervision of C. McMULLEN

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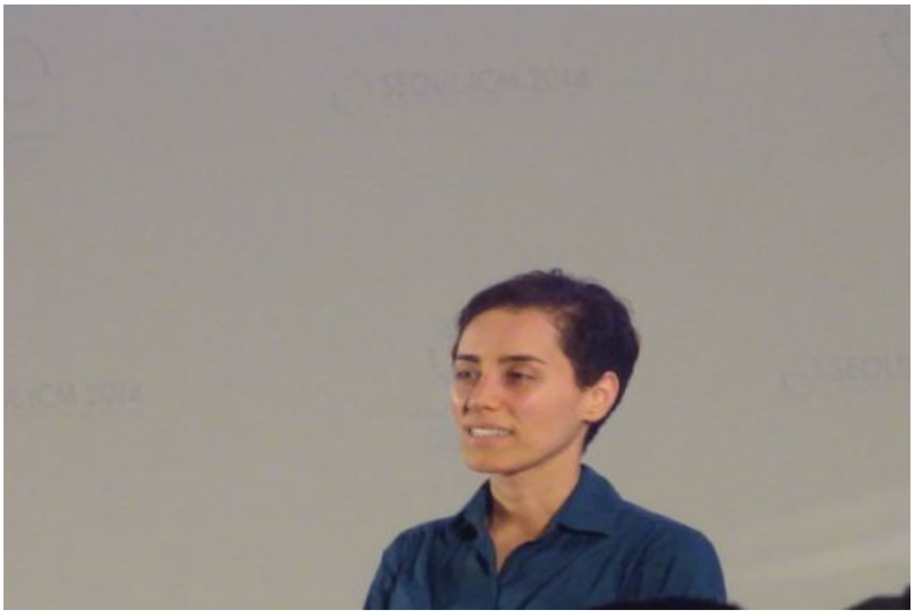
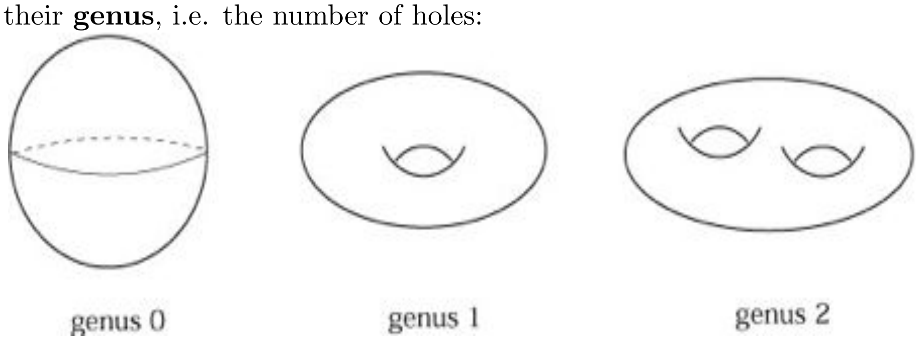
(himself a Fields medallist in 1998). From 2004 to 2008, she was assistant-professor at Princeton, and since 2008 she is full professor at Stanford.

She is the first female laureate of the Fields medal.



The work of Maryam Mirzakhani concerns surfaces of curvature -1, and their moduli spaces.

Let us consider closed, oriented surfaces: these are classified topologically by their **genus**, i.e. the number of holes:

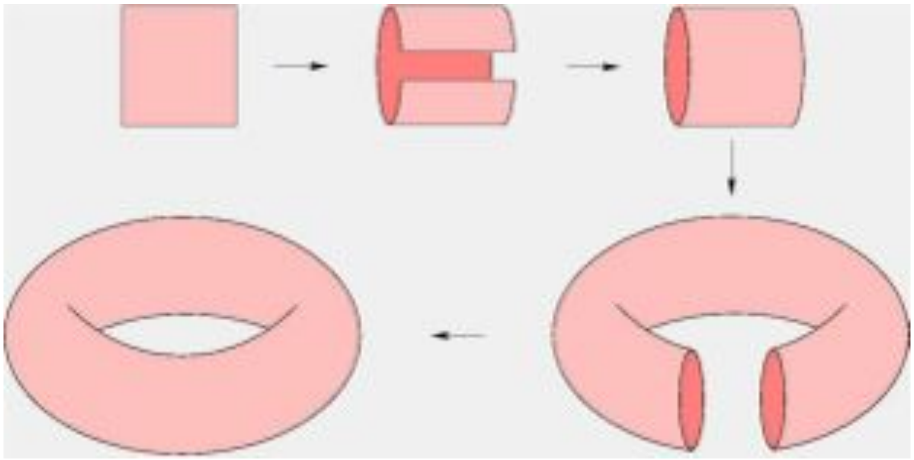


Endow a given surface with a **Riemannian structure:** you may intuitively think of it as a way of measuring angles and curve length on the surface. In particular, you have **geodesic curves**: these are curves on the surface that locally minimize arc length; a more physical way of thinking about them, is to say that geodesics are trajectories of light rays on the surface (examples: straight lines in the plane; great circles on the sphere).

