Singular and strongly mixing MASA's in finite von Neumann algebras

Paul Jolissaint and Yves Stalder

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Abstract

Let Γ be a countable group and let Γ_0 be an infinite abelian subgroup of Γ. We prove that if the pair (Γ_0, Γ) satisfies some combinatorial condition, then the abelian subalgebra $A = L(\Gamma_0)$ is a singular MASA in $M = L(\Gamma)$ and the action of Γ_0 by inner automorphisms on M satisfies the following strong mixing property: for all $x, y \in M$, one has

$$\lim_{\gamma\to\infty,\gamma\in\Gamma_0} \|E_A(\lambda(\gamma)x\lambda(\gamma^{-1})y) - E_A(x)E_A(y)\|_2 = 0.$$

Moreover, we prove that the latter property is a conjugacy invariant of the pair $A \subset M$: if θ is any automorphism of M, if the pair $A \subset M$ has the strong mixing property, then so does the pair $\theta(A) \subset M$. We also exhibit examples of singular MASA's that do not satisfy the above strong mixing condition.

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1 Introduction

If M is a von Neumann algebra, if A is a maximal abelian von Neumann subalgebra of M (henceforth called a MASA), let $\mathcal{N}_M(A)$ be the *normaliser* of A in M: it is the subgroup of the unitary group $U(M)$ of all elements u such that $uAu^* = A$. Then A is *singular* in M if $\mathcal{N}_M(A)$ is as small as possible, namely, if $\mathcal{N}_M(A) = U(A)$. Until recently, it was quite difficult in general to exhibit singular MASA's in von Neumann algebras, though S. Popa proved among others in [9] that all separable type II_1 factors admit singular MASA's.

Motivated by S. Popa's articles [9] and [8], the authors of [11] and [13], introduced sufficient conditions on an abelian von Neumann subalgebra A of a finite von Neumann algebra M that imply that A is even a strongly singular MASA in M. This means that A satisfies the apparently stronger condition: for all $u \in U(M)$, one has

$$\sup_{x \in M, \|x\| \leq 1} \|E_A(x) - E_{uAu^*}(x)\|_2 \geq \|u - E_A(u)\|_2.$$

(In fact, it has been proved recently in [15] that every singular MASA is automatically strongly singular.) They give next several classes of examples of pairs $A \subset M$ where A is a strongly singular MASA in M. In particular, some of them deal with group von Neumann algebras $M = L(\Gamma)$ and $A = L(\Gamma_0)$ where Γ_0 is an abelian subgroup of Γ.

For instance, in order to prove that $L(\Gamma_0)$ is a strongly singular MASA in $L(\Gamma)$, the authors of [11] introduce a combinatorial condition on the pair (Γ_0, Γ) that was called *condition (SS)* in [4]:

For all finite subsets $C, D \subset \Gamma \setminus \Gamma_0$ there exists $g_0 \in \Gamma_0$ such that $gg_0h \notin \Gamma_0$ for all $g \in C$ and all $h \in D$.

In [4], the first-named author proved that Thompson's group F satisfies a stronger property with respect to the subgroup Γ_0 generated by x_0, and called *condition (ST)*:

For all finite subsets $C, D \subset \Gamma \setminus \Gamma_0$ there exists a finite subset $E \subset \Gamma_0$ such that $gg_0h \notin \Gamma_0$ for all $g_0 \in \Gamma_0 \setminus E$, all $g \in C$ and all $h \in D$.

We observe in Section 3.4 that condition (ST) is strictly stronger than condition (SS); examples are borrowed from Section 5 of [14].

In the present article, our goal is twofold: firstly, we add families of examples coming from semi-direct products, HNN extentions and free products that satisfy condition (ST), and secondly, we prove that the MASA $L(\Gamma_0) \subset L(\Gamma)$ satisfies a stronger property than being singular, and that we describe now. More generally, let A be an abelian von Neumann subalgebra in a finite von Neumann algebra M gifted with some normal, faithful, tracial state τ. We say that an infinite subgroup G of the unitary group $U(M)$ is *almost orthonormal* if, for every $\varphi \in M_*$ and for every $\varepsilon > 0$, there exists a finite set $E \subset G$ such that

$$|\varphi(u)| < \varepsilon \quad \forall u \in G \setminus E.$$

Observe that $U(A)$ acts by inner automorphisms on M in an obvious way. Then we say that A is *strongly mixing* in M if, for every almost orthonormal subgroup $G \subset U(A)$, one has for all $a, b \in M$:

$$\lim_{u \to \infty, u \in G} \|E_A(uau^*b) - E_A(a)E_A(b)\|_2 = 0.$$

Notice that it means that $E_A(uau^*b)$ converges to $E_A(a)E_A(b)$ with respect to the σ-strong-$*$ topology. As will be seen later, A is automatically maximal abelian in M, hence E_A is the unique conditional expectation from M onto A, and the above condition is independent of the chosen trace τ. It follows that the strong mixing property is a conjugacy invariant for the pair $A \subset M$: if A is strongly mixing in the finite von Neumann algebra M and if θ is any $*$-isomorphism of M onto N, then $\theta(A)$ is obviously strongly mixing in N, too.

