THE MACROSCOPIC SPECTRUM OF NILMANIFOLDS WITH AN EMPHASIS ON THE HEISENBERG GROUPS

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Abstract

Take a riemanniann nilmanifold, lift its metric on its universal cover. In that way one obtains a metric invariant under the action of some co-compact subgroup. We use it to define metric balls and then study the spectrum of the laplacian for the dirichlet problem on them. We describe the asymptotic behaviour of the spectrum when the radius of these balls goes to infinity. Furthermore we show that the first macroscopic eigenvalue is bounded from above, by an uniform constant for the three dimensional heisenberg group, and by a constant depending on the Albanese's torus for the other nilmanifolds. We also show that the Heisenberg groups belong to a family of nilmanifolds, where the equality characterizes some pseudo left invariant metrics.

1 Introduction and claims

1.1 — In this article we are investigating Riemannian nilmanifolds, these are compact manifolds obtained by taking the quotient of a nilpotent Lie group by one of its subgroups, endowed with a riemannian metric.

Our aim is to find as much informations as we can by just looking at the balls of great radius on the universal covering. One could believe that it is an object too shrewd to explore, but let us recall the theorem due to Brooks [Bro85a] (see also Sunada [Sun89]), which states that if the bottom of the spectrum of the Laplacian acting on functions on the universal cover of a compact manifold is zero then the fundamental group is amenable.

One could transform the statement by just saying that if the bottom of the spectrum on the balls goes to zero as their radius goes to infinity, then the fundamental group is amenable.

Question 1. *Can one precise the speed of convergence to the bottom of the spectrum on the universal cover with respect to the radius and can one extract more geometric informations ?*

The first step with that question in mind is to try to extract more information feeding the problem with some more geometric assumptions. As the nilpotent groups are among the amenable groups the simplest one it is thus logical to try and explore that case first. In this article we answer question 1 in the following way :

Theorem 1. *Let (M^n, g) be nilmanifold, $B_g(\rho)$ the induced Riemannian ball of radius ρ on its universal cover and $\lambda_1\big(B_g(\rho)\big)$ the first eigenvalue of the Laplacian on $B_g(\rho)$ for the Dirichlet problem.*

Then

1. $\lim_{\rho \to +\infty} \rho^2 \lambda_1\big(B_g(\rho)\big) = \lambda_1^\infty \leq \lambda_1(g, Alb)$

2. *in case of equality the stable norm coincides with the albanese metric.*

Where $\lambda_1(g, Alb)$ is the first eigenvalue of the Kohn laplacian arising from the Albanese metric on the unit ball of the carnot-caratheodory ball arising from the same metric. Furthermore for tori and the 3-dimensional Heisenberg group this is a constant independent of the metric g.

Thus we are naturally lead to explore the equality case (see [Ver02] for the case of tori). Feeding with new assumptions on the nilmanifold we get for example the following theorem (see also section 5), introducing a family of metric which are not far from being left invariant (see definition 18).

Theorem 2. *For any 2-step nilmanifold whose center is one dimensional, the albanese metric and the stable norm coincides if and only if the metric is pseudo left invariant.*

In fact it appears that we can say as much for all the eigenvalues, and the asymptotic behaviour of the eigenvalue in theorem 1 is just a particular case of the following theorem :

Theorem 3. *Let (M^n, g) be nilmanifold, $B_g(\rho)$ the induced Riemannian ball of radius ρ on its universal cover and $\lambda_i\big(B_g(\rho)\big)$ the ith eigenvalue of the Laplacian on $B_g(\rho)$ for the Dirichlet problem.*

Then there exists an hypoelliptic operator Δ_∞ (the Kohn Laplacian of a left invariant metric), whose i^{th} eigenvalue for the Dirichlet problem on stable ball is λ_i^∞ and such that

$$\lim_{\rho \to \infty} \rho^2 \lambda_i\big(B_g(\rho)\big) = \lambda_i^\infty$$

1.2 — Instead of studying the Laplacians we could just study the volume of the balls on the universal covering. As surprising at it mays seems this also gives important informations, for example a theorem of M. Gromov [Gro81] states that if the growth of the geodesic balls on the universal covering of a compact manifold is polynomial, then the manifold is almost nilpotent.

Question 2. *Can we describe more precisely the asymptotic behaviour of the volume of balls with respect to their radius and can we extract more geometric informations ?*

For the asymptotic behaviour of the polynomial case, i.e. the nilpotent case, the answer is given by P. Pansu [Pan82] who described precisely the growth of the geodesic balls on the universal cover of nilmanifolds in the following way

Theorem [Pansu]. *Let (M^n, g) be a nilmanifold, $B_g(\rho)$ the induced Riemannian ball of radius ρ on its universal cover then if μ_g is the volume and d_h the homogeneous dimension of M^n, then the asymptotic volume of g is*

$$\mathrm{Asvol}(g) = \lim_{\rho \to \infty} \frac{\mu_g\big(B_g(\rho)\big)}{\rho^{d_h}} = \mu_g(M^n)\frac{\mu\big(B_\infty(1)\big)}{\mu(D_M)}$$

where μ is a Haar measure, $B_\infty(1)$ the stable norm's unit ball and D_M a fundamental domain on the universal cover.

In the particular case of tori D. Burago and S. Ivanov [BI95] showed that the asymptotic volume is bounded from below by the constant arising from the flat cases, which depends only on the dimension, and furthermore that the equality caracterises flat tori. In [Ver02] there is an alternate proof using the macroscopic spectra in the 2-dimensional case.

In this article we also give a lower bound on the asymptotic volume, but even if the lower bound is not uniform (and can not anyway, because except for tori we can find metrics such that the asymptotic volume is as small as we want) we still have some information on the equality case.

Theorem 4. *Let (M^n, g) be a nilmanifold, then its asymptotic volume satisfies the following :*

1. $\mathrm{Asvol(g)} \geq \mathrm{Vol_g}(M^n)\dfrac{\mu_2\big(B_2(1)\big)}{\mu_2(D_M)}$

2. *In case of equality the stable norm coincides with the albanese metric.*

where μ_2 is the (sub-riemannian) measure associated to the sub-riemannian distance obtained from the albanese metric of M^n on the universal cover of M^n, $B_2(1)$ is the unit ball of that distance and D_M is a fundamental domain of the universal cover of M^n.

The last remark is that in the light of the case of tori, this should not be the best inequality we can expect. However even if we find the best one, the case of equality might not characterize left invariant metrics, but only a particular family of metrics (see [NV] for the intuition), just like in the case of the macroscopic spectrum.

2 General facts about nilmanifolds and their geometry

Our main object of study are Nilmanifolds. In all this article a nilmanifold will be a riemanniann compact manifolds (M^n, g) which is obtained by taking the quotient of a Lie group G, whose Lie Algebra is nilpotent, by one of its co-compact subgroups Γ. We would like to stress that unlike in other work, we don't put any restriction on the metric g, i.e. the metric need not be invariant by the action of G on the left (the whole point being to characterize those metrics among all the metric one can put on M).

Notice that following this definition the universal covering of M^n is G. We will let $\tilde{g}$ be the metric g lifted on G.

2.1 Nilpotent Lie Algebras

2.1.1 — Let $\mathfrak{u}$ be a Lie algebra, one says that it is nilpotent if the sequence defined by

$$\mathfrak{u}^1 = \mathfrak{u}, \qquad \mathfrak{u}^{i+1} = [\mathfrak{u}^i, \mathfrak{u}].$$

is such that for some $r \in \mathbb{N}$, $\mathfrak{u}^{r+1} = \{0\}$.