Curvature can be expressed locally in terms of divergence of geodesics:

- sub-linear divergence ⇔ positive curvature (example: the sphere);
- linear divergence ⇔ zero curvature (example: Euclidean plane);
- super-linear divergence ⇔ negative curvature (examples below)

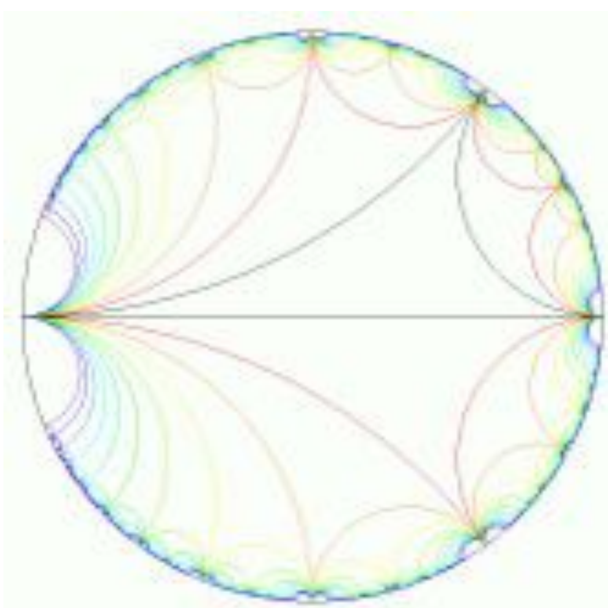
A torus can be endowed with a metric of zero curvature, by viewing it as the quotient of a square in Euclidean plane by identifying opposite pairs of sides; we speak of a **flat torus**.



Observe that, replacing a square by a parallelogram, i.e. varying the angles and side lengths, we obtain a family of **distinct** flat metrics on the torus. This is our first example of a **moduli space**: the space of flat metrics on the torus. You may think of it as the space of parallelograms in Euclidean plane, up to isometry.

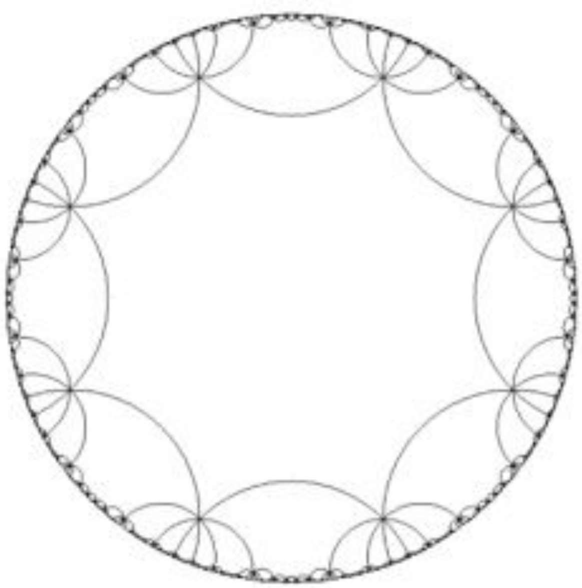
Le us move to curvature -1: the prototype is the **Poincaré disk**, or **hyperbolic plane**, represented by the open disk in which:

- hyperbolic angles are Euclidean angles;
- geodesics are circle arcs orthogonal to the boundary of the disk, together with diameters of the disk.

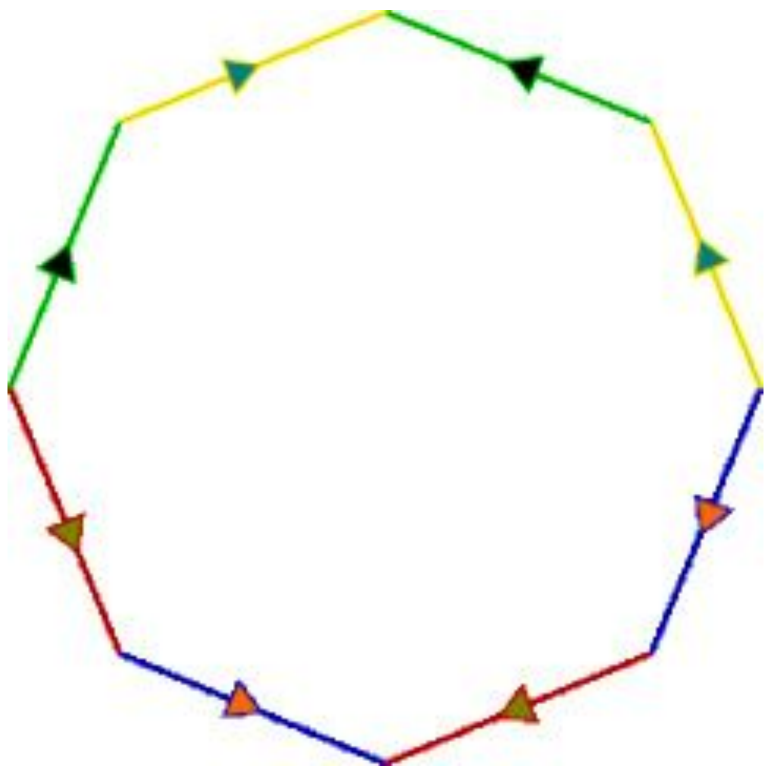


This geometry gave inspiration to artists like M.C. ESCHER:

A result due to POINCARE and KOEBE, but finding its roots in work of RIEMANN, states that every surface of genus $g \geq 2$ is the quotient of Poincaré disk by some tessellation group; in particular every surfce of genus $g \geq 2$ can be endowed with a metric of curvature -1. For example, the genus 2 surface corresponds to the tessellation of Poincaré disk by octagons, with 8 octagons at each vertex:

