Amenability and Margulis super-rigidity

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Abstract

We give a survey of amenability for locally compact groups, and we illustrate the role of amenability in the proof of Margulis' super-rigidity theorem on finite-dimensional representations of lattices in semisimple Lie groups.

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1 Introduction

Amenability was introduced in 1929 by J. von Neumann [vN29] for discrete groups, and in 1950 by M. Day [Day50] for general locally compact groups. Originating from harmonic analysis and representation theory, amenability extended to a well-established body of mathematics, with applications in: dynamical systems, operator algebras, graph theory, metric geometry,... One definite advantage of amenability for groups is the equivalence of various, apparently remote, characterizations. So in Chapter 1 we survey the classical theory of amenability for a locally compact group G, (our basic reference being Appendix E in [BHV]) and we establish the equivalence between:

- G is amenable, in the sense that every action of G by homeomorphisms on a compact space X, fixes some probability measure on X;
- any affine G-action on a compact convex set (in a locally convex Hausdorff space) has a fixed point;
- G admits an invariant mean;
- (Reiter's property (P_1)) For every compact subset $Q \subset G$ and $\epsilon > 0$, there exists $f \in L^1(G)_{1,+}$ such that
$$\max_{x \in Q} \|\lambda_G(x)f - f\| \leq \epsilon;$$
- (Reiter's property (P_2)) The left regular representation λ_G almost has invariant vectors;
- the representation $\infty\lambda_G$ almost has invariant vectors.

In Chapter 2, we digress on ergodic theory for group actions on measure spaces. The goal of the chapter is to establish Moore's ergodicity theorem, stating that if Γ is a lattice in a non-compact simple Lie group G, and H is a non-compact closed subgroup of G, then Γ acts ergodically on G/H. We deduce it from the Howe-Moore vanishing theorem, stating that coefficients of unitary representations of G having no non-zero fixed vector, go to zero at infinity of G. Our basic reference for that chapter is [BM00].

In Chapter 3, we explain how amenability is used in the proof of Margulis'super-rigidity theorem. Although semisimple Lie groups are very far from being amenable, they contain a co-compact amenable subgroup (namely a minimal parabolic subgroup) and this fact, together with Moore's ergodicity theorem, plays a crucial role in super-rigidity. References for this Chapter are [Mar91] and [Zim84].

The presentation follows rather closely the CIME course taught at San Servolo in June 2004. I thank heartily Andrea D'Agnolo and Massimo Picardello for bringing me to that magical place.

2 Amenability for locally compact groups

2.1 Definition, examples, and first characterizations

For a compact space X, we denote by $M(X)$ the space of probability measures on X.

Definition 1 *A locally compact group G is* **amenable** *if, for every compact space X endowed with a G-action, there exists a G-fixed point in $M(X)$ (i.e. G fixes a probability measure on X)*

Example 1 *: Compact groups are amenable.*

Indeed, let dg be normalized Haar measure on the compact group G. Let G act on the compact space X. Pick any $\mu \in M(X)$. Then $\nu = \int_G (g_* \mu)\, dg$ is a fixed point in $M(X)$.

Example 2 *: $SL_2(\mathbf{R})$ is not amenable.*

To see this, let $SL_2(\mathbf{R})$ act by fractional linear transformations on $P^1(\mathbf{R}) = \mathbf{R} \cup \{\infty\}$. Then $SL_2(\mathbf{R})$ fixes no measure at all on $P^1(\mathbf{R})$. Indeed look at subgroups

$$N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \quad (= \textit{ translations on } \mathbf{R});$$

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0 \right\} \quad (= \textit{ dilations on } \mathbf{R});$$

The only N-invariant measures on $P^1(\mathbf{R})$ are of the form $s\, dx + t\, \delta_\infty$ (where dx is Lebesgue measure on $\mathbf{R}$). Among these, the only A -invariant measures are the $t\, \delta_\infty$'s. But those are not w-invariant, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (so that $w(x) = \frac{-1}{x}$).

Proposition 1 *The following are equivalent:*

i) G is amenable;

ii) Any affine action of G on a (non-empty) convex, compact subset of a Hausdorff, locally convex topological vector space, has a fixed point.

Proof: $(ii) \Rightarrow (i)$: If X is a compact space, then $M(X)$ is a convex subset of $C(X)^*$ (space of all Borel measures on X), and $M(X)$ is compact in the weak-* topology.

$(i) \Rightarrow (ii)$: Let C be a compact convex subset in E. To each $\mu \in M(C)$, we associate its barycentre $b(\mu) \in C$: this is the unique point in C such that, for every $f \in E^*$:

$$f(b(\mu)) = \int_C f(c)\, d\mu(c)$$

(formally: $b(\mu) = \int_C c\, d\mu(c)$). If $\mu = \sum_i \lambda_i \delta_{c_i}$ is an atomic measure, i.e. a convex combination of Dirac masses, then $b(\mu) = \sum_i \lambda_i c_i$, and this can be extended to $M(C)$ using density of atomic measures in $M(C)$ (see Theorem 3.27 in [Rud73] for details). Clearly b commutes with affine maps of C: $b(T_*\mu) = T(b(\mu))$. In particular, if μ is a G-fixed probability measure on C, then $b(\mu)$ is a G-fixed point in C. $\square$

Here is the famous Markov-Kakutani theorem.

Theorem 1 *Every abelian group is amenable.*

Proof: Let G be an abelian group, acting on a compact convex subset C in E. For $g \in G$, define $A_n(g) : C \to C$ by

$$A_n(g)x = \frac{1}{n+1} \sum_{i=0}^{n} g^i x.$$

Let $\mathcal{G}$ be the semi-group generated by the $A_n(g)$'s ($n \geq 0$, $g \in G$). For every $\gamma \in \mathcal{G}$, the set $\gamma(C)$ is convex compact.

Claim: $\bigcap_{\gamma \in \mathcal{G}} \gamma(C) \neq \emptyset$

It is enough to see that $\gamma_1(C) \cap \ldots \cap \gamma_m(C) \neq \emptyset$, for $\gamma_1, \ldots, \gamma_m \in \mathcal{G}$. Set $\gamma = \gamma_1\gamma_2 \ldots \gamma_m \in \mathcal{G}$. Since $\mathcal{G}$ is abelian: $\gamma(C) \subset \gamma_i(C)$ for $i = 1, \ldots, m$, proving the claim.

Take $x_0 \in \bigcap_{\gamma \in \mathcal{G}} \gamma(C)$. We claim that x_0 is $\mathcal{G}$-fixed.

For every $n \geq 0, g \in G$, there exists $x \in C$ such that $A_n(g)x = x_0$. For $f \in E^*$:

$$|f(x_0 - gx_0)| = |f(\frac{1}{n+1}(\sum_{i=0}^{n} g^i x\; - \sum_{i=0}^{n} g^{i+1} x))|$$
$$= \frac{1}{n+1} |f(x - g^{n+1}x)| \leq \frac{2K}{n+1}$$

where $K = \max\{|f(c)| : c \in C\}$. So $f(x_0) = f(gx_0)$ for every $f \in E^*$, therefore $x_0 = gx_0$. $\square$

Definition 2 *A* **mean** *on G is a linear form m on $L^\infty(G)$, such that:*

i) $m(1) = 1$*;*

ii) $m(f) \geq 0$ *for every* $f \in L^\infty(G)$, $f \geq 0$

Example 3 *If μ is a Borel probability measure on G, absolutely continuous with respect to Haar measure, then $m(f) = \int_G f \, d\mu$ defines a mean on G.*

There are some important differences between probability measures and means:

1. means make up a convex compact subset in $L^\infty(G)^*$ (for the weak-* topology);
2. for $A \in \mathcal{B}$ (the Borel subsets of G), let χ_A be the characteristic function on G; let m be a mean on G, set $m(A) = m(\chi_A)$. The map $m : \mathcal{B} \to [0,1] : A \mapsto m(A)$ satisfies:
   (i) $m(G) = 1$;
   (ii) If $A_1, \ldots, A_n$ are pairwise disjoint, then $m(A_1 \cup \ldots \cup A_n) = m(A_1) + \ldots + m(A_n)$.

   This second property is **finite additivity** (as opposed to σ-additivity).