2.1.2 — Among the nilpotent Lie algebra, there is a family which distinguish itself, the graded nilpotent Lie algebras, these are the algebras $\mathfrak{u}$ with the following decomposition :

$$\mathfrak{u} = V_1 \oplus \ldots \oplus V_r,$$

such that

1– V_i is a complement of $\mathfrak{u}^{i+1}$ in $\mathfrak{u}^i$;

2– $[V_i, V_j] \subset V_{i+j}$;

2.1.3 — What is quite important in our work is the fact that to such a graduation one can attach a one group of automorphisms $(\tau_\rho)_\rho$ called "Dilatations" such that :

$$\tilde{\tau}_\rho(x) = \rho^i x \quad \text{for all } x \in V_i.$$

In fact the existence of such a family of dilatations is equivalent to the existence of a graduation. These dilatations plays the same role than the dilatation in the Euclidean space.

2.1.4 — All nilpotent Lie algebras are not graded. But to each nilpotent Lie algebras we can associate a graded nilpotent one in the following way :

$$\mathfrak{u}_\infty = \sum_{i \geq 1} \mathfrak{u}_i / \mathfrak{u}_{i+1}$$

the Lie bracket being induced. We will write

$$\tilde{\pi} : \mathfrak{u} \to \mathfrak{u}_\infty$$

and we will call *Homogeneous dimension of* $\mathfrak{u}$ the number

$$d_h = \sum_i i \dim\left(\mathfrak{u}^i/\mathfrak{u}^{i+1}\right)$$

2.1.5 — There is another way to make that graded Lie algebra appear. Let us take a nilpotent Lie algebra $\mathfrak{u}$, remark that for all i, $\mathfrak{u}^{i+1} \subset \mathfrak{u}^i$, and take independent vectors $X_{d_1+\ldots+d_{i-1}+1},\ldots,X_{d_1+\ldots+d_{i-1}+d_i}$ such that the vector space V_i they span is the complement of $\mathfrak{u}^{i+1}$ in $\mathfrak{u}^i$. In that way one gets a basis (X_i) of $\mathfrak{u}$. Now we define an application τ_ρ by

$$\tau_\rho(X_p) = \rho^{\alpha(p)} X_p$$

where $\alpha(p) = i$ if $d_{i-1} < p \le d_i$ with $d_0 = 0$.

2.1.6 — We also define a new Lie algebra $\mathfrak{u}_\rho$ by changing the Lie bracket in the following way : for any X and Y in $\mathfrak{u}_\rho$, $[X,Y]_\rho = \tau_{1/\rho}[\tau_\rho X, \tau_\rho Y]$. Thus τ_ρ becomes a Lie algebra isomorphism from $\mathfrak{u}_\rho = \left(\mathfrak{u}, [\cdot,\cdot]_\rho\right)$ to $\left(\mathfrak{u}, [\cdot,\cdot]\right)$.

Now as ρ goes to Infinity $\mathfrak{u}_\rho$ goes to $\mathfrak{u}_\infty$ in the sense that for $i,j = 1,\ldots,n$ we have

$$[X_i, X_j]_\infty = \mathrm{pr}_{V_{\alpha(i)+\alpha(j)}}[X_i, X_j]$$

Notice that the graded Lie algebra is the same for any member of this family. We will write $\tilde{\pi}_\rho$ the projection from $\mathfrak{u}_\rho$ to $\mathfrak{u}_\infty$ (in fact we could avoid the subscript in $\tilde{\pi}_\rho$, because we can identify the Lie algebras as linear spaces)

2.1.7 — Notice that if the Lie algebra is graded then $[X,Y]_\rho = [X,Y]$ and τ_ρ is a Lie algebra automorphism. Otherwise remark that for all $X \in \mathfrak{u}_\rho$

$$\tilde{\pi}\left(\tau_\rho(X)\right) = \tilde{\tau}_\rho\left(\tilde{\pi}_\rho(X)\right)$$

2.2   Remarks on exponential coordinates

2.2.1 — Let G be the Lie group associated with the nilpotent Lie algebra $\mathfrak{u}$. Thanks to the exponential coordinates we can identify G with some $\mathbb{R}^n$, as a differential manifold :

$$\phi : \mathbb{R}^n \to G, \ \ \phi : x = (x_1,\ldots,x_n) \mapsto \exp x_1 X_1 \ldots \exp x_n X_n$$

because for nilpotent Lie groups, the exponential is a diffeomorphism between the Lie algebra and the Lie group. Let ln be the inverse and X_i^* the dual form of X_i

2.2.2 — Moreover, if we denote by δ_ρ the following family of dilatations

$$\delta_\rho(x_1,\ldots,x_n) = (\rho^{\alpha(1)} x_1,\ldots,\rho^{\alpha(n)} x_n)$$

and we define a family of group products $*_\rho$ by setting

$$x *_\rho y = \delta_{1/\rho}[\delta_\rho(x)\delta_\rho(y)]$$

and also

$$x *_\infty y = \lim_{\rho \to \infty} x *_\rho y$$

we get a family of nilpotent Lie groups $G_\rho = (G, *_\rho)$, $\rho \in \overline{\mathbb{R}}$, whose associated family of Lie algebras are isomorphic to the family $\mathfrak{u}_\rho$. We also write $\pi_\rho : G_\rho \to G_\infty$ the application which sends $x \in G_\rho$ to $x \in G_\infty$. Notice also that $d\delta_\rho = \tau_\rho$.

2.2.3 — Let us define $\varphi_i : G \mapsto \mathbb{R}$ by $\varphi_i(g) = X_i^* \ln(g)$, then using the Campbell-Haussdorff formula, i.e.

$$\ln(x * y) = \ln(x) + \ln(y) + \frac{1}{2}\big[\ln(x), \ln(y)\big] + C(x, y)$$

where $C(x, y) \in \mathfrak{u}^3$, for $i = 1, \dots, d_1$ we get :

$$X_i^* \ln(x * y) = X_i^* \ln(x) + X_i^* \ln(y)$$

which means that for $i = 1, \dots, d_1$ the function φ_i is a group morphism hence $d\varphi_i$ is left invariant, i.e. $d\varphi_{i|\gamma*g} \cdot dl_{\gamma|g} = d\varphi_{j|g}$ thus

$$X_i \cdot \varphi_j = d\varphi_{j|g} \cdot X_i(g) = d\varphi_{j|e} \cdot X_i(e) = X_j^* \cdot X_i = \delta_{ij}$$

2.2.4 — In other words, if we use exponential coordinates, taking as a maximum family of independent vector in $V_1 = \mathfrak{u}^1 \backslash \mathfrak{u}^2$ the vectors $X_1, \dots, X_{d_1}$, the previous calculation says that for $j = 1, \dots, d_1$ and all i,

$$X_i \cdot x_j = \delta_{ij}$$

2.3 Horizontal distribution and the Stable norm

2.3.1 — On the graded nilpotent Lie group G_∞ associated to G we obtain a natural distribution by left multiplication of $V_1 = \mathfrak{u}_1/\mathfrak{u}_2 \subset \mathfrak{u}_\infty$, we shall call that distribution *Horizontal* and write it $\mathcal{H}$.

2.3.2 — Let us remark that because of the nilpotency and the graduation of the Lie algebra $\mathfrak{u}_\infty$, a base of V_1 satisfies the so called Hörmander conditions in the Lie group G_∞.

2.3.3 — We will say that a function f is periodic with respect to Γ (the cocompact subgroup) if for every $\gamma \in \Gamma$ and $x \in G$ we have $f(\gamma * x) = f(x)$. Thus $\tilde{g}$ is periodic with respect to Γ.