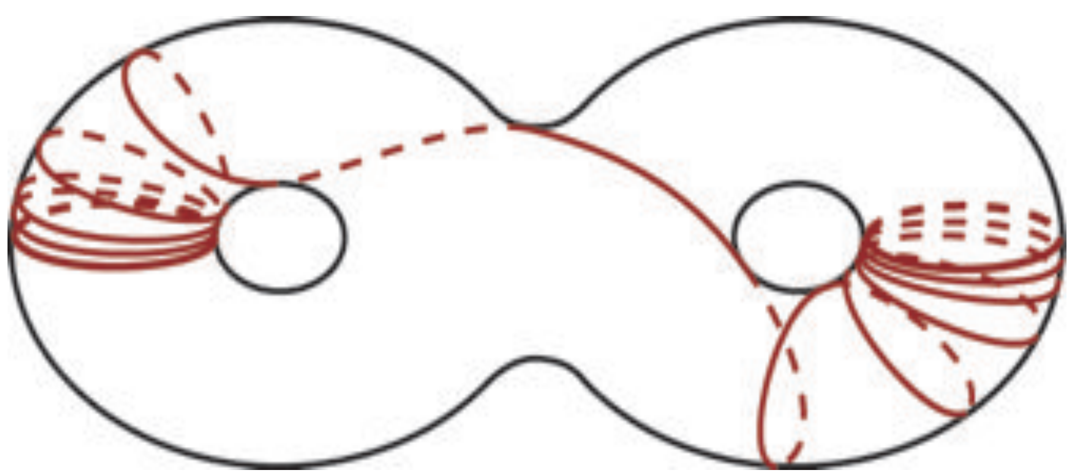


where identification of sides of the fundamental octagon is made according to:

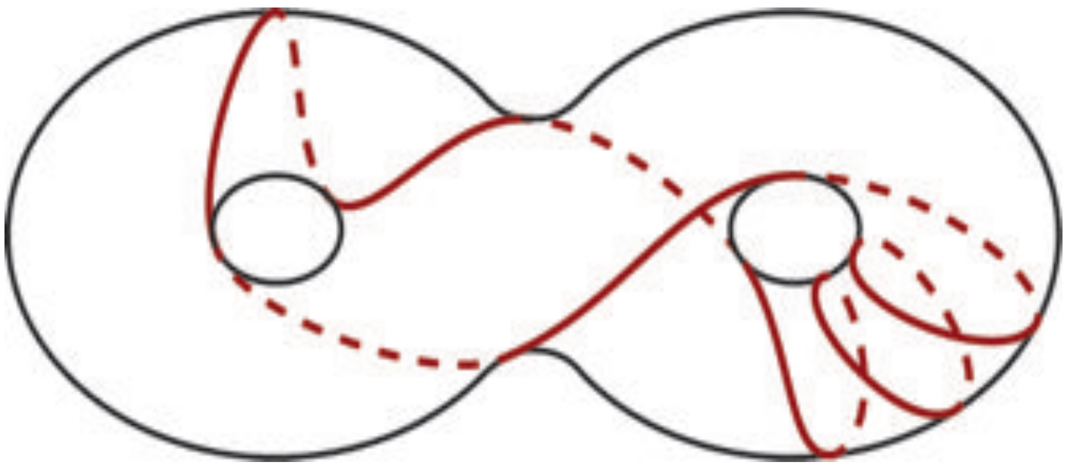


Varying the parameters of the octagon (angles, side lengths) you get the **moduli space** M_2 of the surface of genus 2. More generally, for a surface Σ_g of genus $g \geq 2$,obtained by glueing from a $4g$-gon, the moduli space M_g is the set of curvature -1 metrics on Σ_g; it is the set of tilings of Poincaré disk by $4g$-gons, where $4g$ faces meet at each vertex (up to isometries of the disk). TEICHMUELLER has shown, at the end of the 1930's, that M_g is a space of dimension $6g - 6$.

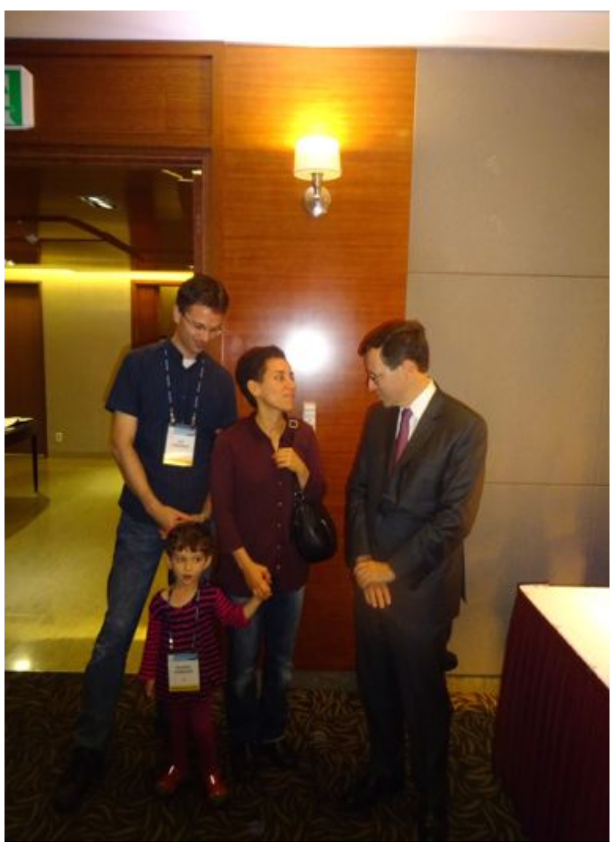
For a given metric of curvature -1 on σ_g, geodesics can be non-closed:



or they can be closed:



It was known since the 1940's that the number of closed geodesics of length $\leq L$ on a surface, asymptotically behaves like $\frac{e^L}{L}$. The problem of counting **simple** closed geodesics (those which do not self-intersect, see the above figure) is much more subtle, and was solved by Mirzakhani in 2004: she shows that there exists a constant $C > 0$, depending on the given metric, such that the number of simple closed geodesics of length $\leq L$ on the surface, asymptotically behaves as $C.L^{6g-6}$. A remarkable feature of the proof is to consider not only the given metric, but also neighboring metrics in M_g.