In other words, we may think of a mean as a probability measure which is only finitely additive.

Proposition 2 *The following are equivalent:*

i) G is amenable;

ii) G admits an invariant mean.

Proof: $(i) \Rightarrow (ii)$ Follows from compactness and convexity of the set of means.

$(ii) \Rightarrow (i)$ Let G act on a compact space X. Fix $x_0 \in X$. For $f \in C(X)$, set $\phi_f(g) = f(gx_0)$. The map

$$\phi : C(X) \to L^\infty(G) : f \mapsto \phi_f$$

is G-equivariant. So if m is an invariant mean on G, then $\mu(f) = m(\phi_f)$ defines a G-invariant linear functional on $C(X)$. This μ is positive, unital, so by the Riesz representation theorem it is a probability measure on X. $\square$

Example 4 *The free group $\mathbf{F}_2$ on two generators a, b is not amenable.*

To see it, assume by contradiction that $\mathbf{F}_2$ is amenable. Let m be an invariant mean. Set

$$A = \{w \in \mathbf{F}_2 : w \,\text{starts with a non-zero power (positive or negative)of}\, a\}.$$

Then $A \cup aA = \mathbf{F}_2$, so $m(A) + m(aA) \geq 1$ and $m(A) = m(aA)$, so $m(A) \geq \frac{1}{2}$. On the other hand A, bA, b^2A are pairwise disjoint, so $m(A) + m(bA) + m(b^2A) \leq 1$; with $m(A) = m(bA) = m(b^2A)$, this gives $m(A) \leq \frac{1}{3}$, a contradiction.

2.2 Stability properties

Proposition 3 *Every closed subgroup of an amenable group is amenable.*

We postpone the proof until the end of section 2.5.

Proposition 4 *:*

i) Every quotient of an amenable group is amenable.

ii) Let $1 \to N \to G \to G/N \to 1$ be a short exact sequence, with N closed, amenable in G. The following are equivalent:

- *G is amenable;*
- *G/N is amenable.*

Proof: (i) Every action of G/N can be seen as an action of G.

(ii) Assume that G/N is amenable. Let G act affinely on a non-empty, compact convex subset C. Since N is amenable, the set C^N of N-fixed points is convex, compact and *non-empty*. Since N is normal, the set C^N is G-invariant, and the G-action factors through G/N. We conclude by amenability of G/N. $\square$

Example 5 *Solvable groups are amenable.*

This is proved by induction on the length of the derived series.

Example 6 *Borel subgroups are amenable. More precisely, if $G = KAN$ is a semisimple Lie group, and $P = MAN$ is a minimal parabolic subgroup, then P is amenable.*

Example 7 *Non-compact semisimple Lie groups are not amenable.*

Indeed, upon replacing G by $G/Z(G)$, we may assume that G has trivial centre. By root theory, G has a closed subgroup isomorphic to $(P)SL_2(\mathbf{R})$, so G is not amenable.

Proposition 5 *A connected Lie group G is amenable if and only if G is an extension of a solvable group by a compact group.*

Proof: Let $G = RS$ be a Levi decomposition (with R closed, normal, solvable, and S semisimple). Then G is amenable if and only if $S/(R \cap S)$ is amenable, if and only if S is compact. $\square$

2.3 Lattices in locally compact groups

Definition 3 *A discrete subgroup $\Gamma \subset G$ is a* **lattice** *if G/Γ carries a G-invariant probability measure. A lattice Γ is* **uniform**, *or* **co-compact**, *if G/Γ is compact.*

Example 8 *1) $\Gamma = \mathbf{Z}^n$ is a uniform lattice in $G = \mathbf{R}^n$;*

2) The discrete Heisenberg group

$$\Gamma = H(\mathbf{Z}) = \left\{ \begin{pmatrix} 1 & m & p \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} : m, n, p \in \mathbf{Z} \right\}$$

is a uniform lattice in the Heisenberg group $G = H(\mathbf{R})$;

3) $\Gamma = \mathbf{Z}^2 \rtimes_{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n} \mathbf{Z}$ is a uniform lattice in

$$SOL = \mathbf{R}^2 \rtimes_{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}} \mathbf{R};$$

4) $\Gamma = SL_n(\mathbf{Z})$ $(n \geq 2)$ is a non-uniform lattice in $G = SL_n(\mathbf{R})$;

5) $\Gamma = Sp_n(\mathbf{Z})$ $(n \geq 1)$ is a non-uniform lattice in $G = Sp_n(\mathbf{R})$;

6) The free group $\mathbf{F}_2$ can be embedded as a non-uniform lattice in $SL_2(\mathbf{R})$. E.g.,

$$\mathbf{F}_2 \simeq \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle$$

is of index 12 in $SL_2(\mathbf{Z})$;

7) Let Γ_g be the fundamental group of a closed Riemann surface of genus $g \geq 2$. Then Γ_g embeds as a uniform lattice in $G = PSL_2(\mathbf{R})$.

More examples of lattices will be given in section 4.5.

Proposition 6 *Let Γ be a lattice in G. The following are equivalent:*

i) G is amenable;

ii) Γ is amenable.

Proof of $(ii) \Rightarrow (i)$: Let G act affinely on a compact convex subset C. Let μ be an invariant probability measure on G/Γ. Since Γ is amenable, the set C^Γ is closed, convex, non-empty. For $x_0 \in C^\Gamma$, the orbit map $G \to C : g \mapsto g.x_0$ is (right) Γ-invariant, so factors through G/Γ. So $x = \int_{G/\Gamma} y.x_0 \, d\mu(y) \in C$, and this x is a G-fixed point, by G-invariance of μ. $\square$

2.4 Reiter's property (P_1)

We denote by λ_G the left regular representation of G on the space of all functions $G \to \mathbf{C}$. We set $L^1(G)_{1,+} = \{f \in L^1(G) : f \geq 0, \|f\|_1 = 1\}$.

Theorem 2 *The following are equivalent:*

1. G is amenable;

2. (Reiter's property (P_1), see [Rei52]) For every compact subset $Q \subset G$ and $\epsilon > 0$, there exists $f \in L^1(G)_{1,+}$ such that

$$\max_{x \in Q} \|\lambda_G(x)f - f\|_1 \leq \epsilon.$$

Proof: $(ii) \Rightarrow (i)$ Using the assumption, we find a net $(f_i)_{i \in I}$ in $L^1(G)_{1,+}$ such that $\lim_{i \in I} \|\lambda_G(x)f_i - f_i\|_1 = 0$ for every $x \in G$. Let m be any weak-* limit point of the f_i's in the set of means on G. Then m is a G-invariant mean on G.

$(i) \Rightarrow (ii)$ Recall that, for $f \in L^1(G)$, $\phi \in L^\infty(G)$:

$$(f \star \phi)(x) = \int_G f(y)\phi(y^{-1}x) \, dy = \int_G f(y)(\lambda_G(y)\phi)(x) \, dy$$

so that $f \star \phi \in L^\infty(G)$.

Let m be an invariant mean on G. Since m is continuous on $L^\infty(G)$, we have

$$m(f \star \phi) = m(\phi) \tag{1}$$

if $f \in L^1(G)_{1,+}$.

Since $L^1(G)_{1,+}$ is weak-* dense in the space of all means on G, there exists a net $(f_i)_{i\in I}$ such that for every $\phi \in L^\infty(G)$:

$$\lim_{i\in I} \int_G f_i(y)\phi(y)\, dy = m(\phi).$$

For every $f \in L^1(G)_{1,+}$, we also have, because of (1):

$$\lim_{i\in I} \int_G f_i(y)(f \star \phi)(y)\, dy = m(\phi).$$

From this, we deduce $\lim_{i\in I}(f \star f_i - f_i) = 0$ in the weak topology of $L^1(G)$.