2.3.4 — We recall what the *stable norm* is :

Definition 5. *Let $||\cdot||_\infty^*$ be the quotient of the sup norm on 1-forms, arising from the metric g, on the cohomology $H^1(M^n, \mathbb{R})$. Then its dual norm on the homology $H_1(M^n, \mathbb{R})$, is called the stable norm.*

2.3.5 — By a theorem of K. Nomizu [Nom54], $H_1(M^n, \mathbb{R}) \equiv V_1$, hence we will call stable ball $B_\infty(1)$, the unit ball for the carnot-caratheodory metric d_∞ induced by the stable norm of (M^n, g) on G_∞.

2.4 The horizontal distribution and the metric

2.4.1 — We will need to consider the family $\{G_\rho\}_{\rho \in \mathbb{R}}$ of simple connected Lie groups who corresponds to the family $\{\mathfrak{u}_\rho\}_\rho$ of Lie algebra associated to $\mathfrak{u}$. On each G_ρ we are going to pull back the metric of G, $\tilde{g}$ and rescale it in the following way

$$g_\rho = \frac{1}{\rho^2}(\delta_\rho)^* \tilde{g}$$

thus we are also able to focus on the riemanniann spaces (G_ρ, g_ρ). The Laplacian in that space will be Δ_ρ.

2.4.2 — If $e \in G$ is the unit element and $X \in \mathfrak{u}$ then for $\rho \in \overline{\mathbb{R}}$, X^ρ will be the $*_\rho$ left invariant field in G_ρ such that $X^\rho(e) = X(e)$. Thus to the base (X_i) defined in 2.1.5 we will associate the $*_\rho$ left invariant fields (X_i^ρ). Notice also that

$$d\delta_\rho(X_i^\rho) = \tau_\rho(X_i^\rho) = \rho^{\alpha(i)} X_i$$

2.4.3 — Let us write the metric $\tilde{g}$ in the basis (X_i).

$$\tilde{g} = (g_{ij})$$

We can distinguish two distincts parts when writing the laplacian in coordinates

$$-\det \tilde{g} \Delta = \sum_{1 \leq i,j \leq d_1} X_i\big(\det g \cdot g^{ij} X_j\big) + \sum_{\alpha(i)+\alpha(j)>2} \big(X_i \det g \cdot g^{ij} X_j\big)$$

which will be of significant importance in what follows.

2.4.4 — It is a straightforward calculation to find that the metric g_ρ in the coordinates (X_i^ρ) is written (for $x \in G_\rho$ and $\rho \in \mathbb{R}$):

$$-\det \tilde{g}(\delta_\rho x)\Delta_\rho = \sum_{1 \leq i,j \leq d_1} X_i^\rho\big(\det g \; g^{ij}(\delta_\rho x) X_j^\rho\big) + \\ \sum_{\alpha(i)+\alpha(j)>2} \rho^{2-\alpha(i)-\alpha(j)} X_i^\rho\big(\det \tilde{g} \; g^{ij}(\delta_\rho x) X_j^\rho\big) \tag{1}$$

In this formula the whole difference between the two parts becomes clear, indeed it is quite apparent now that the second part vanishes when ρ goes to infinity.

2.4.5 — That is the reason why we introduce $\nabla_{\mathcal{H}}$ by :

$$\nabla_{\mathcal{H}} f = (X_1^\infty \cdot f, \ldots, X_{d_1}^\infty \cdot f)$$

2.5 Gromov-Haussdorff convergence of balls

2.5.1 — In what follows $B_g(\rho)$ will always mean the geodesic ball of radius ρ on the universal cover of $(M^n = \Gamma \backslash G, g)$, and $B_\rho(1)$ will be the geodesic ball of radius one on (G_ρ, g_ρ). We will also need to define μ_ρ (resp. μ_g) the Riemannian volume associated to g_ρ (resp. g) and μ_∞ defined as follows : Let D_Γ be a fundamental domain in G and μ a Haar measure on G_∞ then

$$\mu_\infty = \frac{\mu_g(D_\Gamma)}{\mu\big(\pi(D_\Gamma)\big)} \; \mu$$

Theorem 6. *Let (M^n, g) be a riemanniann nilmanifold, d_g the induced distance on its universal cover and $(\delta_\rho)_{\rho \in \mathbb{R}}$ the familly of isomorphisme between G_ρ and G if one writes for any $x, y \in G_\rho$:*

$$d_\rho(x, y) = \frac{d_g(\delta_\rho x, \delta_\rho y)}{\rho}$$

and $B_\rho(1)$ the unit ball for each of this rescaled distances, then the family of metric spaces $(B_\rho(1), d_\rho, \mu_\rho)$ converges in the Gromov-Haussdorff measure topology to $(B_\infty(1), d_\infty, \mu_\infty)$ as ρ goes to infinity.

Proof. The Gromov-Haussdorff convergence comes from P. Pansu work [Pan82], which implies

$$\lim_{\rho \to \infty} \frac{d_\infty\big(\pi_\rho(x), \pi_\rho(y)\big)}{d_\rho(x, y)} = \lim_{\rho \to \infty} \frac{d_\infty\big(\pi \circ \delta_\rho(x), \pi \circ \delta_\rho(y)\big)}{d_g(\delta_\rho x, \delta_\rho y)} = 1$$

It remain to show the measure part.

Claim. *for any compact domain A in G_∞, whose boundary is of haar measure 0, and any function $f \in L^1(A, \mu_\infty)$ we have*

$$\lim_{\rho \to \infty} \int_{\pi_\rho^{-1}(A)} f\big(\pi_\rho(x)\big) d\mu_\rho(x) = \int_A f d\mu_\infty$$

Indeed, let A be a domain in G_∞, then $\pi_\rho^{-1}(A)$ belongs to G_ρ and $\delta_\rho \circ \pi_\rho^{-1}(A)$ belongs to G. We will write $*$ the law group of G. Let $z_1, \ldots, z_k$ and $\zeta_1, \ldots, \zeta_l$ be elements of Γ such that $\zeta_j * D_\Gamma \cap \delta_\rho \circ \pi_\rho^{-1}(A) \neq \emptyset$ for any j and

$$\bigcup_i z_i * D_\Gamma \subset \delta_\rho \circ \pi_\rho^{-1}(A) \subset \bigcup_j \zeta_j * D_\Gamma$$

Let us notice that

$$\mu_g(D_\Gamma) = \frac{\mu_g(D_\Gamma)}{\mu\big(\pi(D_\Gamma)\big)} \mu\big(\pi(D_\Gamma)\big) = \mu_\infty\big(\pi(D_\Gamma)\big)$$

then we get

$$\sum_i \inf_{\delta_\rho \circ \pi_\rho^{-1}(x) \in z_i * D_\Gamma} f(x) \mu_\infty\big(\pi(D_\Gamma)\big) \leq$$

$$\int_{\delta_\rho \circ \pi_\rho^{-1}(A)} f\big(\tilde{\delta}_{1/\rho} \circ \pi(x)\big) d\mu_g(x) \leq \sum_j \sup_{\delta_\rho \circ \pi_\rho^{-1}(x) \in \zeta_j * D_\Gamma} f(x) \mu_\infty\big(\pi(D_\Gamma)\big)$$

divide all members by ρ^{d_h} (see 2.1.4) we get :

$$\sum_i \inf_{x \in \pi_\rho \circ \delta_{1/\rho}(z_i * D_\Gamma)} f(x) \mu_\infty\big(\tilde{\delta}_{1/\rho} \circ \pi(D_\Gamma)\big) \leq$$

$$\int_{\pi_\rho^{-1}(A)} f\big(\pi_\rho(x)\big) d\mu_\rho \leq \sum_j \sup_{x \in \pi_\rho \circ \delta_{1/\rho}(\zeta_j * D_\Gamma)} f(x) \mu_\infty\big(\tilde{\delta}_{1/\rho} \circ \pi(D_\Gamma)\big)$$

then the extremal terms are riemann's sums which converge toward $\int_A f d\mu_\infty$.