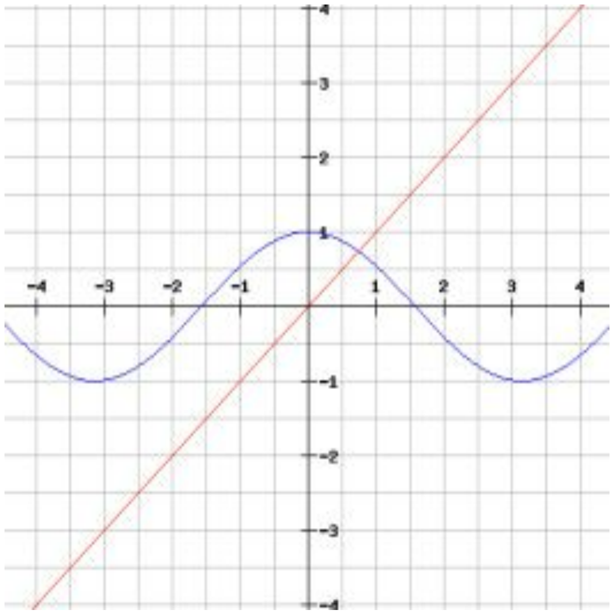
2 Artur AVILA

Citation: *his profound contributions to dynamical systems theory have changed the face of the field, using the powerful idea of renormalization as a unifying principle.*

Born in Brasil in 1979, Artur Avila is a Brasilian/French citizen. He got his thesis in 2001 from Instituto Nacional de Matematica Pura e Aplicada (IMPA) in Rio de Janeiro, under the supervision of Welington DE MELO. Since 2003, he is on a research position of the French CNRS at Paris; he became Directeur de Recherches in 2009. Since 2009 he shares his time between Paris and Rio. In 2008 he got the prize of the European Mathematical Society.

2.1 Unimodal transformations

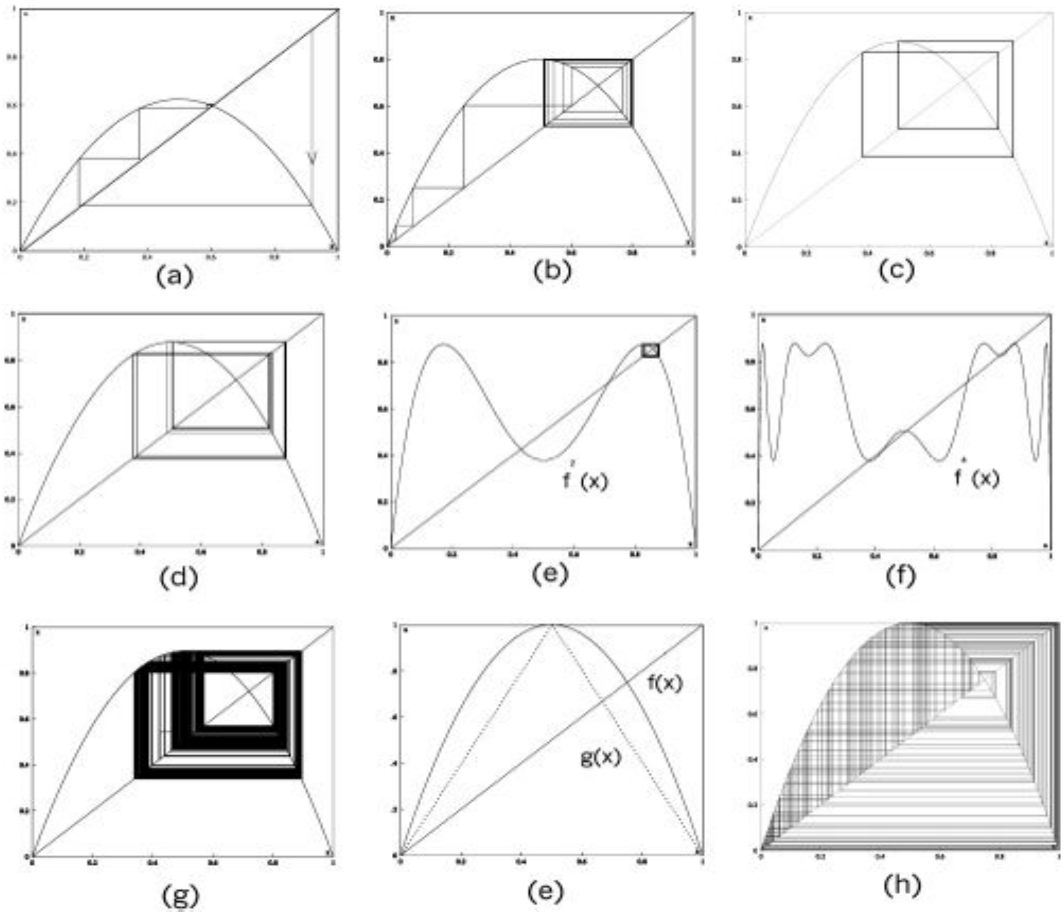
On a pocket calculator, enter $x \in [0,1]$ and press several times the cos key (in radians): the sequence $x, \cos x, \cos(\cos x), \cos(\cos(\cos x)), \ldots$ converges fairly quickly to 0.739085..., which is the unique solution of $x = \cos x$:



A **discrete dynamical system** is given by a function $f : [0,1] \to [0,1]$, and the sequence of iterates $f(x), f \circ f(x), f \circ f \circ f(x), ...$; the question is to describe the behavior of the n-th iterate $f^n(x) = f \circ ... \circ f(x)$ for $n \gg 0$.