Consider now the space E of all functions $L^1(G)_{1,+} \to L^1(G)$, endowed with the pointwise norm topology. The set

$$\Sigma = \{(f \star g - g)_{f\in L^1(G)_{1,+}} : g \in L^1(G)_{1,+}\}$$

is convex in E, and its weak closure contains 0, by the previous observation. Since the weak closure of Σ coincides with its closure in the pointwise topology (a general fact from functional analysis, see Theorem 3.12 in [Rud73]), there exists a net $(g_j)_{j\in J}$ in $L^1(G)_{1,+}$ such that $\lim_{j\in J} \|f \star g_j - g_j\|_1 = 0$ for every $f \in L^1(G)_{1,+}$. Since $\|g_j\|_1 = 1$, this convergence is uniform on norm-compact subsets of $L^1(G)_{1,+}$. One such subset is $\{\lambda_G(x)f : x \in Q\}$. So fix $f_0 \in L^1(G)_{1,+}$, and find $j \in G$ large enough so that $\|\lambda(x)(f_0 \star g_j) - g_j\|_1 \leq \frac{\epsilon}{2}$ for $x \in Q \cup \{1\}$. Set $f_{Q,\epsilon} = f_0 \star g_j$: then $f_{Q,\epsilon} \in L^1(G)_{1,+}$ and

$$\begin{aligned} &\|\lambda_G(x) f_{Q,\epsilon} - f_{Q,\epsilon}\|_1 \\ \leq\ & \|\lambda_G(x)(f_0 \star g_j) - g_j\|_1 + \|g_j - (f_0 \star g_j)\|_1 \leq \epsilon \end{aligned}$$

for every $x \in Q$. □

2.5 Reiter's property (P_2)

Definition 4 *A unitary representation π of G* **almost has invariant vectors**, *or* **weakly contains the trivial representation** *if, for every compact subset Q of G and every $\epsilon > 0$, there exists a non-zero $\xi \in \mathcal{H}_\pi$ such that*

$$\max_{g\in Q} \|\pi(g)\xi - \xi\| \leq \epsilon\|\xi\|.$$

For a unitary representation π, we denote by $\infty\pi$ the (Hilbert) direct sum of countably many copies of π.

Theorem 3 *The following are equivalent:*

i) G is amenable;

ii) (Reiter's property (P_2), see [Rei64]) The left regular representation λ_G almost has invariant vectors.

iii) The representation $\infty\lambda_G$ almost has invariant vectors.

Proof: $(i) \Rightarrow (ii)$ Fix a compact subset $Q \subset G$ and $\epsilon > 0$. As G is amenable, by Reiter's property we find $f \in L^1(G)_{1,+}$ such that $\|\lambda_G(x)f - f\|_1 \leq \epsilon$. Set $g = \sqrt{f}$. Then $g \in L^2(G)$ and $\|g\|_2 = 1$. Moreover, using $|a-b|^2 \leq |a^2 - b^2|$ for $a, b \geq 0$, we get

$$\begin{aligned}\|\lambda_G(x)g - g\|_2^2 &\leq \int_G |g(x^{-1}y)^2 - g(y)^2|\, dy \\ &= \|\lambda_G(x)f - f\|_1 \leq \epsilon.\end{aligned}$$

$(ii) \Rightarrow (iii)$ Obvious, since λ_G is a subrepresentation of $\infty\lambda_G$.

$(iii) \Rightarrow (i)$ We assume that $\infty\lambda_G$ almost has invariant vectors and prove in 3 steps that G satisfies Reiter's property (P_1), hence is amenable. So fix a compact subset $Q \subset G$, and $\epsilon > 0$; find a sequence $(f_n)_{n\geq 1}$ of functions, $f_n \in L^2(G)$, $\sum_{n=1}^{\infty} \|f_n\|_2^2 = 1$, such that $\sum_{n=1}^{\infty} \|\lambda_G(x)f_n - f_n\|_2^2 < \frac{\epsilon^2}{4}$ for $x \in Q$.

1) Replacing f_n with $|f_n|$, we may assume that $f_n \geq 0$.

2) Set $g_n = f_n^2$, so that $g_n \in L^1(G)$, $\sum_{n=1}^{\infty} \|g_n\|_1 = 1$, $g_n \geq 0$. For $x \in Q$, we have:

$$\begin{aligned}\sum_{n=1}^{\infty} \|\lambda_G(x)g_n - g_n\|_1 &= \sum_{n=1}^{\infty} \int_G |f_n(x^{-1}y)^2 - f_n(y)^2|\, dy \\ &= \sum_{n=1}^{\infty} \int_G |f_n(x^{-1}y) - f_n(y)|(f_n(x^{-1}y) + f_n(y))\, dy \\ &\leq \left(\sum_{n=1}^{\infty} \int_G |f_n(x^{-1}y) - f_n(y)|^2\, dy\right)^{\frac{1}{2}} \times \left(\sum_{n=1}^{\infty} \int_G (f_n(x^{-1}y) + f_n(y))^2\, dy\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq \left(\sum_{n=1}^{\infty} \|\lambda_G(x) f_n - f_n\|_2^2 \right)^{\frac{1}{2}} \times \left(2 \sum_{n=1}^{\infty} \int_G (f_n(x^{-1}y)^2 + f_n(y)^2)\, dy \right)^{\frac{1}{2}}$$

$$= 2 \left(\sum_{n=1}^{\infty} \|\lambda_G(x) f_n - f_n\|_2^2) \right)^{\frac{1}{2}} < \epsilon$$

where we have used consecutively the Cauchy-Schwarz inequality, $(a+b)^2 \leq 2(a^2+b^2)$ for $a, b > 0$, and the fact that $\sum_{n=1}^{\infty} \|f_n\|_2^2 = 1$.

3) Set $F = \sum_{n=1}^{\infty} g_n$. Then $F \geq 0$ and $\|F\|_1 = \sum_{n=1}^{\infty} \|g_n\|_1 = 1$. Moreover, for $x \in Q$:

$$\|\lambda_G(x) F - F\|_1 \leq \sum_{n=1}^{\infty} \|\lambda_G(x) g_n - g_n\|_1 < \epsilon$$

by the previous step. This establishes property (P_1) for H. □

Finally we reach a result left unproved in section 2.2:

Corollary 1 *Closed subgroups of amenable groups are amenable.*

Proof: Let H be a closed subgroup of the amenable group G. Choose a measurable section s for $G \to H\backslash G$; so every $g \in G$ is written uniquely $g = hs(y)$, with $h \in H$, $y \in H\backslash G$. This gives an H-equivariant measurable identification $G \simeq H \times H\backslash G$, inducing a unitary map $L^2(G) \to L^2(H) \otimes L^2(H\backslash G)$ intertwining $\lambda_G|_H$ and $\lambda_H \otimes 1$. Choosing an orthonormal basis of $L^2(H\backslash G)$, we identify $\lambda_H \otimes 1$ with the direct sum of $[G : H]$ copies of λ_H, which we embed as a subrepresentation in $\infty\lambda_H$. This means that $\infty\lambda_H$ almost has invariant vectors, hence H is amenable by Theorem 3. □

2.6 Amenability in Riemannian geometry

In the Introduction, we mentioned that amenability became relevant in various fields of mathematics. In this section, independent of the rest of the Chapter, we substantiate this claim and indicate how amenability enters Riemannian geometry.

Let N be a complete Riemannian manifold. It carries a *Laplace operator*

$$\Delta_N = d^* d = -div \circ grad.$$

This operator is self-adjoint on $L^2(N)$, so it has a non-negative spectrum, and we denote by $\lambda_0(N)$ the bottom of its spectrum:

$$\lambda_O(N) = \inf\{\lambda \geq 0 : \lambda \in Spec_{L^2(N)}\Delta_N\}.$$

The following result was obtained by R. Brooks [Bro81].

Theorem 4 *Let M be a compact Riemannian manifold. Let $\tilde{M}$ be the universal cover of M, and $\pi_1(M)$ its fundamental group. The following are equivalent:*

i) $\pi_1(M)$ *is amenable;*

ii) $\lambda_0(\tilde{M}) = 0$. □

Since $\pi_1(M)$ only depends of the topological structure of M, this shows in particular that the property $\lambda_0(\tilde{M}) = 0$ does *not* depend on the choice of a Riemannian structure on M.

Example 9 *1.* $\lambda_0(\mathbf{R}^n) = 0$*, which gives another proof of the amenability of* $\mathbf{Z}^n$.

2. If $\mathbb{H}^2$ *denotes the Poincaré disk, with the metric of constant curvature* -1*, then* $\lambda_0(\mathbb{H}^2) = \frac{1}{4}$*, which gives another proof of the non-amenability of the surface group* Γ_g*,* $g \geq 2$.