Using the claim and the fact that the functionals $x \mapsto d_\rho(0, \pi_\rho^{-1}(x))$ converges simply toward the functional $x \mapsto d_\infty(0, x)$ on $B_\infty(1) \backslash \partial B_\infty(1)$ we can conclude. ❏

2.5.2 — Let us introduce the asymptotic volume as

$$\text{Asvol}(g) = \lim_{\rho \to \infty} \frac{\mu_g\big(B_g(\rho)\big)}{\rho^{d_h}}$$

following our last statement it is straightforward that

$$\text{Asvol}(g) = \mu_\infty\big(B_\infty(1)\big)$$

3 Looking for convergences

For the proof of theorem 3 we will a few more definitions. Indeed we are going to look at a family of L^2-spaces which are not defined on the same space. More exactly we are going to look at $L^2(B_\rho(1), \mu_\rho)$. However we are going to show that in some sens (see below 2.5) these spaces converge towards $L^2(B_\infty(1), \mu_\infty)$. So now we would like to give a precise meaning to the fact that a net composed of functions in $L^2(B_\rho(1), \mu_\rho)$ converges toward a function in $L^2(B_\infty(1), \mu_\infty)$.

Once this will be done, we will be able to give a meaning to the convergence of a net of operators (the resolvent of the Laplacian for example).

3.1 Convergence on a net of Hilbert spaces

In what follows $\mathcal{A}$ and $\mathcal{B}$ are directed set.

3.1.1 — Let $(X_\alpha, d_\alpha, m_\alpha)_{\alpha \in \mathcal{A}}$ be a net of compact measured metric spaces converging in the Gromov-Hausdorff measured topology to $(X_\infty, d_\infty, m_\infty)$. We will write $L^2_\alpha = L^2(X_\alpha, m_\alpha)$ (resp. $L^2_\infty(X_\infty, m_\infty)$) for the square integrable function spaces. Their respective scalar product will be $\langle \cdot, \cdot \rangle_\alpha$ (resp. $\langle \cdot, \cdot \rangle_\infty$) and $||\cdot||_\alpha$ (resp. $||\cdot||_\infty$).

3.1.2 — Furthermore we suppose that in every L^2_α the continuous functions form a dense subset $C^0(X_\alpha)$.

Definition 7. *We say that a net $(u_\alpha)_{\alpha \in \mathcal{A}}$ of functions $u_\alpha \in L^2_\alpha$ strongly converges to $u \in L^2_\infty$ if there exists a net $(v_\beta)_{\beta \in \mathcal{B}} \subset C^0(X_\infty)$ converging to u in L^2_∞ such that*

$$\lim_\beta \limsup_\alpha ||f^*_\alpha v_\beta - u_\alpha||_\alpha = 0;$$

where (f_α) is the net of Hausdorff approximations. We will also talk of strong convergence in $\mathcal{L}^2$.

Definition 8. *We say that a net $(u_\alpha)_{\alpha \in \mathcal{A}}$ of functions $u_\alpha \in L^2_\alpha$ weakly converges to $u \in L^2_\infty$ if and only if for every net $(v_\alpha)_{\alpha \in \mathcal{A}}$ strongly converging to $v \in L^2_\infty$ we have*

$$\lim_\alpha \langle u_\alpha, v_\alpha \rangle_\alpha = \langle u, v \rangle_\infty \tag{2}$$

We will also talk of weak convergence in $\mathcal{L}^2$.

The following claim (whose proof we don't give, see [Ver01] pages 32—33) justifies those two definitions

Lemma 9. *Let $(u_\alpha)_{\alpha \in \mathcal{A}}$ be a net of functions $u_\alpha \in L^2_\alpha$. If $(||u_\alpha||_\alpha)$ is uniformly bounded, then there exists a weakly converging subnet. Furthermore every weakly converging net is uniformly bounded.*

3.1.3 — Now that we gave sense to the convergence of a net of functions, we are going to define convergences of a net of operators. Let $B_\infty \in \mathcal{L}(L^2_\infty)$ and $B_\alpha \in \mathcal{L}(L^2_\alpha)$ for every $\alpha \in \mathcal{A}$.

Theorem and Definition 10. *Let $u, v \in L_\infty^2$ and $(u_\alpha)_{\alpha \in \mathcal{A}}$, $(v_\alpha)_{\alpha \in \mathcal{A}}$ two nets such that $u_\alpha, v_\alpha \in L_\alpha^2$. We say that the net of operators $(B_\alpha)_{\alpha \in \mathcal{A}}$ strongly (resp. weakly, compactly) converges to B if $B_\alpha u_\alpha \to Bu$ strongly (resp. weakly, strongly) for every net (u_α) strongly (resp. weakly, weakly) converging to u $\iff$*

$$\lim_\alpha \langle B_\alpha u_\alpha, v_\alpha \rangle_\alpha = \langle Bu, v \rangle_\infty \tag{3}$$

for every (u_α), (v_α), u and v such that $u_\alpha \to u$ strongly (resp. weakly, weakly) and $v_\alpha \to v$ weakly (resp. strongly, weakly), (See [Ver01] page 35 for the justifications).

3.2   The importance of being compactly convergent

The theorem and definition of the previous section is useful, for the goal we would like to achieve thanks to the following one (whose proof can be found in [Ver02] for example), which links the convergence of the resolvents R_ζ^α associated to a family of functionals A_α, with their spectra.

Theorem 11. *Let $R_\zeta^\alpha \to R_\zeta$ compactly for all ζ outside the spectra of (A_α). Assume that all resolvents R_ζ^α are compact. Let λ_k (resp. λ_k^α) be the kth eigenvalue of A (resp. A_α) with multiplicity. We take $\lambda_k = +\infty$ if $k > \dim L_\infty^2 + 1$ when $\dim L_\infty^2 < \infty$ and $\lambda_k^\alpha = +\infty$ if $k > \dim L_\alpha^2 + 1$ when $\dim L_\alpha^2 < \infty$. Then for every k*

$$\lim_\alpha \lambda_k^\alpha = \lambda_k$$

Furthermore let $\{\varphi_k^\alpha \mid k = 1, \ldots, \dim L_\alpha^2\}$ be an orthonormal bases of L_α^2 such that φ_k^α is an eigenfunction of A_α for λ_k^α. Then there is a sub-net such that for all $k \leq \dim L_\infty^2$ the net $(\varphi_k^\alpha)_\alpha$ strongly converges to the eigenfunction φ_k of A for the eigenvalue λ_k, and such that the family $\{\varphi_k \mid k = 1, \ldots, \dim L_\alpha^2\}$ is an orthonormal basis of L_∞^2.

3.3   What shall we finally study ?

We are now going to focus on the spectrum of the balls $B_g(\rho)$, and we want to show that the eigenvalues of the laplacian are converging to zero with a $1/\rho^2$ speed, to be more specific we want to find a precise equivalent.

For this let recall that Δ_ρ is the Laplacian (or Laplace-Beltrami operator) associated to the rescaled metrics $g_\rho = 1/\rho^2 (\delta_\rho)^* g$ on G_ρ, and for any function f from $B_g(\rho)$ to $\mathbb{R}$ lets associate a function f_ρ on $B_\rho(1)$ by $f_\rho(x) = f(\delta_\rho \cdot x)$. Then it is an easy calculation to see that for any $x \in B_\rho(1)$:

$$\rho^2 (\Delta f)(\delta_\rho \cdot x) = (\Delta_\rho f_\rho)(x)$$

hence the eigenvalues of Δ_ρ on $B_\rho(1)$ are exactly the eigenvalues of Δ on $B_g(\rho)$ multiplied by ρ^2 and our problems becomes the study of the spectrum of the laplacian Δ_ρ on $B_\rho(1)$.