There are obvious relationships between strongly mixing MASA's and the notion of relative mixing action of a discrete group as in Definition 2.9 of [6] on the one hand, and the notion of *asymptotic homomorphism conditional expectation* of [13] on the other hand, but we prefer our definition because of its similarity with the classical notion of strongly mixing actions, and because of the following result that is proved in Section 3:

Theorem 1.1 *Let Γ_0 be an abelian infinite group that acts on a finite von Neumann algebra N and which preserves some normal, faithful finite trace τ. Let M be the associated crossed product $N \rtimes_\alpha \Gamma_0$, and let A be the abelian von Neumann subalgebra generated by Γ_0. Then A is a strongly mixing MASA in M if and only if the action α is strongly mixing.*

As a consequence, we will see that the hyperfinite II_1 factor R contains singular MASA's that are not strongly mixing.

We also consider pairs $A \subset M$ where $M = L(\Gamma)$ is a group von Neumann algebra and $A = L(\Gamma_0)$ for some abelian subgroup Γ_0 of Γ. It turns out that the strong mixing property of $L(\Gamma_0)$ in $L(\Gamma)$ is characterised completely by condition (ST):

Theorem 1.2 *Let $M = L(\Gamma)$ and $A = L(\Gamma_0)$ be as above. The following conditions are equivalent:*

(1) for all $x, y \in M$, we have

$$\lim_{\gamma \to \infty, \gamma \in \Gamma_0} \|E_A(\lambda(\gamma)x\lambda(\gamma^{-1})y) - E_A(x)E_A(y)\|_2 = 0;$$

(2) the pair (Γ_0, Γ) satisfies condition (ST): for all finite subsets $C, D \subset \Gamma \setminus \Gamma_0$ there exists a finite subset $E \subset \Gamma_0$ such that $gg_0h \notin \Gamma_0$ for all $g_0 \in \Gamma_0 \setminus E$, all $g \in C$ and all $h \in D$;

(3) A is strongly mixing in M.

Notational conventions and the proof of Theorem 1.2 are contained in Section 2. Section 3 is subdivided into four subsections: in the first one, we prove Theorem 1.1, Subsections 3.2 and 3.3 are devoted to families of examples of pairs (Γ_0, Γ) that satisfy condition (ST), and the final subsection discusses relationships between conditions (SS), (ST) and the case where Γ_0 is a malnormal subgroup of Γ, namely

for every $g \in \Gamma \setminus \Gamma_0$, one has $g\Gamma_0 g^{-1} \cap \Gamma_0 = \{1\}$.

Such pairs have been considered first in the pioneering article [8] to control normalizers (in particular relative commutants) of $L(\Gamma_0)$ and of its diffuse subalgebras, and, as a byproduct, to produce singular MASA's $L(\Gamma_0)$. They were also used in [10], [13] to provide more examples of (strongly) singular MASA's in type II_1 factors that fit Popa's criteria of Proposition 4.1 in [8].

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2 A mixing property relative to abelian subalgebras

In the rest of the article, M denotes a finite von Neumann algebra, and τ is some normal, faithful, finite, normalised trace on M (henceforth simply called a *trace* on M). It defines a scalar product on M: $\langle a,b\rangle = \tau(b^*a) = \tau(ab^*)$, and the corresponding completion is the Hilbert space $L^2(M,\tau)$ on which M acts by left multiplication extending the analogous operation on M. As usual, we denote by $\|\cdot\|_2$ the corresponding Hilbert norm. When the trace τ must be specified, we write $\|\cdot\|_{2,\tau}$. We denote also by M_* the predual of M, i.e. the Banach space of all normal linear functionals on M. We will always assume for convenience that M_* is separable, or equivalently, that $L^2(M,\tau)$ is a separable Hilbert space. Recall that, for every $\varphi \in M_*$, there exist $\xi,\eta \in L^2(M,\tau)$ such that $\varphi(x) = \langle x\xi,\eta\rangle$ for all $x \in M$.