3 Measurable ergodic theory

3.1 Definitions and examples

In this section, the context will be the following:

- G is a locally compact, σ-compact group;
- (X, μ) is a standard measure space;
- G is acting on (X, μ), i.e. we are given a measurable map $G \times X \to X : (g, x) \mapsto gx$ which is an action such that, for every $g \in G$, the measure $g_*\mu$ is equivalent to μ (i.e. they have the same null sets); when this happens, we say that μ is **quasi-invariant**.

Definition 5 *The measure μ is* **invariant** *if $g_*\mu = \mu$ for every $g \in G$;*

Definition 6 *The action of G on (X, μ) is* **ergodic** *if every G-invariant measurable subset A is either null or co-null (i.e. $\mu(A) = 0$ or $\mu(X - A) = 0$).*

Example 10 *(invariant measures on homogeneous spaces, see section 9 in [Wei65]) Let L be a closed subgroup of G; there always exists a quasi-invariant measure on G/L; there exists an invariant measure on G/L if and only if the restriction to L of the modular function of G, coincides with the modular function of L. Since the action of G on G/L is transitive, it is trivially ergodic.*

Example 11 *(irrational rotation) Take $X = S^1$, μ = normalized Lebesgue measure. Fix $\theta \in \mathbf{R} - \mathbf{Q}$. Let $G = \mathbf{Z}$ act on S^1 by powers of the irrational rotation T of angle $2\pi\theta$:*

$$T(z) = e^{2\pi i\theta} z.$$

This action is measure-preserving, and ergodic.

To check ergodicity, it is convenient to appeal to Fourier series: if $A \subset S^1$ is T-invariant, let χ_A be its characteristic function, and

$$\chi_A(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$$

be its Fourier expansion in $L^2(S^1)$. Then

$$T^*(\chi_A)(z) = \chi_A(T^{-1}z) = \sum_{n=-\infty}^{+\infty} e^{-2\pi i n\theta} a_n z^n.$$

By T-invariance, we must have $a_n = e^{-2\pi i n\theta} a_n$ for every $n \in \mathbf{Z}$, so $a_n = 0$ for $n \neq 0$, as θ is irrational; thus χ_A is constant, i.e. either $\chi_A = 0$ and $\mu(A) = 0$, or $\chi_A = 1$ and $\mu(A) = 1$.

Example 12 *(linear action on tori) The linear action of $SL_n(\mathbf{Z})$ on $\mathbf{R}^n$ leaves $\mathbf{Z}^n$ invariant, so descends to an action on the n-torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. Let μ be normalized Lebesgue measure on $\mathbf{T}^n$: since Lebesgue measure is $SL_n(\mathbf{R})$-invariant on $\mathbf{R}^n$, the measure μ is $SL_n(\mathbf{Z})$-invariant on $\mathbf{T}^n$. This action is ergodic.*

To check ergodicity of this action, we use n-variable Fourier series: if $A \subset \mathbf{T}^n$ is $SL_n(\mathbf{Z})$-invariant, let χ_A be its characteristic function, and

$$\chi_A(z_1, \ldots, z_n) = \sum_{r \in \mathbf{Z}^n} a_r z^r$$

(where $z^r = z_1^{r_1} \dots z_n^{r_n}$) be its Fourier expansion in $L^2(\mathbf{T}^n)$. For $g \in SL_n(\mathbf{Z})$, one has

$$(g\chi_A)(z) = \chi_A(g^{-1}z) = \sum_{r \in \mathbf{Z}^n} a_r z^{{}^t g^{-1} r} = \sum_{r \in \mathbf{Z}^n} a_{{}^t g r} z^r.$$

Since A is $SL_n(\mathbf{Z})$-invariant, we have $a_{{}^t g r} = a_r$ for every $r \in \mathbf{Z}^n$ and $g \in SL_n(\mathbf{Z})$; i.e. a_r is constant on $SL_n(\mathbf{Z})$-orbits on $\mathbf{Z}^n$. Notice that non-trivial orbits are infinite. Since $(a_r)_{r \in \mathbf{Z}^n} \in \ell^2(\mathbf{Z}^n)$, we must have $a_r = 0$ for $r \neq 0$. So χ_A is constant and we conclude as in Example 11.

Note that in example 12, $SL_n(\mathbf{Z})$ can be replaced by any subgroup with infinite non-trivial orbits on $\mathbf{Z}^n$. For $n = 2$, one can for example take the infinite cyclic subgroup generated by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Definition 7 *A Borel space is* **countably separated** *if there exists a countable family of Borel subsets separating points (i.e. two distinct points can be put in two disjoint subsets of the countable family).*

For example, $\mathbf{R}^n$ is countably separated since we can take the collection of balls with rational centres and rational radii as a countable family separating points.

Proposition 7 *Let S be countably separated. If the G-action on (X, μ) is ergodic, any measurable G-invariant map $f : X \to S$, is almost everywhere constant.*

Proof: Let $(A_j)_{j \geq 1}$ be a sequence of Borel subsets separating points in S. Let $\mathcal{P}_n$ be the partition of S generated by $A_1, \dots, A_n$. For every $B \in \mathcal{P}_n$, the set $f^{-1}(B)$ is G-invariant, so it is either null or co-null; moreover there exists a unique $B_n \in \mathcal{P}_n$ such that $f^{-1}(B_n)$ is co-null. The sequence $(B_n)_{n \geq 1}$ is decreasing, and the intersection $\bigcap_{n=1}^{\infty} B_n$ is reduced to one point s, as the A_j's separate points in S. So $f(x) = s$ almost everywhere. $\square$

The converse of Proposition 7 holds in the finite measure-preserving case. The following result, due to Koopman [Koo31], is known today under the name "Koopmanism".

Proposition 8 *Let G act on the probability space (X, μ), preserving μ. The following are equivalent:*

i) The action is ergodic;

ii) Any G-invariant function in $L^2(X, \mu)$ is constant almost everywhere.

iii) *Set* $L^2_0(X,\mu) = \{f \in L^2(X,\mu) : \int_X f\, d\mu = 0\}$ *(the orthogonal of constants in* $L^2(X,\mu)$*). The representation of* G *on* $L^2_0(X,\mu)$ *has no non-zero fixed vector.*

Proof: $(i) \Rightarrow (ii)$: Follows from Proposition 7.
$(ii) \Rightarrow (iii)$: Obvious.
$(iii) \Rightarrow (i)$: If a Borel subset A is G-invariant, set

$$\xi_A(x) = \begin{cases} 1-\mu(A) & if \quad x \in A \\ -\mu(A) & if \quad x \notin A \end{cases}$$

ξ_A is a G-invariant function in $L^2_0(X,\mu)$, so $\xi_A = 0$; this gives the result. □

Example 13 *Proposition 8 false in infinite measure. Indeed, let* $\mathbf{Z}$ *act by translation on* $\mathbf{R}$*: the action is not ergodic (why?). However the only* $\mathbf{Z}$*-invariant function in* $L^2(\mathbf{R})$ *is the constant* 0.

3.2 Moore's ergodicity theorem

We have seen that, if L is a closed subgroup of G, then the action of G on G/L is trivially ergodic. A more interesting situation arises by considering H, L, two closed subgroups of G. Question: when is the H-action on G/L ergodic? Moore's theorem gives the answer.

Theorem 5 *The following are equivalent:*

i) *The* H*-action on* G/L *is ergodic;*

ii) *The* L*-action on* $H\backslash G$ *is ergodic.*

Example 14 *Let* $\Gamma = SL_2(\mathbf{Z})$ *act by fractional linear transformations on the real projective line* $\mathbf{P}^1(\mathbf{R}) \simeq S^1$*. Is the action ergodic? Write* $\mathbf{P}^1(\mathbf{R}) = G/P$*, where* $G = SL_2(\mathbf{R})$ *and* $P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$*. By Theorem 5, ergodicity of* Γ *on* G/P *(situation with no invariant measure) is equivalent to ergodicity of* P *on* $\Gamma\backslash G$ *(situation with a finite invariant measure).*

Later we will see that these actions *are* ergodic.