Enlightened by what happens on tori we would like to show that there is some operator Δ_∞ acting on $B_\infty(1)$ (see 2.3.5) such that, in a good sense, the

net of laplacian (Δ_ρ) converges towards Δ_∞ and the spectra also converge to the spectrum of Δ_∞.

In the light of theorem 11, the good sense is the compact convergence of the resolvent like in our paper on the macroscopical sound of tori [Ver02]. The proof is quite similar, but needs some adaptation to the geometry of nilmanifolds. The following section gives the proof.

4 Homogenisation and proof of theorem 3

The first step consisted in showing the convergence of the metric geodesic balls w.r.t. the Gromov-Haussdorff measure topology (see 2.5).

The next step will consist in introducing some functions linked with the Albanese metric (see 4.1) which will lead us to the definitions of the functional spaces involved and the compact inclusion we can deduce (see 4.2). Finaly we prove the compact convergence of the resolvent (see 4.3), which thanks to theorem 11 finishes the proof.

4.1 Homogenisation of the Laplacian and Albanese's Torus

In this section we are going to "built" the operator Δ_∞ of theorem 3 and give the proof of theorem 4.

4.1.1 — Let D_Γ be a fundamental domain. Let χ^i be the unique periodic with respect to Γ solution (up to an additive constant) of (for $1 \le i \le r$)

$$\Delta\chi^i = \Delta x_i \text{ on } D_\Gamma$$

The operator Δ_∞ is then defined by (we use Einstein's summation convention)

$$\Delta_\infty f = -\frac{1}{\text{Vol}(g)} \sum_{1 \le i,j \le d_1} \left(\int_{D_\Gamma} g^{ij} - g^{ik} X_k \cdot \chi^j \, d\mu_g \right) X_i^\infty \cdot X_j^\infty f \qquad (4)$$

Now let us write $\eta_j(x) = \chi^j(x) - x_j$ the induced harmonic function and

$$q^{ij} = \frac{1}{\text{Vol}(g)} \left(\int_{D_\Gamma} g^{ij} - g^{ik} X_k \cdot \chi^j \, d\mu_g \right)$$

we can notice that the $d\eta_i$ are harmonic 1-forms on the nilmanifold. It is not difficult now to show that

Proposition 12. *Let $\langle\cdot,\cdot\rangle_2$ be the scalar product induced on 1-forms by the Riemannian metric g. Then*

$$q^{ij} = \frac{1}{\text{Vol}(g)} \langle d\eta_i, d\eta_j \rangle_2 = q^{ji}$$

thus Δ_∞ is an Hyppoelliptic operator.

4.1.2 — In fact we can say more, (q^{ij}) induces a scalar product on harmonic 1-forms (whose norm will be written $\| \cdot \|_2$) and then to $H^1(M^n, \mathbb{R})$, which can

be identified with the horizontal $\mathcal{H}$ following a theorem of K. Nomizu [Nom54]. Indeed, as mentioned earlier, we can see the $(d\eta_i)$ as 1-forms over the nilmanifold. Being a free family they can be seen as a basis of $H^1(M^n, \mathbb{R})$ (Hodge's theorem). Thus by duality this yields also a scalar product (q_{ij}) over $H_1(M^n, \mathbb{R})$ (whose induced norm will be written $|| \cdot ||_2^*$).

      4.1.3 — The question naturally arising is to know how is this norm related to the stable norm. To understand their link we have to go back on $H^1(M^n, \mathbb{R})$. Indeed the stable norm is the dual of the norm obtained by quotient of the sup norm on 1-forms (see Pansu [Pan99] lemma 17), which we write $|| \cdot ||_\infty^*$, and the norm $||\cdot||_2$ comes from the normalised L^2 norm. Thus mixing the Hölder inequality and the Hodge-de Rham theorem we get :

Proposition 13. *For every 1-form α we have*

$$||\alpha||_2 \leq ||\alpha||_\infty^*$$

thus by duality, for every $\gamma \in H_1(M^n, \mathbb{R})$ we have

$$||\gamma||_\infty \leq ||\gamma||_2^*$$

in other words the unit ball of the sub-riemannian metric arising from $|| \cdot ||_2^*$ is included in $B_\infty(1)$.

      4.1.4 — Let us recall that the manifold $H_1(M^n, \mathbb{R})/H_1(M^n, \mathbb{Z})$ with the flat metric induced by $||\cdot||_2^*$ is usually called the Jacobi manifold or the Albanese torus of (M^n, g). This last proposition also implies the following inequality, regarding the asymptotic volume

Corollary 14. *Let (M^n, g) be a nilmanifold, then its asymptotic volume satisfies the following inequality :*

$$\mathrm{Asvol(g)} \geq \mu_{\mathrm{g}}(M^n)\frac{\mu_2\big(B_2(1)\big)}{\mu_2(D_\Gamma)}$$

where μ_2 is the (sub-riemanniann) measure associated to the sub-riemanniann distance obtained from the albanese metric of M^n on the universal cover of M^n, $B_2(1)$ is the unit ball of that distance and D_Γ is a fundamental domain of the universal cover of M^n.

Proof. Following Nomizu [Nom54] we can identify the horizontal space $\mathcal{H}$ with $H_1(M^n, \mathbb{R})$. This allows us to get two sub-riemannian distances d_2 and d_∞ from $|| \cdot ||_2^*$ and $|| \cdot ||_\infty$ respectively. The previous proposition implies that the ball of d_2 is inside the stable ball. Thus for any Haar measure μ one gets the following inequality :

$$\mu\big(B_2(1)\big) \leq \mu\big(B_\infty(1)\big)$$

now taking for μ the haar measure μ_∞ (see section 2.5) giving the asymptotic volume we can conclude. ❑

      Remark that theorem 4 is now a simple corollary of that last corollary.

4.2 Asymptotic compactness

4.2.1 — Let us now define the various functional spaces involved. For $\rho \in \overline{\mathbb{R}}$, $L_\rho^2 = L^2\big(B_\rho(1), d\mu_\rho\big)$ will be the space of square integrable functions over the ball $B_\rho(1)$, which is a Hilbert space with the scalar product

$$(u, v)_\rho = \int_{B_\rho(1)} uv \, d\mu_\rho$$

whose norm will be $|\cdot|_\rho$. Hence $\mathcal{L}^2$ will be the net of spaces (L_ρ^2) with either the strong or weak topoly induced by the definitions 7 and 8.

4.2.2 — Following the usual nomenclature we will be interested in the following spaces

$$H_\rho^1\big(B_\rho(1)\big) = \left\{ v \,\Big|\, v, X_i^\rho \cdot v \in L^2\big(B_\rho(1), d\mu_\rho\big), \ 1 \leq \alpha(i) \leq r \right\}$$

$$\left(\text{resp.} \qquad H_\infty^1\big(B_\infty(1)\big) = \left\{ v \,\Big|\, v, X_i^\infty \cdot v \in L^2\big(B_\infty(1), d\mu_\infty\big), \ 1 \leq i \leq d_1 \right\} \right)$$

which becomes Hilbert spaces with the norm $||\cdot||_\rho$ defined by

$$||v||_\rho^2 = |v|_\rho^2 + \sum_{1 \leq \alpha(i) \leq r}^{n} \big|X_i^\rho \cdot v\big|_\rho^2$$

$$\left(\text{resp.} \qquad ||v||_\infty^2 = |v|_\infty^2 + \sum_{1 \leq i \leq d_1}^{n} \big|X_i^\infty \cdot v\big|_\infty^2 \right)$$

4.2.3 — Hence $H_{\rho,0}^1\big(B_\rho(1)\big)$ will be the closure of the $C^\infty\big(B_\rho(1)\big)$ functions with compact support, in $H_\rho^1\big(B_\rho(1)\big)$ for the norm $||\cdot||_\rho$.