For given x, the sequence of iterates $(f^n(x))_{n \geq 0}$ is the **orbit** of x. For $f(x) = \cos x$, every orbit converges to a fixed point.

Consider now the function $f_r(x) = rx(1-x)$, where $r \in [1,4]$ is a parameter. For $1 \leq r < 3$, all orbits converge to a fixed point. When r reaches 3, a cycle of length 2 appears. Then, approximately at $r = 3.44949$, the cycle of length 2 bifurcates into a cycle of length 4. At approximately $r = 3.54409$, that cycle bifurcates into a cycle of length 8, and these period-doubling bifurcations occur faster and faster, until approximately $r = 3.56995$, the **onset of chaos**: the system becomes unpredictable,orbits being seemingly distributed in $[0,1]$ in a completely random way: look at graphs (a), (b), (c), (d), (g) and (h) in the picture below.



Say that a map $f : [0,1] \to [0,1]$ is **unimodal** if $f(0) = f(1) = 0$ and $f'' < 0$ (so: f has a unique maximum).

Theorem 2.1 (Avila-Lyubich-de Melo 2003)**.** *In a real analytic family $(f_r)_{r \in I}$ of unimodal maps, for almost every r there is the following dichotomy:*

- *either f_r is* **regular***, i.e. almost every orbit converges to some periodic orbit;*
- *or f_r is* **stochastic***: there exists a probability measure ν_r on $[0,1]$, absolutely continuous with respect to Lebesgue measure, which is invariant under f_r and almost all orbits of f_r are equi-distributed according to ν_r, in the sense that for every continuous function ϕ on $[0,1]$:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \phi(f_r^n(x)) = \int_0^1 \phi(x) d\nu_r(x)$$

(the "time average" on the LHS is equal to the "space average" on the RHS).

"By now, we have reached a full probabilistic understanding of real analytic unimodal dynamics, and Artur Avila has been the key player in the final stage of the story." (M. Lyubich, 2012).

2.2 The 10 Martini problem

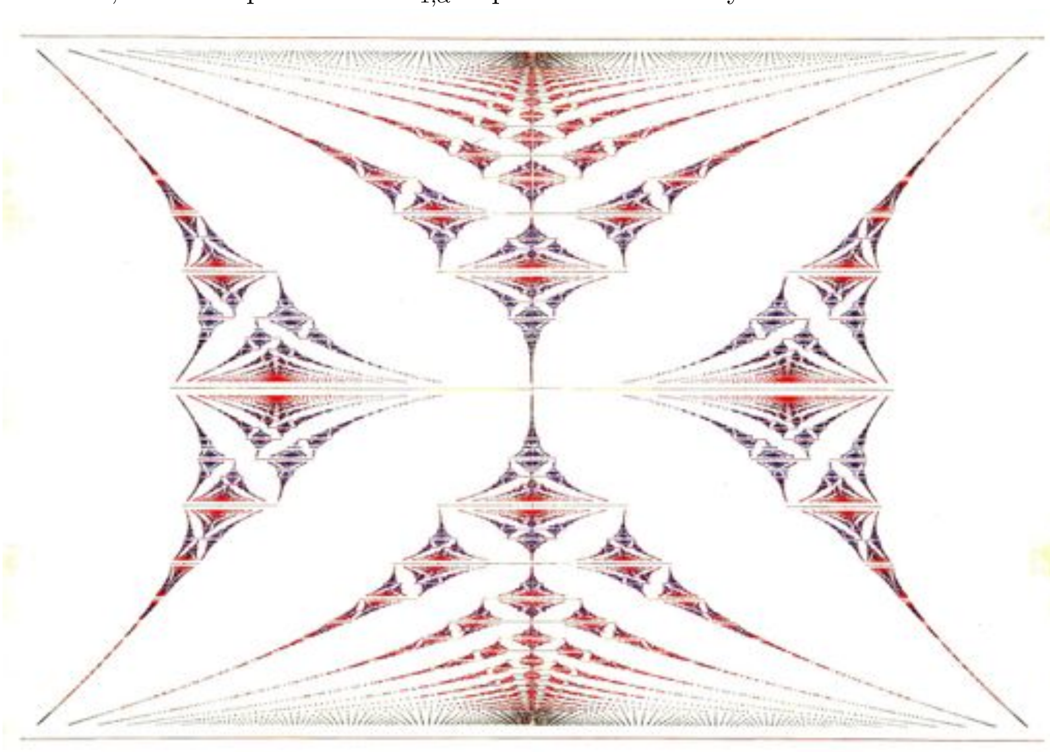
Consider the following situation from physics: an electron is allowed to move on a one-dimensional crystal lattice, submitted to an electro-magnetic potential V. The electron may jump one step either to the left or to the right. In the quantum-mechanics formulation of the situation, the state of the particle is described by a function f on $\mathbb{Z}$, such that $|f(n)|^2$ is the probability that the electron can be found at $n \in \mathbb{Z}$ (so that $\sum_{n\in\mathbb{Z}} |f(n)|^2 = 1$). The evolution of the system is described by the Hamiltonian operator H acting on the Hilbert space $\ell^2(\mathbb{Z})$:

$$(Hf)(n) = f(n+1) + f(n-1) + V(n)f(n).$$

The spectrum of H provides the possible energy levels for the moving particle.