Let Γ be a countable group and let Γ_0 be an abelian subgroup of Γ. Denote by M (respectively A) the von Neumann algebra $L(\Gamma)$ (respectively $L(\Gamma_0)$) generated by the left regular representation λ of Γ (respectively Γ_0). Recall that $\lambda : \Gamma \to U(\ell^2(\Gamma))$ is defined by $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$ for all $g,h \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. It extends linearly to the group algebra $\mathbb{C}\Gamma$, and $L(\Gamma)$ is the weak-operator closure of $\lambda(\mathbb{C}\Gamma)$, the latter being denoted by $L_f(\Gamma)$. Its elements are thus convolution operators with finite support. The normal functional $\tau(x) = \langle x\delta_1,\delta_1\rangle$ is a faithful trace on M. For $x \in M$, denote by $\sum_{g\in\Gamma} x(g)\lambda(g)$ its "Fourier expansion": $x(g) = \tau(x\lambda(g^{-1}))$ for every $g \in \Gamma$, and the series $\sum_g x(g)\lambda(g)$ converges to x in the $\|\cdot\|_2$-sense so that $\sum_{g\in\Gamma} |x(g)|^2 = \|x\|_2^2$.

Let $1 \in B$ be a unital von Neumann subalgebra of M gifted with some trace τ as above and let E_B be the τ-preserving conditional expectation of M onto B. E_B is characterised by the following two conditions: $E_B(x) \in B$ for all $x \in M$ and $\tau(E_B(x)b) = \tau(xb)$ for all $x \in M$ and all $b \in B$. It enjoys the well-known properties:

(1) $E_B(b_1xb_2) = b_1E_B(x)b_2$ for all $x \in M$ and all $b_1,b_2 \in B$;

(2) $\tau \circ E_B = \tau$.

(3) If $M = L(\Gamma)$ is the von Neumann algebra associated to the countable group Γ, if H is a subgroup of Γ, if $B = L(H)$ and if $x \in M$, then $E_B(x) = \sum_{h\in H} x(h)\lambda(h)$.

Let us motivate our definition below by recalling the following classical situation: suppose that G is an infinite group that acts by Borel and measure-preserving transformations on some standard probability space (X, μ). The action is called *strongly mixing* if, for all Borel subsets $A, B \subset X$ one has

$$\lim_{g \to \infty} \mu((gA) \cap B) = \mu(A)\mu(B).$$

As is well-known, the latter property can be stated in terms of the W^*-dynamical system $(L^\infty(X, \mu), G)$: the latter is gifted with the trace $\tau(a) = \int_X a(x) d\mu(x)$, and the corresponding action α of G on $L^\infty(X, \mu)$ is given by $\alpha_g(a)(x) = a(g^{-1}x)$ for $a \in L^\infty(X, \mu)$, $g \in G$ and $x \in X$. Then the action of G is strongly mixing if and only if, for all $a, b \in L^\infty(X, \mu)$, one has

$$(\star) \quad \lim_{g \to \infty} \tau(\alpha_g(a)b) = \tau(a)\tau(b).$$

Thus, a τ-preserving action α on the finite von Neumann algebra M is said to be *strongly mixing* if it satisfies $(\star)$ for all $a, b \in M$.

We will also need in Section 3 the following weaker notion: the action α on M is *weakly mixing* if, for all $a, b \in M$, for every $\varepsilon > 0$, there exists $g \in G$ such that

$$|\tau(\alpha_g(a)b) - \tau(a)\tau(b)| < \varepsilon.$$

Consider now an abelian von Neumann subalgebra $1 \in A$ of M. Let also G be a countable subgroup of the unitary group $U(A)$; the action of G by inner automorphisms on M is given by $(u, x) \mapsto uxu^*$.

Definition 2.1 *With hypotheses and notations above, we say that the action of G is **strongly mixing** relative to A if, for all $x, y \in M$, one has:*

$$\lim_{u \to \infty, u \in G} \|E_A(uxu^{-1}y) - E_A(x)E_A(y)\|_2 = 0.$$

Remarks. (1) Using standard approximation arguments, the above limit also holds with respect to the σ-strong-$*$ topology. Moreover, it will be proved in Proposition 2.3 below that A is a singular MASA in M. Thus E_A is the unique conditional expectation onto A, and the strong mixing property of the action of G is independent of the chosen trace τ on M.

(2) The above notion is obviously related to the notion of *asymptotic homomorphism conditional expectation* introduced in [13]: recall that the conditional expectation E_A onto the abelian von Neumann subalgebra A of M is an asymptotic homomorphism if there exists a unitary element $u \in A$ such that

$$\lim_{|k| \to \infty} \|E_A(xu^k y) - E_A(x)E_A(y)u^k\|_2 = 0$$

for all $x, y \in M$. This means that the action of the cyclic group generated by u is strongly mixing relative to A. Furthermore, it is also related to Section 3 and Definition 4.2 of [7], and to the notion of weakly mixing actions appearing in [6]. Indeed, to be more precise, let us recall Definition 2.9 of [6]: Let (Q, τ) be a finite von Neumann algebra, let $N \subset Q$ be a von Neumann subalgebra and let σ be a τ-preserving action of some discrete group G on Q which leaves N globally invariant. Then σ is said to be *weakly mixing relative to* N if, for any finite set $F \subset L^2(Q) \ominus L^2(N)$ with $E_N(\eta^*\eta)$ bounded $\forall \eta \in F$, and for any $\varepsilon > 0$, there exists $g \in G$ such that $\|E_N(\eta^*\sigma_g(\eta'))\|_2 \leq \varepsilon$ for all $\eta, \eta' \in F$. Lemma 2.10 of [6] contains alternative characterisations of that property.