4.2.4 — We can define a "spectral structure" on L_ρ^2 by expanding the Laplacian (sub-laplacian for Δ_∞) defined on $H_{\rho,0}^1\big(B_\rho(1)\big)$ thanks to the following quadratic form

$$||v||_{\rho,0}^2 = |v|_\rho^2 + (v, \Delta_\rho v)_\rho$$

Now let us see what can we say of a bounded net in $H_{\rho,0}^1\big(B_\rho(1)\big)$ for this quadratic form.

Lemma 15. *Let $(u_\rho)_\rho$ be a net with $u_\rho \in H_{\rho,0}^1\big(B_\rho(1)\big)$ for every ρ, if there is a constant C such that for all $\rho > 0$ we have*

$$||u_\rho||_{\rho,0} \leq C$$

then there is a strongly converging sub-net in $\mathcal{L}^2$.

Proof. Let B a compact set such that $\bigcup_\rho \pi_\rho\big(B_\rho(1)\big) \subset B \subset G_\infty$ we are going to show that the strong convergence in $L^2(B, \mu_\infty)$ implies the strong convergence in $\mathcal{L}^2$. Then the compact embedding of $H_\infty^1\big(B\big)$ in $L^2\big(B, \mu_\infty\big)$ will conclude the proof.

Let us first notice that the periodicity with respect to Γ, and the co-compactness of Γ gives the existence of two constants α and β such that (we suppose the norms defined on B, and identify B et $\pi_\rho^{-1}B$)

$$\alpha|v|_\infty \leq |v|_\rho \leq \beta|v|_\infty \ .$$

Let us start by taking a net (v_ρ) strongly converging in $L^2(B, \mu_\infty)$ to v_∞ we also assume $v_\rho \circ \pi_\rho \in H^1_{\rho,0}\big(B_\rho(1)\big)$ for every ρ (because it is all we need). The first remark to be done is that thanks to the Gromov Haussdorff convergence, $v_\infty \in L^2_\infty$ (we mean that v_∞ can be considered equal to zero outside $B_\infty(1)$). Thus, let us take $c_p \in C^\infty_0\big(B_\infty(1)\big)$ be a sequence of functions strongly converging to v_∞ in L^2_∞. We have

$$|c_p \circ \pi_\rho - v_\rho \circ \pi_\rho|_\rho \leq \beta|c_p - v_\infty|_\infty + \gamma|v_\infty - v_\rho|_\infty$$

now let $\varepsilon > 0$ then for p large enough $\beta|c_p - v_\infty|_\infty \leq \varepsilon$. We fix p large enough and take ρ large enough for the second term to converge to 0 which gives us the strong convergence needed (see 7).

Now to conclude observe that from the assumptions the net $(u_\rho \circ \pi_\rho^{-1})$ (if need be we extend this function by zero outside $B_\rho(1)$) is bounded in $H^1_\infty(B)$, hence using the compact embedding of $H^1_\infty(B)$ in $L^2(B, \mu_\infty)$ (with the right regularity assumption on the boundary of B) we can extract a strongly converging net in $L^2(B, \mu_\infty)$ an by what we just did in $\mathcal{L}^2$. ❏

4.3   Compact convergence of the resolvents

4.3.1 — Let $\lambda > 0$, $a^\rho_\lambda(u,v) = (\Delta_\rho u, v)_\rho + \lambda(u,v)_\rho$ and G^ρ_λ be the operator from L^2_ρ to $H^1_{\rho,0} \subset L^2_\rho$ such that

$$a^\rho_\lambda(G^\rho_\lambda f, \phi) = (f, \phi)_\rho \quad \forall \phi \in H^1_{\rho,0}. \tag{5}$$

4.3.2 — We want to show that the net of operators (G^ρ_λ) converges compactly to G_λ the operator corresponding to the homogenised problem:

$$a^\infty_\lambda(G_\lambda F, \Phi) = (F, \Phi)_\infty \quad \forall \Phi \in H^1_{\infty,0} \tag{6}$$

with $(F, \Phi)_\infty = \int_{B_\infty(1)} F\Phi \, d\mu_\infty$ and

$$a^\infty_\lambda(u,v) = \int_{B_\infty(1)} q^{ij} X^\infty_i u \, X^\infty_j v \, d\mu_\infty + \lambda(u,v)_\infty$$

in other word we want to show the following theorem

Theorem 16. *For every $\lambda < 0$, the net of resolvents $(R^\rho_\lambda)_\rho$ of the Laplacian (Δ_ρ) converges compactly to R^∞_λ, the resolvent of Δ_∞ from the homogenised problem. Thus the net (Σ_ρ) compactly converges to Σ_∞.*

Proof. This comes from the fact that $R^\rho_\lambda = -G^\rho_{-\lambda}$ and $R^\infty_\lambda = -G_{-\lambda}$.

First step :

Let f_ρ be a weakly convergent net to f in $\mathcal{L}^2$, lemma 9 tells us that this net is uniformly bounded in $\mathcal{L}^2$ and in H^{-1}_ρ, the dual space of $H^1_{\rho,0}$.

Let $f_\rho \in H^1_{\rho,0}$ then thanks to (5) we have :

$$\alpha ||G^\rho_\lambda f_\rho||^2_{\rho,0} \leq (f_\rho, G^\rho_\lambda f_\rho)_\rho \leq K||f_\rho||_{H^{-1}_\rho}||G^\rho_\lambda f_\rho||_{\rho,0}$$

thus

$$||G^\rho_\lambda f_\rho||_{\rho,0} \leq C||f_\rho||_{H^{-1}_\rho}$$

the net $(G^\rho_\lambda f_\rho)$ being uniformly bounded for the norms $|| \cdot ||_{\rho,0}$, using lemma 15 there is a subnet strongly converging in $\mathcal{L}^2$. i.e.

$$u_\rho = G^\rho_\lambda f_\rho \to \tilde{u}_\lambda \text{ strongly in } \mathcal{L}^2 \tag{7}$$

Furthermore $P_\rho = (g^{ij}_\rho) \nabla G^\rho_\lambda f_\rho$ is also bounded in $\mathcal{L}^2$ thus there is a subnet of the net P_ρ weakly converging in $\mathcal{L}^2$ to $\tilde{P}_\lambda \in L^2_\infty$, moreover $\tilde{P}_\lambda$ is horizontal for if we write P^i_ρ and P^i_λ the coordinates of P_ρ and $\tilde{P}_\lambda$ then we have $P^i_\rho = (g^{ij}_\rho) \nabla G^\rho_\lambda f_\rho = \rho^{2-\alpha(i)-\alpha(j)} (g^{ij}(\delta_\rho x)) \nabla G^\rho_\lambda f_\rho$, so if $\alpha(i) \geq 2$ then this coordinates strongly converges to 0 in $\mathcal{L}^2$, because $(g^{ij}(\delta_\rho x)) \nabla G^\rho_\lambda f_\rho$ is also bounded.