One case much studied since the 1970's, is the **almost periodic** potential $V(n) = 2\lambda\cos(2\pi n\alpha)$. The operator $(H_{\lambda,\alpha}f)(n) = f(n+1) + f(n-1) + 2\lambda\cos(2\pi n\alpha)f(n)$ is then called the **almost Mathieu operator**, the constant $\lambda \in \mathbb{R}$ being the **coupling constant** ($\lambda = 0$ correspondds to the free electron).

The first numerical study of the spectrum of $H_{\lambda,\alpha}$ was made by Douglas HOFSTADTER in the 1970's, for $\lambda = 1$. He obtained this incredibly fractal image, nowadays known as **Hofstadter's butterfly** (where $\alpha \in [0, 1]$ is on the vertical axis, and the spectrum of $H_{1,\alpha}$ is plotted horizontally:



Analogous images are obtained for $\lambda \neq 0$. It was rather quickly shown that, for rational values of α, there is **band spectrum**, i.e. the spectrum of $H_{\lambda,\alpha}$ is a finite union of intervals. This led the famous a american probabilist Marc KAC, in 1981, to offer 10 Martinis for a proof that, for irrational α, the spectrum of $H_{\lambda,\alpha}$ is a **Cantor set**, i.e. a closed bounded subset of $\mathbb{R}$, totally disconnected, without isolated points. At about the same time, the physicists ANDRE and AUBRY conjectured that the Lebesgue measure (i.e. the "length") of the spectrum of $H_{\lambda,\alpha}$ is $4|1-\lambda|$ for α irrational.

In 2004, various methods of mathematical physics allowed to establish the 10 Martini conjecture for a set of parameters (λ, α) of measure 1 in the square $[0,1] \times [0,1]$; it became clear that new ideas were needed for further progress. Introducing ideas coming from dynamical systems, Avila obtained:

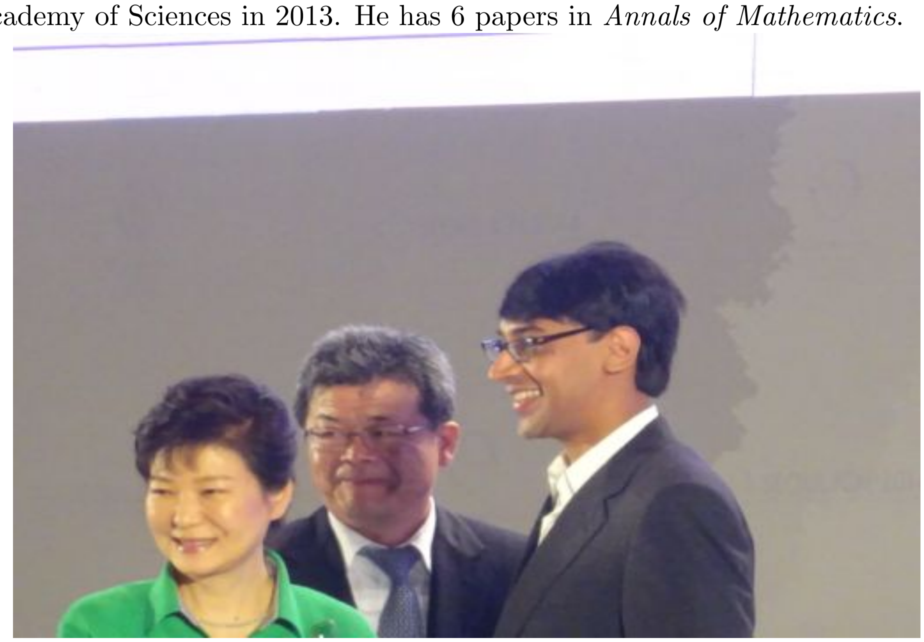
Theorem 2.2.
- *(Avila-Krikorian 2006) The Audré-Aubry conjecture is true;*
- *(Avila-Jitomirskaya 2009) The 10 Martini conjecture is true.*

Sadly Kac passed in 1984, so he couldn't share the 10 Martinis with Avila...

3 Manjul BHARGAVA

Citation: *for developing powerful new methods in the geometry of numbers and applying them to count rings of small rank and to bound the average rank of elliptic curves.*

Born in 1974 in Canada, Bhargava grew up in India and the US. He got his PhD in 2001 from Princeton University under the supervision of Andrew WILES[1]. He became professor in Princeton in 2003, and was elected at the US Academy of Sciences in 2013. He has 6 papers in *Annals of Mathematics.*



[1]Of Fermat's last theorem fame... remember?

Bhargava's mathematics are rooted in classical number theory - he is said to be the person in the world who best understood C.F. GAUSS' *Disquisitiones Arithmeticae.*

3.1 The 15 theorem, and the 290 theorem

In the 17th century, FERMAT claimed that an integer n is a sum of two squares, i.e. $n = x^2 + y^2$ with $x, y \in \mathbb{Z}$, if and only every prime number congruent to 3 modulo 4 appears with even exponent in the prime number factorization of n. He also claimed that every positive integer is a sum of 4 squares. These claims where proved one century later, the former by EULER, the latter by LAGRANGE.

One common feature between these two statements is the appearance of positive definite quadratic forms with integer coefficients:

Quadratic form in k variables: $Q(\mathbf{x})=\sum_{1\leq i\leq j\leq k} a_{ij}x_ix_j$, where $\mathbf{x} = (x_1, ..., x_k)$;
With integer coefficients: $a_{ij} \in \mathbb{Z}$;
Positive definite: $\forall \mathbf{x} \in \mathbb{R}^k : Q(\mathbf{x}) \geq 0$, with equality if and only if $\mathbf{x} = 0$.