Lemma 1 *Let* H *be a closed subgroup of* G*; let* (X,μ) *be a* G*-space. The following are equivalent:*

i) H acts ergodically on X;

ii) G acts ergodically (via the diagonal action) on $X \times G/H$.

Proof: $(i) \Rightarrow (ii)$ Contraposing, assume that G is not ergodic on $X \times G/H$. So find $A \subset X \times G/H$, neither null nor co-null, G-invariant. Let $p : X \times G/H \to G/H$ be the second projection, set $A_y = p^{-1}(y) \cap A$ for $y \in G/H$. Since p is G-equivariant, one has $A_{gy} = gA_y$. Since G acts transitively on G/H, this implies (by Fubini) that A_{eH} is neither null nor co-null. But A_{eH} is an H-invariant subset in X, so H is not ergodic on X.

$(ii) \Rightarrow (i)$ Contraposing, assume that H is not ergodic on X. So find $B \subset X$, neither null nor co-null, H-invariant. Choosing a measurable section s for $G \to G/H$, we may define

$$A = \{(x, y) \in X \times G/H : x \in s(y)B\}.$$

The set A is then G-invariant: indeed, for $g \in G$ and $(x, y) \in A$, we must check that $gx \in s(gy)B$. But $s(gy) = gs(y)h$ for some $h \in H$ so $gx \in gs(y)B = gs(y)hB = s(gy)B$. One sees easily that A is neither null or co-null. □

Proof of Theorem 5: By lemma 1, the action of H on G/L is ergodic if and only if the action of G on $G/L \times H\backslash G$ is ergodic, if and only if the action of L on $H\backslash G$ is ergodic. □

Definition 8 *Let Γ be a lattice in a semisimple Lie group G. Say that Γ is* **irreducible** *if, for any normal subgroup N of positive dimension in G, the image of Γ in G/N is dense.*

This definition is designed to eliminate examples of the form $\Gamma = \Gamma_1 \times \Gamma_2$ in $G = G_1 \times G_2$, with Γ_i a lattice in G_i $(i = 1, 2)$.

Example 15 *Let σ be the non-trivial element of $Gal(\mathbf{Q}(\sqrt{2})/\mathbf{Q})$:*

$$\sigma(r + s\sqrt{2}) = r - s\sqrt{2}.$$

Then $SL_n(\mathbf{Z}[\sqrt{2}])$ sits as a non-uniform irreducible lattice in $SL_n(\mathbf{R}) \times SL_n(\mathbf{R})$, via the embedding $g \mapsto (g, \sigma(g))$.

Here is Moore's ergodicity theorem [Moo66]:

Theorem 6 *Let G be a connected, semisimple Lie group with finite centre. Let Γ be an irreducible lattice in G, and H a closed subgroup in G. If H is not compact, then Γ acts ergodically on G/H.*

In the next section, we will deduce this result from the Howe-Moore vanishing theorem. Notice that Theorem 6 implies, in Example 14, that the action of Γ on G/P is ergodic.

3.3 The Howe-Moore vanishing theorem

Let π be a (strongly continuous) unitary representation of a locally compact group G, on a Hilbert space $\mathcal{H}$. Denote by $\mathcal{H}^{\pi(G)}$ the space of fixed vectors in $\mathcal{H}$.

Definition 9 *π is a C_0-**representation** if all coefficients of π vanish at infinity on G, i.e. $\lim_{g\to\infty}\langle\pi(g)\xi|\eta\rangle = 0$ for every $\xi,\eta\in\mathcal{H}$.*

Example 16 *The left regular representation λ_G of G on $L^2(G)$ is C_0.*

Indeed, if $\xi,\eta\in L^2(G)$ have compact support in G, then so does $g\mapsto\langle\lambda_G(g)\xi|\eta\rangle$. By density of $C_c(G)$ in $L^2(G)$, we conclude that every coefficient vanishes at infinity.

Example 17 *C_0-representations have no finite-dimension subrepresentation.*

The reason is: if σ is a finite-dimensional unitary representation, then the identity $1 = |\det\sigma(g)|$ prevents σ from being C_0). Observe that this implies in particular that a C_0-representation has no non-zero fixed vector.

Here is the Howe-Moore theorem ([HM79], Theorem 5.1).

Theorem 7 *Let $G = \prod_i G_i$ be a semisimple Lie group with finite centre and simple factors G_i's, and let π be a unitary representation of G. Assume that $\mathcal{H}^{\pi(G_i)} = 0$ for every i. Then π is C_0.*

From this we deduce Moore ergodicity.

Proof of Theorem 6: Let H be a closed non-compact subgroup of G. To prove that Γ is ergodic on G/H, by lemma 1 it is enough to prove that H is ergodic on G/Γ. So let π be the representation of G on $L^2_0(G/\Gamma)$. Take a function on G/Γ which is G_i-invariant, lift to a left-G_i, right-Γ invariant function on G, and project to a right-Γ invariant function on $G_i\backslash G$. Since

Γ is irreducible, the image of Γ in $G_i \backslash G$ is dense, hence this function must be a.e. constant. This shows that $\pi(G_i)$ has no non-zero invariant function in $L^2_0(G/\Gamma)$. By Howe-Moore (Theorem 7), π is a C_0-representation. So $\pi|_H$ is C_0 as well, in particular (by Example 17) $\pi|_H$ has no non-zero invariant vector. By Proposition 8 (valid since we have a finite invariant measure on G/Γ), H is ergodic on G/Γ. $\square$

We will give a complete proof of the Howe-Moore theorem in the case of $SL_2(\mathbf{R})$, and then indicate briefly how to pass from $SL_2(\mathbf{R})$ to a more general semisimple Lie group.

Let G be a locally compact group, and let $\alpha = (a_n)_{n\geq 1}$ be a sequence in G. Set

$$U_\alpha^+ = \{g \in G : 1 \text{ is an accumulation point of } (a_n^{-1} g a_n)_{n\geq 1}\}$$

and let N_α^+ be the subgroup generated by U_α^+.

The following result is known as Mautner's phenomenon.

Proposition 9 *Let π be a (strongly continuous) unitary representation of G. Let ξ, ξ_0 be vectors in $\mathcal{H}$ such that $\lim_{n\to\infty} \pi(a_n)\xi = \xi_0$ in the weak topology. Then $\pi(x)\xi_0 = \xi_0$ for every $x \in N_\alpha^+$.*

Proof: Fix $x \in U_\alpha^+$. Let $(a_{n_k})_{k\geq 1}$ be a subsequence of α such that $\lim_{n\to\infty} a_{n_k}^{-1} x a_{n_k} = 1$. For every $\eta \in \mathcal{H}$:

$$\begin{aligned}
|\langle(\pi(x)\xi_0 - \xi_0)|\eta\rangle| &= |\langle\pi(x)\xi_0|\eta\rangle - \langle\xi_0|\eta\rangle| \\
&= \lim_{k\to\infty} |\langle\pi(x a_{n_k})\xi|\eta\rangle - \langle\pi(a_{n_k})\xi|\eta\rangle| \\
&= \lim_{k\to\infty} |\langle\pi(a_{n_k}^{-1} x a_{n_k})\xi|\pi(a_{n_k}^{-1})\eta\rangle - \langle\xi|\pi(a_{n_k}^{-1})\eta\rangle| \\
&\leq \lim_{k\to\infty} \|\pi(a_{n_k}^{-1} x a_{n_k})\xi - \xi\|.\|\eta\| \;=\; 0
\end{aligned}$$

by Cauchy-Schwarz. $\square$

Example 18 *Let G be $SL_2(\mathbf{R})$, $a_n = \begin{pmatrix} e^{t_n} & 0 \\ 0 & e^{-t_n} \end{pmatrix}$, with $t_n \to +\infty$. It is easy to see that $N_\alpha^+ = N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$.*

Lemma 2 *Let π be a unitary representation of $G = SL_2(\mathbf{R})$. If a vector $\xi \in \mathcal{H}$ is N-invariant, then it is G-invariant.*

Proof: For a vector $\eta \in \mathcal{H}$ of norm 1, consider $\phi_\eta(g) = \langle \pi(g)\eta|\eta\rangle$, the associated coefficient function. For any closed subgroup $H \subset G$, the vector η is H-fixed if and only if $\phi|_H = 1$, if and only if ϕ is H-bi-invariant (by the equality case of the Cauchy-Schwarz inequality).