Now for any $\phi_\infty \in L^2_\infty$ let ϕ_ρ be a strongly converging net to ϕ_∞ in $\mathcal{L}^2$ then

$$\int_{B_\rho(1)} P_\rho \cdot \nabla \phi_\rho \, d\mu_\rho + \lambda(G^\rho_\lambda f_\rho, \phi_\rho)_\rho = (f_\rho, \phi_\rho)_\rho \to$$
$$\int_{B_\infty(1)} \tilde{P}_\lambda \cdot \nabla_{\mathcal{H}} \phi_\infty \, d\mu_\infty + \lambda(u^*_\lambda, \phi_\infty)_\infty = (f, \phi_\infty)_\infty. \tag{8}$$

Thus it is enough to show that $\tilde{P}_\lambda = (q^{ij}) \nabla_{\mathcal{H}} \tilde{u}_\lambda$ on $B_\infty(1)$ because it induces $\tilde{u}_\lambda = G_\lambda f$.

Second step :

We first take $\chi^k(y)$ (see 4.1) such that $\mathcal{M}(\chi^k) = 0$ and we define

$$w_\rho(x) = x_k - \frac{1}{\rho} \chi^k(\delta_\rho x) \tag{9}$$

for every $k = 1, \ldots, d_1$. Then

$$w_\rho \to x_k \text{ strongly in } \mathcal{L}^2. \tag{10}$$

and by construction of χ^k (see 4.1) we have

$$-X^\rho_i \left(\det \tilde{g}(\delta_\rho x) g^{ij}_\rho \, X^\rho_j w_\rho \right) = 0 \text{ on } B_\rho(1). \tag{11}$$

We multiply this equation by a test function $\phi_\rho \in V_\rho$ and after an integration we get

$$\int_{B_\rho(1)} g^{ij}_\rho \, X^\rho_j w_\rho \, X^\rho_i \phi_\rho \, d\mu_\rho = 0 \tag{12}$$

Let $\varphi \in C^\infty_0(B_\infty(1))$ (notice that for ρ large enough the support of φ will be in $\pi_\rho(B_\rho(1))$) and $\phi_\rho = \varphi \circ \pi_\rho w_\rho$ which we put into the equation (5) and into the equation (12) we put $\phi_\rho = \varphi \circ \pi_\rho u_\rho$, and then we subtract the results

$$\int_{B_\rho(1)} g^{ij}_\rho \left(X^\rho_j u_\rho \left(X^\rho_i(\varphi \circ \pi_\rho) \right) w_\rho - X^\rho_j w_\rho \left(X^\rho_i(\varphi \circ \pi_\rho) \right) u_\rho \right) d\mu_\rho$$
$$= \int_{B_\rho(1)} f_\rho w_\rho \varphi \circ \pi_\rho \, d\mu_\rho - \lambda \int_{B_\rho(1)} \varphi \circ \pi_\rho u_\rho w_\rho \, d\mu_\rho \tag{13}$$

Now let $\rho \to \infty$ in (13), all terms converge because they are product of one strongly converging net and one weakly converging net in $\mathcal{L}^2$. More precisely,

(i) P_ρ, remember that $P_\rho^i = g_\rho^{ij} X_j^\rho u_\rho$, weakly converges to $\tilde{P}_\lambda$ in $\mathcal{L}^2$ following (8);

(ii) $\big(X_i^\rho(\varphi \circ \pi_\rho)\big)w_\rho$ strongly converges to $(X_i^\infty \varphi)x_k$ in $\mathcal{L}^2$ from (10) and because, writing l_x^ρ the function left multiplication by x in G_ρ:

$$X_i^\rho(\varphi \circ \pi_\rho)_{|x} = d\varphi_{\pi_\rho \circ l_x^\rho(e)} \circ d\pi_{\rho|l_x^\rho(e)} \circ dl_x^\rho \cdot X_i^\rho(e)$$

now by definition $l_x^\rho \to l_x^\infty$ and $\pi_\rho \to id_{G_\infty}$ which explains why

$$X_i^\rho(\varphi \circ \pi_\rho) \to X_i^\infty \varphi$$

pointwise (and weakly $\mathcal{L}^2$ from the claim in the proof of section 2.5).

(iii) for $1 \le i, j \le d_1$, $g_\rho^{ij} X_i^\rho w_\rho$ is periodic with respect to $\delta_{1/\rho}\Gamma$ and weakly converges in $\mathcal{L}^2$ towards the mean value

$$q^{jk} = \frac{1}{\mu_g(D_\Gamma)} \int_{D_\Gamma} \left(g^{ij}(y)\Big(\delta_{ik} - X_i \chi^k(y)\Big)\right) d\mu_g$$

this comes from the following claim

Lemma 17. *Let h be a function periodic with respect to Γ on G. Let h_ρ be defined on G_ρ by $h_\rho(x) = h(\delta_\rho x)$. Then (h_ρ) weakly converges in $\mathcal{L}^2$ toward*

$$h_\infty = \frac{1}{\mu_g(D_\Gamma)} \int_{D_\Gamma} h d\mu_g$$

i.e. for any $u_\rho \to u_\infty$ strongly in $\mathcal{L}^2$ we have

$$\int_{B_\rho(1)} u_\rho h_\rho d\mu_\rho \to h_\infty \int_{B_\infty(1)} u_\infty d\mu_\infty$$

To see this apply the proof in section 2.5 to $h^{1/n}g$ instead of g (even if it is not a measure, what makes everything work in that proof is the fact that $\det g$ in the coordinates (X_i) is periodic with respect to Γ).

(iv) for $\alpha(i) + \alpha(j) > 2$, $g_\rho^{ij} X_i^\rho w_\rho = \rho^{2-\alpha(i)-\alpha(j)} g^{ij}(\delta_\rho x) X_i^\rho w_\rho$ thus this term weakly converges in $\mathcal{L}^2$ towards 0.

(v) $\big(X_j^\rho(\varphi \circ \pi_\rho)\big)u_\rho$ strongly converges to $(X_j^\infty \varphi)\tilde{u}_\lambda$ by (7), because φ has compact support.

(vi) Now for the right side, w_ρ strongly converges as u_ρ does and f_ρ weakly converges to f.

To summarise (13) converges to (remember that $\tilde{P}_\lambda^i$ are the coordinates of $\tilde{P}_\lambda$ which is horizontal)

$$\int_{B_\infty(1)} \big(\tilde{P}_\lambda^j x_k - q^{jk}\tilde{u}_\lambda\big) X_j^\infty \varphi \, d\mu_\infty = \int_{B_\infty(1)} f x_k \varphi \, d\mu_\infty - \lambda \int_{B_\infty(1)} \varphi u_\lambda x_k \, d\mu_\infty \quad (14)$$

furthermore if we put into equation (8), $\phi_\infty = \varphi x_k$ it gives

$$\int_{B_\infty(1)} f x_k \varphi \, d\mu_\infty - \lambda \int_{B_\infty(1)} \varphi \tilde{u}_\lambda x_k \, d\mu_\infty = \int_{B_\infty(1)} \tilde{P}_\lambda^j X_j^\infty(\varphi x_k) \, d\mu_\infty \quad (15)$$

and by mixing (14) and (15) we get for every $\varphi \in \mathcal{C}_c^\infty(B_\infty(1))$ the following equality :

$$\int_{B_\infty(1)} \left(\tilde{P}_\lambda^j x_k - q^{jk}\tilde{u}_\lambda\right) X_j^\infty \varphi \, d\mu_\infty = \int_{B_\infty(1)} \tilde{P}_\lambda^j X_j^\infty(\varphi x_k) \, d\mu_\infty$$

which in terms of distribution can be translated into :

$$-\sum_{j=1}^{d_1} X_j^\infty \left(\tilde{P}_\lambda^j x_k - q^{jk}\tilde{u}_\lambda\right) = -\sum_{j=1}^{d_1} X_j^\infty \tilde{P}_\lambda^j \, x_k \iff \tilde{P}_\lambda^k = \sum_{j=1}^{d_1} q^{jk} \, X_j^\infty \tilde{u}_\lambda$$

which allow us to conclude that $\tilde{u}_\lambda = G_\lambda f$. ❑

4.3.3 — It is now easy to finish the proof, for theorem 16 gives the compact convergence of the resolvents fact which thanks to theorem 11 gives the convergence of the spectrum which is what theorem 3 is all about.