Definition 3.1. The quadratic form Q **represents the integer** n if there exists $\mathbf{x} \in \mathbb{Z}^k$ such that $Q(\mathbf{x}) = n$, i.e. the equation $Q(\mathbf{x}) = n$ admits integer solutions.

A classical problem is then: which integers are represented by a given quadratic form? For binary forms ($k = 2$), the problem goes back to LAGRANGE, LEGENDRE and GAUSS.

In 1993, J.H. CONWAY and W. SCHNEEBERGER prove: let Q be a positive definite quadratic form, with a_{ij} even[2] for $i < j$: if Q represents every integer in $\{1, 2, ..., 15\}$, then Q represents every positive integer. They do not publish the proof (apparently there were too many cases to consider). In 2000 Bhargava enters the game and provides a 6 page proof of a more precise result:

Theorem 3.2. *Let Q be a positive definite quadratic form with a_{ij} even for $i < j$. If Q represents every integer in* $\{1, 2, 3, 5, 6, 7, 10, 14, 15\}$*, then Q represents every positive integer. Moreover, for every integer t in that list, there exists a form in 4 variables not representing t but representing every other positive integer.*

Exemple 3.3. The form $x^2 + 2y^2 + 5z^2 + 5t^2$ represents every positive integer..

In 2005, in collaboration with J. HANKE, Bhargava strikes back by removing the evenness assumption on a_{ij}, and proves the 290 theorem, conjectured by Conway:

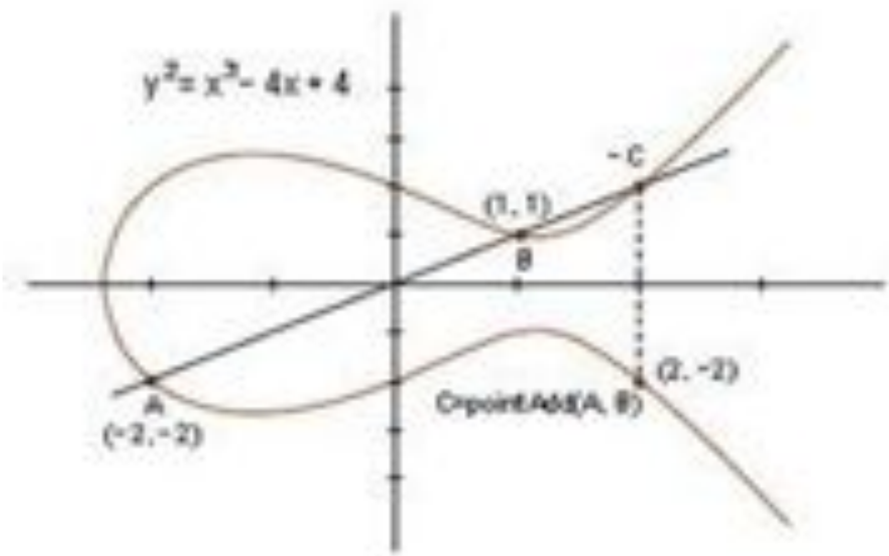
Theorem 3.4. *Let Q be a positive definite quadratic form with integer coefficients. If Q represents the 29 integers* $1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290$*, then Q represents all positive integers. Moreover, for every integer t in that list, there exists a form not representing t, but representing every other positive integer.*

[2]This condition means that the bilinear form associated with Q has integer coefficients.

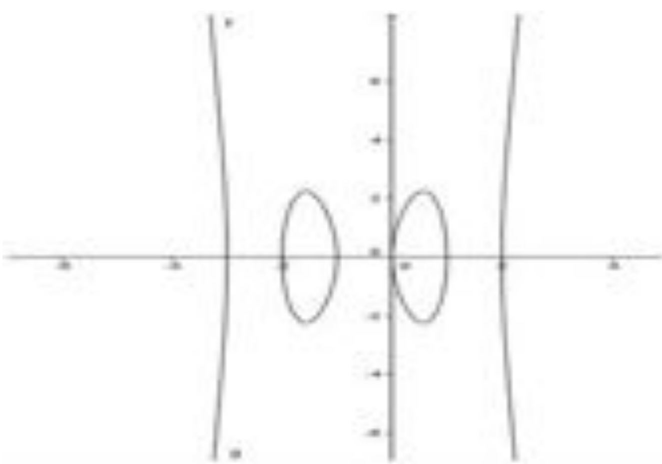
3.1.1 Rational points on algebraic curves

Bhargava devoted much attention to the question: let $P(x)$ be a polynomial with integer coefficients, of degree n: does P take square values on some integers? Or: does the equation $y^2 = P(x)$ admit integer solutions? Or, more geometrically: does the plane algebraic curve with equation $y^2 = P(x)$ have integer points? More generally, does the same curve have rational points? Here some assumptions are needed to avoid trivialities; e.g. if P vanishes at $\alpha \in \mathbb{Q}$, clearly the point $(\alpha, 0)$ is a rational point on the curve. So we assume that P is irreducible over $\mathbb{Q}$.

For $n = 3, 4$ we get an **elliptic curve**:



For $n \geq 5$ we get a **hyperelliptic curve**:



It was previously known that:

- For $n = 1, 2$ the set of rational points is either empty or infinite;

- For $n \geq 5$, the number of rational points is finite[3].