Here, ξ is N-invariant, so $\phi_\xi =: \phi$ is N-bi-invariant.

1st step: ξ is P-invariant. Indeed, by right N-invariance, ϕ descends to a function $\tilde{\phi}$ on $G/N \simeq \mathbf{R}^2 - \{0\}$, which is continuous and constant on orbits of N. In particular, $\tilde{\phi}$ is constant on lines parallel to the horizontal axis, and distinct from this axis. By continuity, $\tilde{\phi}$ is equal to 1 on the horizontal axis. Observing that this axis (minus $\{0\}$) is the P-orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get that $\phi|_P = 1$, i.e. ξ is P-fixed.

2nd step: Since ϕ is P-bi-invariant, ϕ descends to a function $\overline{\phi}$ on $G/P \simeq \mathbf{P}^1(\mathbf{R})$, which is continuous and constant on P-orbits. But there are exactly two P-orbits, namely $\{0\}$ and its complement. By continuity we have $\overline{\phi} \equiv 1$, so $\phi \equiv 1$ and ξ is G-fixed. $\square$

Proof of Theorem 7, case $G = SL_2(\mathbf{R})$: Set

$$A^+ = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \geq 0 \right\}$$

and $K = SO(2)$. In view of the Cartan decomposition $G = KA^+K$, to show vanishing of coefficients it is enough to show that, for every $\xi, \eta \in \mathcal{H}$ one has $\lim_{t\to+\infty} \langle \pi(a_t)\xi|\eta\rangle = 0$. By compactness of closed balls in Hilbert spaces for the weak topology, we find an accumulation point ξ_0 of the $\pi(a_t)\xi$'s:

$$\lim_{n\to\infty} \pi(a_{t_n})\xi = \xi_0$$

in the weak topology. By Mautner's phenomenon (Proposition 9) and example 18, ξ_0 must be N-fixed. By lemma 2, the vector ξ_0 is also G-fixed. By assumption this implies $\xi_0 = 0$, so the only weak accumulation point of the $\pi(a_t)\xi$'s is 0. In other words $w - \lim_{t\to\infty} \pi(a_t)\xi = 0$, which amounts to the desired result. $\square$

Let us conclude by indicating how one can pass from $SL_2(\mathbf{R})$ to more general semisimple groups, say $SL_3(\mathbf{R})$. Here

$$A = \left\{ \begin{pmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{pmatrix} : t_1 + t_2 + t_3 = 0 \right\}.$$

Let π be a unitary representation of $SL_3(\mathbf{R})$, without non-zero fixed vector. Embed $SL_2(\mathbf{R})$ into $SL_3(\mathbf{R})$ in the three standard ways:

$$\begin{pmatrix} * & * & 0 \\ & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ & 0 & * \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Claim: For each of these 3 copies of $SL_2(\mathbf{R})$, the restriction $\pi|_{SL_2(\mathbf{R})}$ has no non-zero fixed vector.

Taking this claim for granted, we see (by the Howe-Moore theorem in the case of $SL_2(\mathbf{R})$) that $\pi|_{SL_2(\mathbf{R})}$ is a C_0-representation. Since A is generated by its intersections with the three embeddings of $SL_2(\mathbf{R})$, we get for $a \in A$:

$$\lim_{a\to\infty} \langle \pi(a)\xi|\eta\rangle = 0$$

i.e. π is C_0.

Proof of the Claim: Assume that $\pi|_{SL_2(\mathbf{R})}$ has a non-zero fixed vector ξ, say for the first embedding of $SL_2(\mathbf{R})$. We are going to show that ξ is fixed under $SL_3(\mathbf{R})$.

We use the fact that $SL_3(\mathbf{R})$ is generated by elementary matrices $U_{ij}(t)$ ($t \in \mathbf{R}, i \neq j$). Let us show that ξ is $U_{13}(\mathbf{R})$-invariant. Take $u = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $\alpha = (a_n)_{n\geq 1}$, with $a_n = \begin{pmatrix} e^{t_n} & 0 & 0 \\ 0 & e^{-t_n} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, for some sequence $t_n \to +\infty$. Then $u \in N_\alpha^+$. Since $\pi(a_n)\xi = \xi$, we have $\pi(u)\xi = \xi$ by the Mautner phenomenon (Proposition 9). $\square$

4 Margulis'super-rigidity theorem

4.1 Statement

Recall that the **real rank** of a semisimple Lie group G, denoted by $\mathbf{R}-rk(G)$, is the dimension of a maximal split torus in G. For example:

$$\mathbf{R} - rk(SL_n(\mathbf{R})) = n - 1;$$

$$\mathbf{R} - rk(SO(p, q)) = \min\{p, q\}.$$

Theorem 8 *(see [Mar91], Theorem 5.6) Take:*

- *G, a connected, semisimple real algebraic group, with no compact factor, and $\mathbf{R} - rk(G) \geq 2$;*
- *Γ an irreducible lattice in $G(\mathbf{R})$;*
- *k a local field of characteristic 0 (i.e. $k = \mathbf{R}, \mathbf{C}$ or a finite extension of $\mathbf{Q}_p$) and H a simple, connected, algebraic k-group.*

Assume that $\pi : \Gamma \to H(k)$ is a homomorphism with Zariski dense image. Then:

i) If $k = \mathbf{R}$ and $H(\mathbf{R})$ is not compact, then π extends to a rational homomorphism $G \to H$ defined over $\mathbf{R}$ (hence induces $G(\mathbf{R}) \to H(\mathbf{R})$);

ii) If $k = \mathbf{C}$, then either $\overline{\pi(\Gamma)}$ is compact, or π extends to a rational homomorphism $G \to H$;

iii) If k is totally disconnected, then $\overline{\pi(\Gamma)}$ is compact.

4.2 Mostow rigidity

One of the most spectacular applications of Theorem 8 is Mostow's rigidity theorem [Mos73].

Theorem 9 *Let G, G' be connected semi-simple Lie groups with trivial centre, no compact factors, and suppose $\Gamma \subset G$, $\Gamma' \subset G'$ are lattices. Assume Γ irreducible in G and $\mathbf{R} - rk(G) \geq 2$. Let $\pi : \Gamma \to \Gamma'$ be an isomorphism. Then π extends to an isomorphism $G \to G'$.*

In other words, the lattice determines the ambient Lie group.

Proof, from Theorem 8: For each simple factor H_i' of G', find a structure of a simple real algebraic group such that $H_i' = H_i'(\mathbf{R})^o$. By the Borel density theorem [Bor60], Γ' is Zariski-dense in $G' = \mathbb{G}'(\mathbf{R})^o = \prod_i H_i'(\mathbf{R})^o$. Similarly, write $G = \mathbb{G}(\mathbf{R})^o$. By Theorem 8 (case $k = \mathbf{R}$), applied to each factor H_i', we may extend π to a rational homomorphism $\mathbb{G} \to \mathbb{G}'$. Since $\pi(\mathbb{G})$ is an algebraic subgroup of $\mathbb{G}'$, by Zariski-density we deduce $\pi(\mathbb{G}) = \mathbb{G}'$ and from that: $\dim_{\mathbf{R}} \pi(G) = \dim_{\mathbf{R}} G'$, so $\pi(G) = G'$ by connectedness. Set $N = Ker\,\pi$. Assume $N \neq \{1\}$. Since G has no center, then $\dim_{\mathbf{R}} N > 0$. Since Γ is an irreducible lattice, the image of Γ is dense in G/N, which implies that $\pi(\Gamma)$ is dense in G', contradicting discreteness of Γ'. $\square$

We may rephrase the Mostow rigidity theorem as follows.

Let M, M' be locally symmetric Riemannian manifolds, with finite volume, irreducible (in the sense that neither M nor M' is locally a Riemannian product), with rank ≥ 2. If $\pi_1(M) \simeq \pi_1(M')$, then M is isometric to M' (up to a rescaling of metrics).