5   Macroscopic spectral rigidity

In this section we give the proof of the upper bound on the first eigenvalue and of theorem 2.

5.1   The upper bound

By proposition 13 we have $B_\infty(1) \supset B_2(1)$ thus by the minmax property, we have

$$\lambda_i(B_\infty(1)) \leq \lambda_i(B_2(1)) \tag{16}$$

for any i and equality holds, by the maximum principle (see J.-M. Bony [Bon69]) if and only if the two balls coincides.

5.2   Nilmanifolds having a one dimensional center

The aim of this part is to characterize the case of equality in (16) for a class of nilmanifold, which contains the Heisenberg groups.

Let us first introduce the description of the metrics involved.

Definition 18. *Let N^{n+1} be a 2-step nilmanifold whose kernel is one dimensional. Suppose that there is a submersion p of N^{n+1} onto a flat torus $\mathbb{T}^n$. Let $(\alpha_1, \ldots, \alpha_n)$ be the lift of an orthonormal base of harmonic 1-forms over the torus. Choose a 1-form ϑ of N^{n+1} such that $d\vartheta = p^*b$ where b is a closed 2-form over the torus (in other words we chose a connection). Let g_ϑ be the Riemanniann metric such that the dual base of $(\alpha_1, \ldots, \alpha_n, \vartheta)$ is orthonormal. Thus p becomes a Riemannian submersion. We will call such a metric* pseudo left invariant*.*

The idea is that if your choice of connections gives a left invariant base of vector fields then the above construction gives a left invariant metric. Thus this pseudo left invariant metric can be seen as perturbation of a left invariant metric, by perturbation of a left invariant base of vector fields.

We are now able to give our precise claim

Lemma 19. *Let $(\mathbb{H}_{2n+1}, g)$ be the $2n+1$-dimensional Heisenberg group, then its stable norm coincides with its albanese Metric if and only if g is pseudo left invariant.*

which thanks to theorem 1 allows us to obtain the following corollary :

Corollary 20. *Let $(\mathbb{H}_3, g)$ be the 3-dimensional Heisenberg group, $B_g(\rho)$ the induced Riemannian ball of radius ρ on its universal cover and $\lambda_1(B_g(\rho))$ the first eigenvalue of the Laplacian on $B_g(\rho)$ for the Dirichlet problem.*

Then

1. $\lim_{\rho \to +\infty} \rho^2 \lambda_1(B_g(\rho)) = \lambda_1^\infty \leq \lambda_1^{\mathcal{H}_3}$

2. *in case of equality the metric is pseudo left invariant.*

Where $\lambda_1^{\mathcal{H}_3}$ is the first eigenvalue of the Kohn laplacian arising from the Albanese metric on the unit ball of the carnot-caratheodory ball arising from the same metric.

In fact we are going to show a little more, we are going to focus on 2-step nilmanifolds for which the center of their Lie algebra is of dimension 1.

Lemma 21. *Let (M^{n+1}, g) be a 2-step nilmanifold whose center is of dimension 1, then its stable norm and its Albanese metric coincides if and only if the metric is pseudo left invariant.*

Proof. The main idea comes from the fact that the albanese metric and the stable norm coincides if and only if every harmonic 1-form has constant length.

Now take an orthonormal base of Harmonic 1-forms $\alpha_1, \ldots, \alpha_n$ (which can be seen as lift of harmonic forms over the Albanese torus), and consider their dual vector fields with respect to the metric $X_1, \ldots, X_n$, they span $\mathcal{H}$. Using the fact that for a closed one form α and any vector fields X and Y we have:

$$\alpha([X, Y]) = X \cdot \alpha(Y) - Y \cdot \alpha(X)$$

we see that for any i, j

$$[X_i, X_j] \in \mathcal{H}^\perp$$

(remark that we can also deduce from that fact that $[X_i, X_j] = 2\nabla_{X_i} X_j$) Now look at Z the dual vector field to the 1-form $Z^\flat = *(\alpha_1 \wedge \ldots \wedge \alpha_n)$ ($*$ is the Hodge operator, thus this form is co-closed), its length is constant by construction. Furthermore Z belongs and spans $\mathcal{H}^\perp$.

For a co-closed one form α we have

$$\sum_i \nabla_{X_i} \alpha \cdot \alpha_i + \nabla_Z \alpha \cdot Z^\flat = 0$$

which implies that for $i = 1, \ldots, n$

$$[X_i, Z] = 0$$

which also implies that Z is a killing field.

hence we have

$$[X_i, X_j] = f_{ij} Z$$

where f_{ij} are some functions, which are not all zero, otherwise our nilmanifold will be locally a flat torus.

Let us remark that (Z being a killing field) we have

$$dZ^\flat(X, Y) = 2g(\nabla_X Z, Y) \qquad (17)$$

thus if we decompose $dZ^\flat$ in the base given by $\alpha_i \wedge \alpha_j$, $Z^\flat \wedge \alpha_i$ and $Z^\flat \wedge Z^\flat$ for all i, j then thanks to and (17) we get that

$$dZ^\flat = \sum_{i<j} f_{ij} \alpha_i \wedge \alpha_j$$

In other words $dZ^\flat$ is horizontal and there exists some 2-form β on the albanese torus such that $dZ^\flat = \pi^* \beta$.

Hence g is pseudo left invariant.

❏

5.3   On the higher dimensional case

The case of tori, which is quite exceptional has been treated in [Ver02].

For other, non abelian nilmanifolds, it seems that things are not that simple, as in the case of the Heisenberg groups, it seems that there can be metrics which are not left invariant, but for which the equality between the stable norm and the albanese metric holds.

The interested reader will be able to consult [NV], where this question is studied in details, and where we explain the rigidity involved.

5.4   Graded nilmanifolds with totally geodesic fibers over a Torus

There is one last particular case we would like to study, the case where the nilmanifold is graded (i.e. its algebra is nilpotent and graded as defined in section 2.1.2) and the metric on (M^n, g) is as follows : We suppose that the first betti number $b_1(M^n) = k$, we recall that $\mathcal{H}$ is the horizontal distribution coming from V_1 (see sections 2.3.1 and 2.1.2). Moreover we assume that we have the following riemanniann submersion, with totally geodesics fibers and with a metric equivariant on the fibers :

$$[M, M] \hookrightarrow (M^n, g) \xrightarrow{p} (\mathbb{T}^k, \check{g})$$

where dp_x is an isometry (we write $\hat{g} = g_{|\mathcal{H}}$) from $(\mathcal{H}_x, \hat{g}_x)$ to $(T_{p(x)} \mathbb{T}^k, \check{g}_{p(x)})$.

Then, in case of equality in the theorems 1 and 4, the albanese map is a riemanniann submersion, which implies that $\check{g}$ is flat. Which in turn, using our assumptions implies that the metric g is left invariant (indeed see chapter 9 section F in [Bes87]). In other words :

Proposition 22. *Let (M, g) satisfying the above assumptions. The albanese metric and the stable norm coincides if and only if the metric is left invariant.*

In other words, we could say heuristically that for sub-riemannian metrics the equality case in theorems 1 (which holds in that context too, see [Ver01] for the convergence of the spectrum) characterises the left-invariant sub-riemannian metrics.

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