Bhargava had the idea of getting statistical results on curves. For this, he orders curves according to the **height** of the polynomial P: if $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, the height of P is $h(P) = \max_{0 \leq i \leq n} |a_i|$. For a given degree n, the **density** of curves of degree n without rational point is

$$d_n = \liminf_{h \to \infty} \frac{\text{Number of curves of height} \leq h \text{ without rational point}}{\text{Number of curves of height} \leq h}.$$

Theorem 3.5 (Bhargava 2013)**.** *Most hyperelliptic curves have no rational point: for $n > 5$, one has $d_n > \frac{1}{2}$. Moreover, for $n \to \infty$, the density d_n converges to 1 exponentially fast.*

For $n = 3, 4$ (elliptic curves):

Theorem 3.6 (Bhargava-Shankar-Skinner)**.** *A positive proportion of elliptic curves has no rational point. A positive proportion of elliptic curves has infinitely many rational points.*

4 Martin HAIRER

Citation: *for his outstanding contributions to the theory of stochastic partial differential equations; in particular he created a theory of regularity structures for such equations.*

[3]This was proved in 1983 by G. FALTINGS, who got the Fields medal in 1986.

An austrian citizen, Martin Hairer was born in Geneva in 1975[4]. He got his thesis in 2001 from University Geneva, under the supervision of Jean-Pierre ECKMANN, a renowned mathematical physicist. As a PhD student, he made himself famous for creating and marketing the software *Amadeus* for sound processing, *"the swiss army knife of sound editing"*, used in particular for digitalizing vinyl records by eliminating cracks. Hairer is full professor at Warwick University (England).

In physics, time-dependent phenomena are described by differential equations:

$$y'(t) = f(y(t))$$

(where $y(t)$ is the unknown function). If the function f is not precisely known, one can be led to add an error term, describing some uncertainty in the model:

$$y'(t) = f(y(t)) + \sigma(y(t))n(t),$$

else:

$$y(t) = y(0) + \int_0^t (f(y(s)) + \sigma(y(s))n(s))\,ds.$$

If f, σ, n are smooth enough, you will probably find in your toolkit a fixed point theorem that will apply to guarantee existence and uniqueness of a solution (even if this solution cannot be written explicitly).

If n is very irregular, one idea is to approximate n by smooth approximations n_k, look at the family of integral equations

$$y(t) = y(0) + \int_0^t (f(y(s)) + \sigma(y(s))n_k(s))\,ds,$$

and pray that everything goes well, i.e. the solution y_k of the approximate equation converges for $k \to \infty$ towards the solution y of the starting equation.

[4]His father, Ernst Hairer, is an emeritus professor in numerical analysis at University Geneva.

Ideally the limit function y should not depend on the approximations n_k chosen for n (universality property).

If $n(t)$ is a **noise**, i.e a realization of a stochastic process, the differential equation becomes a **stochastic differential equation**.

A well-studied model is the *white noise*: $\mathbb{E}(n(t)) = 0$, $\mathbb{E}(n(s)n(t)) = 0$ for $s \neq t$ (non-correlated in time). If $n(t)$ is a white noise, then $W(t) = \int_0^t n(s)\, ds$ is a **brownian motion** - first observed by biologist R. BROWN in 1827, formalized by G. BACHELIER in 1900 (the birth of financial mathematics) then A. EINSTEIN in 1905 (kinetic theory of gases), then put on rigorous mathematical grounds by N. WIENER (1930) and P. LEVY (1948)... A difficulty is that it is proved that (almost) every brownian trajectory is everywhere non-differentiable, so the "derivative" $n(t)$ does not make sense as a function, but only as a distribution in the sense of Laurent SCHWARTZ. A rigorous theory for stochastic differential equations was developed by ITO since 1948:

$$Y(t) = Y(0) + \int_0^t (f(Y(s))\, ds + \text{“}\int_0^t \sigma(Y(s))\, dW(s)''$$

(where the term in quotation marks is Ito's stochastic integral.)

Physical phenomena depending both on space and time are described by **partial differential equations** (PDE's): e.g. the behavior of heat in a thin bar is governed by:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2},$$

where $u(x,t)$ is temperature at time t at the point with abscissa x. If a noise term is added, one gets a **stochastic partial differential equation** (SPDE).

Martin Hairer has been especially interested in the KPZ equation (from the physicists KHARDAR, PARISI and ZHANG who devised it in 1986), that models rough interface phenomena and growth phenomena:

$$\frac{\partial h(x,t)}{\partial t} = \frac{\partial^2 h(x,t)}{\partial x^2} + (\frac{\partial h(x,t)}{\partial x})^2 + n(t)$$

where $n(t)$ is a white noise. One serious difficulty is that partial derivatives of $h(x,t)$ must be taken in the distributional sense... and that squaring a distribution has no *a priori* meaning!

Martin Hairer developed a **theory of regularity structures** (incorporating as a particular case the theory of "rough paths" of T. LYONS, 1998), allowing him to give rigorous meaning to the solutions of some very singular SPDE's from mathematical physics. He proves that considering a family of equations:

$$\partial_t h_\varepsilon = \partial_x^2 h_\varepsilon + (\partial_x h_\varepsilon)^2 - C_\varepsilon + n_\varepsilon$$

where n_ε is smooth and converges in some appropriate sense to the white noise n, and C_ε is a well-chosen constant, then the solution h_ε converges to some universal limit.

According to experts, Hairer's regularity structures can be applied well beyond the 1-dimensional KPZ equation, and offers a new set of tools to explicitly construct good approximations to singular equations. It is apparently a revolutionary approach that provides a new viewpoint on several fundamental equations of mathematical physics, for which it was long believed that there were impossible to handle in a mathematically rigorous way.

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