4.3 Ideas to prove super-rigidity, $k = \mathbf{R}$

Lemma 3 *Suppose $P \subset G$ and $L \subset H$ are proper real algebraic subgroups, and there exists a rational Γ-equivariant map $\phi : G/P \to H/L$ defined over* $\mathbf{R}$ *(where Γ acts on H/L via π - explicitly: $\phi(\gamma.x) = \pi(\gamma).\phi(x)$). Then π extends to a rational homomorphism $G \to H$ defined over* $\mathbf{R}$. $\square$

Proof: Idea: look at the graph of π:

$$gr(\pi) = \{(\gamma, \pi(\gamma)) : \gamma \in \Gamma\} \subset G \times H,$$

and show that the Zariski closure $\overline{gr(\pi)}^Z$ is the graph of a homomorphism.

1st step: The projection of $\overline{gr(\pi)}^Z$ on the first factor G, is *onto*: this follows from the Borel density theorem.

2nd step: We have to show that, if $(g, h_1), (g, h_2) \in \overline{gr(\pi)}^Z$, then $h_1 = h_2$.

For this, let $R(G/P, H/L)$ be the set of all rational maps $G/P \to H/L$. Let $G \times H$ act on $R(G/P, H/L)$ by

$$((g, h).\psi)(x) = h.\psi(g^{-1}x)$$

($\psi \in R(G/P, H/L)$). By assumption, our ϕ is Γ-equivariant. In terms of the $G \times H$-action, this means that ϕ is $gr(\pi)$-invariant. Since ϕ is rational, this implies that ϕ is $\overline{gr(\pi)}^Z$-invariant. In particular, for every $x \in G/P$:

$$h_1.\phi(g^{-1}x) = h_2.\phi(g^{-1}x)$$

i.e. $h_1^{-1}h_2$ fixes $\phi(G/P)$ pointwise.

On the other hand, $\pi(\Gamma)$ stabilizes $\phi(G/P)$, so $\pi(\Gamma)$ also stabilizes $\overline{\phi(G/P)}^Z$. By Zariski density of $\pi(\Gamma)$, we deduce that H stabilizes $\overline{\phi(G/P)}^Z$. Since H acts transitively on H/L, this implies that $\overline{\phi(G/P)}^Z = H/L$ (i.e. $\phi(G/P)$ is Zariski-dense in H/L).

As a consequence, $h_1^{-1}h_2$ fixes H/L pointwise, so $h_1^{-1}h_2 \in \bigcap_{h \in H} hLh^{-1}$. The latter is a proper normal subgroup of H. Since H is assumed to be simple, this subgroup is $\{1\}$, i.e. $h_1^{-1}h_2 = 1$. This concludes the proof. $\square$

We will apply this when P is a minimal parabolic subgroup of G, defined over $\mathbf{R}$.

Example 19 $G = SL_n,\ G(\mathbf{R}) = SL_n(\mathbf{R}),\ P = \left\{ \begin{pmatrix} * & \dots & \dots & * \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & * \end{pmatrix} \right\}.$

Set $P_0 = P \cap G(\mathbf{R})^o$. Then $G(\mathbf{R})^o/P_0$ is Zariski dense in the flag variety G/P. So, if we have a rational map $\phi : G/P \to H/L$ defined on a Zariski dense subset of $G(\mathbf{R})^o/P_0$, which is Γ-equivariant as a map $G(\mathbf{R})^o/P_0 \to H/L$, then ϕ is Γ-equivariant.

It is therefore enough to find a proper real algebraic subgroup $L \subset H$ and a rational Γ-equivariant map $G(\mathbf{R})^o/P_0 \to H/L$, defined on a Zariski-dense subset of $G(\mathbf{R})^o/P_0$. This will be done in two steps.

1st step: There is a proper real algebraic subgroup $L \subset H$ and a measurable Γ-equivariant map $\phi : G(\mathbf{R})^o/P_0 \to H(\mathbf{R})/L(\mathbf{R})$.

2nd step: Any such measurable Γ-equivariant map agrees almost everywhere with a rational map.

We shall *not* elaborate on the second step, and refer instead to Chapter 5 in [Zim84]. Note however that *it is here that* $\mathbf{R} - rk(G) \geq 2$ *is used!* The proof of the 1st step appeals to the following result of Furstenberg (Theorem 15.1 in [Fur73]), which will be proved in the next subsection.

Recall that, for a compact space X, we denote by $M(X)$ the set of probability measures on X.

Proposition 10 *Let X be a compact, metrizable Γ-space. There exists a measurable Γ-equivariant map $\omega : G/P \to M(X)$, i.e. $\omega(\gamma x) = \gamma\omega(x)$ for all $\gamma \in \Gamma$ and almost all $x \in G/P$.*

Proof of the 1st step above: Let Q be a proper parabolic subgroup of H, defined over $\mathbf{R}$. Then $X = H(\mathbf{R})/Q(\mathbf{R})$ is a compact metrizable Γ-space. By Proposition 10, we find $\omega : G/P \to M(X)$ measurable and Γ-equivariant. We then argue as follows:

- the orbit space $H(\mathbf{R})\backslash M(X)$ is countably separated (this is a result of Zimmer [Zim78]);
- Let $\overline{\omega} : G/P \to H(\mathbf{R})\backslash M(X)$ be the composition of ω with the quotient map $M(X) \to H(\mathbf{R})\backslash M(X)$. Then $\overline{\omega}(\gamma x) = \overline{\omega}(x)$ a.e. in $x \in G/P$. By

Moore's ergodicity theorem 6, Γ acts ergodically on G/P. By Proposition 7, the map $\overline{\omega}$ is almost everywhere constant on G/P. This means that ω takes values essentially in a unique orbit $H(\mathbf{R})\mu_0 \in M(X)$.

- For every $\mu \in M(X)$, the stabilizer of μ in $H(\mathbf{R})$ is the set of real points of a proper, real algebraic subgroup of H. This is another result of Furstenberg [Fur63].

Set then $L = Stab_H(\mu_0)$. The map $H(\mathbf{R})\mu_0 \to H(\mathbf{R})/L(\mathbf{R})$ is $H(\mathbf{R})$-equivariant. Composing, we get a Γ-equivariant measurable map $\phi : G/P \to H(\mathbf{R})/L(\mathbf{R})$, defined on a Γ-invariant co-null set. □

4.4 Proof of Furstemberg's Proposition 10 - use of amenability

We give Margulis'proof (see Theorem 4.5 in [Mar91]).

Denote by dg the Haar measure on G. Let $\Gamma \times G$ act on $G \times X$ by

$$(\gamma, g)(h, x) = (\gamma h g^{-1}, \gamma x).$$

The projection $p : G \times X \to G$ is $(\Gamma \times G)$-equivariant. Let Q be the set of non-negative Borel measures μ on $G \times X$ such that $p_*(\mu) = dg$ and $(\gamma, 1)_*\mu = \mu$ for every $\gamma \in \Gamma$. We make 3 observations.

- Q is non-empty. Indeed, fix D a Borel fundamental domain for Γ on G: for every $g \in G$, there exists a unique $\gamma_g \in \Gamma$ such that $g \in \gamma_g D$. Fix $x_0 \in X$ and define $\phi : G \to G \times X : g \mapsto (g, \gamma_g x_0)$. Then $(\gamma, 1)\phi(g) = (\gamma g, \gamma\gamma_g x_0) = \phi((\gamma, 1)g)$. So ϕ is measurable, Γ-equivariant, and $p \circ \phi = Id_G$. So $\phi_*(dg) \in Q$.

- Q is convex (clear) and compact in the weak-* topology. Indeed, if $(K_n)_{n\geq 1}$ is an increasing sequence of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$. Since X is compact, so is $K_n \times X$, and therefore elements in Q are uniformly bounded on $K_n \times X$, namely $\mu(K_n \times X) \leq \int_{K_n} dg$ for $\mu \in Q$. So Q is bounded; since it is also weak-* closed, by Tychonov it is weak-* compact.

- Q is $(\Gamma \times G)$-invariant. This is because $(\gamma, h)_*(dg) = dg$, since G is unimodular.

Since P is amenable, there exists $\tau \in Q$ which is $(\{1\} \times P)$-invariant, hence also $(\Gamma \times P)$-invariant, by definition. As $p_*(\tau) = dg$, we may disintegrate τ over G:

$$\tau = \int_G (\delta_g \otimes \nu_g)\, dg$$

where $\nu_g \in M(X)$ and the field $g \mapsto \nu_g$ is measurable, and unique up to modification on a null set. Now

$$\begin{aligned}(\gamma, p)_*(\tau) &= \int_G (\delta_{\gamma g p^{-1}} \otimes \gamma_* \nu_g)\, dg \qquad (h = \gamma g p^{-1}) \\ &= \int_G (\delta_h \otimes \gamma_* \nu_{\gamma^{-1} h p})\, dh.\end{aligned}$$

By uniqueness: $\nu_g = \gamma_* \nu_{\gamma^{-1} g p}$ for almost every $g \in G$. In particular $\nu_{gp} = \nu_g$ for almost every $g \in G$ and every $p \in P$. So we may define a measurable map

$$\omega : G/P \to M(X) : gP \mapsto \nu_g$$

which is Γ-equivariant. □

4.5 Margulis'arithmeticity theorem

Recall that two subgroups H_1, H_2 in the same group, are *commensurable* if their intersection $H_1 \cap H_2$ has finite index both in H_1 and H_2.

Definition 10 *Let G be a real, linear, semisimple Lie group with finite centre. A lattice Γ in G is* **arithmetic** *if there exists a semisimple algebraic $\mathbf{Q}$-group H and a surjective continuous homomorphism $\phi : H(\mathbf{R})^0 \to G$, with compact kernel, such that $\phi(H(\mathbf{Z}) \cap H(\mathbf{R})^0)$ is commensurable with Γ (here $H(\mathbf{R})^0$ is the connected component of identity in $H(\mathbf{R})$).*

Example 20 *Let Φ be a quadratic form in $n+1$ variables, with signature $(n, 1)$, and coefficients in a number field $k \subset \mathbf{R}$. We denote by SO_Φ the special orthogonal group of Φ: this is a simple algebraic group defined over k. Set $\Gamma = SO_\Phi(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of k.*

a) $\Phi = x_1^2 + \ldots + x_n^2 - x_{n+1}^2$; here $k = \mathbf{Q}$ and $H = SO_\Phi$, so that $\Gamma = SO(n,1)(\mathbf{Z})$ is a non-uniform arithmetic lattice in $SO_\Phi(\mathbf{R}) = SO(n, 1)$.

b) $\Phi = x_1^2 + \ldots + x_n^2 - \sqrt{2} x_{n+1}^2$; here $k = \mathbf{Q}(\sqrt{2})$ and $H = SO_\Phi \times SO_{\sigma(\Phi)}$, where σ is the non-trivial element of $Gal(k/\mathbf{Q})$. Then $\Gamma = SO_\Phi(\mathbf{Z}[\sqrt{2}])$ is a uniform arithmetic lattice in $SO_\Phi(\mathbf{R}) \simeq SO(n, 1)$.

c) $\Phi = x_1^2 + \ldots + x_n^2 - \delta x_{n+1}^2$ where $\delta > 0$ is a root of a cubic irreducible polynomial over $\mathbf{Q}$, having two positive roots δ, δ' and one negative root δ''. Here $k = \mathbf{Q}(\delta)$; let σ, τ be the embeddings of k into $\mathbf{R}$ defined by $\sigma(\delta) = \delta'$ and $\tau(\delta) = \delta''$. Then $H = SO_\Phi \times SO_{\sigma(\Phi)} \times SO_{\tau(\Phi)}$ and Γ is an irreducible, uniform, arithmetic lattice in $SO_\Phi(\mathbf{R}) \times SO_{\sigma(\Phi)}(\mathbf{R}) \simeq SO(n, 1) \times SO(n, 1)$.

Margulis'arithmeticity theorem is another spectacular application of super-rigidity (Theorem 8).

Theorem 10 *(see Chapter IX in [Mar91]) Let G be a connected semisimple Lie group with trivial centre, no compact factors, and $\mathbf{R} - rk(G) \geq 2$. Let $\Gamma \subset G$ be an irreducible lattice. Then Γ is arithmetic.* $\square$

Non-arithmetic lattices are known to exist in $SO(n,1)$ for every $n \geq 2$ (Gromov - Piatetskii-Shapiro [GPS88]), and in $SU(n,1)$ for $1 \leq n \leq 3$ (Deligne-Mostow [DM86]).

For other rank 1 groups, i.e. $Sp(n,1)$ and the exceptional group $F_{4(-20)}$, super-rigidity and arithmeticity of lattices have been established by Corlette [Cor92] and Gromov-Schoen [GS92].

For a wealth of material on arithmetic groups, see [Bor69] and [WM].

References

[BHV] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's Property (T).* Forthcoming book, currently available at `http://poncelet.sciences.univ-metz.fr/~bekka/`, 2004.

[BM00] B. Bekka and M. Mayer. *Ergodic theory and topological dynamics of group actions on homogeneous spaces.* Cambridge Univ. Press, London Math. Soc. Lect. Note Ser. 269, 2000.

[Bor60] A. Borel. Density properties for certain subgroups of semi-simple groups without compact components. *Ann. Math.*, 72:62–74, 1960.

[Bor69] A. Borel. *Introduction aux groupes arithmétiques.* Hermann, Actu. sci. et industr. 1341, 1969.

[Bro81] R. Brooks The fundamental group and the spectrum of the Laplacian. *Comment. Math. Helv.* 56:581–598, 1981.

[Cor92] K. Corlette. Archimedean superrigidity and hyperbolic rigidity. *Ann. of Math.*, 135:165–182, 1992.

[Day50] M. Day. Amenable groups. *Bull. Amer. Math. Soc.*, 56: 46, 1950.

[DM86] P. Deligne and G.D. Mostow. Monodromy of hypergeometric functions and non-lattice integral monodromy. *Publ. Math. IHES*, 63:5–89, 1986.

[Fur63] H. Furstenberg A Poisson formula for semisimple Lie groups *Annals of Math.* 77: 335-383, 1963

[Fur73] H. Furstenberg. Boundary theory and stochastic processes in homogeneous spaces. in: *Harmonic analysis on homogeneous spaces*, Symposia on Pure and Applied Math., Williamstown, Mass. 1972, Proceedings, 26: 193–229, 1973.

[GPS88] M. Gromov and I. Piatetski-Shapiro. Nonarithmetic groups in Lobachevsky spaces. *Publ. Math. IHES*, 66:93–103, 1988.

[GS92] M. Gromov and R. Schoen. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Publ. Math. IHES*, 76:165–246, 1992.

[HM79] R.E. Howe and C.C. Moore. Asymptotic properties of unitary representations. *Journal of Functional Analysis*, 32: 72–96, 1979.

[Koo31] B.O. Koopman. Hamiltonian systems and transformations in Hilbert spaces. *Proc. Nat. Acad. Sci. (USA)*, 17: 315–318, 1931.

[Mar91] G.A. Margulis. *Discrete subgroups of semisimple Lie groups.* Springer-Verlag, Ergeb. Math. Grenzgeb. 3 Folge, Bd. 17, 1991.

[Moo66] C.C. Moore. Ergodicity of flows on homogeneous spaces. *Amer. J. Math.*, 88:154–178, 1966.

[Mos73] G.D. Mostow. *Strong rigidity of locally symmetric spaces.* Annals of Math. studies 78, Princeton Univ. Press, 1973.

[vN29] J. von Neumann. Zur allgemeinen Theorie des Masses. *Fund. Math.*, 13:73–116, 1929.

[Rei52] H. Reiter Investigations in harmonic analysis. *Trans. Amer. Math. Soc.*, 73:401–427, 1952.

[Rei64] H. Reiter Sur la propriété (P_1) et les fonctions de type positif. *C.R.Acad. Sci. Paris*, 258A:5134–5135, 1964.

[Rud73] W. Rudin *Functional analysis.* McGraw Hill, 1973.

[Wei65] A. Weil *L'intégration dans les groupes topologiques et ses applications.* Hermann, Paris, 1965.

[WM] D. Witte-Morris. *Introduction to arithmetic groups.* Pre-book, february 2003.

[Zim78] R.J. Zimmer Induced and amenable actions of Lie groups. *Ann. Sci. Ec. Norm. Sup.* 11: 407–428, 1978.

[Zim84] R.J. Zimmer. *Ergodic theory and semisimple groups.* Birkhauser, 1984.

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