In Proposition 3.3 of [4], it is proved that the conditional expectation E_A from the Thompson's group factor $L(F)$ onto its von Neumann subalgebra A generated by the generator x_0 satisfies a particular ergodic property; it is a special case of the above mixing property, as the next result shows.

Proposition 2.2 *If there is a subgroup G of $U(A)$ whose action on M is strongly mixing relative to A, then one has for every $x \in M$:*

$$\lim_{u \to \infty, u \in G} uxu^* = E_A(x)$$

with respect to the weak-operator topology.

Proof. Fix $x, y \in M$. Then

$$\tau(uxu^*y) = \tau(E_A(uxu^*y)) \to \tau(E_A(x)E_A(y)) = \tau(E_A(x)y)$$

as $u \to \infty$, $u \in G$. □

Finally, we prove that if $A \subset M$ is as in Definition 2.1, then it is strongly singular in M.

Let M be a finite von Neumann algebra, let τ be some trace on M as above and let A be an abelian, unital von Neumann subalgebra of M. Recall from [13] or [11] that A is *strongly singular* if the following inequality holds for every unitary element $u \in M$:

$$\|E_{uAu^*} - E_A\|_{\infty,2} \geq \|u - E_A(u)\|_2,$$

where, for $\phi : M \to M$ linear,

$$\|\phi\|_{\infty,2} = \sup\{\|\phi(x)\|_2 \ ; \ \|x\| \leq 1\}.$$

Notice that such an algebra is automatically a MASA of M. When $M = L(\Gamma)$ is the von Neumann algebra associated to a countable group Γ and $A = L(\Gamma_0)$ where Γ_0 is an abelian subgroup of Γ, Lemma 4.1 of [11] gives a sufficient condition in order that A be a strongly singular MASA in M; if there is a subgroup G of $U(A)$ such that its action is strongly mixing relative to A, then A is a strongly singular MASA in M. We recall a particular case of Lemma 2.1 of [11], and we give its proof for the sake of completeness.

Proposition 2.3 *Let M be a finite von Neumann algebra gifted with some finite trace τ, and let A be an abelian von Neumann subalgebra. Assume that there is some subgroup G of $U(A)$ such that its action on M by inner automorphisms is strongly mixing relative to A. Then A is a strongly singular MASA in M.*

Proof. Let $u \in U(M)$ and let $\varepsilon > 0$. Observe that $E_{uAu^*}(x) = uE_A(u^*xu)u^*$ for all $x \in M$. As the action of G is strongly mixing relative to A, one can find a unitary element $v \in G \subset U(A)$ such that

$$\|E_A(v^*u^*vu) - E_A(u^*)E_A(u)\|_2 = \|E_A(u^*vu) - E_A(u^*)vE_A(u)\|_2 \leq \varepsilon.$$

We get then:

$$\begin{aligned}
\|E_A - E_{uAu^*}\|_{\infty,2}^2 &\geq \|v - uE_A(u^*vu)u^*\|_2^2 \\
&= \|u^*vu - E_A(u^*vu)\|_2^2 \\
&= 1 - \|E_A(u^*vu)\|_2^2 \\
&\geq 1 - (\|E_A(u^*)vE_A(u)\|_2 + \varepsilon)^2 \\
&\geq 1 - (\|E_A(u)\|_2 + \varepsilon)^2 \\
&= \|u - E_A(u)\|_2^2 - 2\varepsilon\|E_A(u)\|_2 - \varepsilon^2.
\end{aligned}$$

As ε is arbitrary, we get the desired inequality. □

We prepare now the main result of the present section. To do that, let M and τ be as above. Let us say that a subset S of M is *τ-orthonormal* if $\tau(xy^*) = \delta_{x,y}$ for all $x, y \in S$. We will need a weaker notion which is independent of the chosen trace τ.

Proposition 2.4 *Let M be a finite von Neumann algebra, let τ be a finite trace on M as above and let S be an infinite subset of the unitary group $U(M)$. The following conditions are equivalent:*

(1) for every $\varphi \in M_*$ and for every $\varepsilon > 0$, there exists a finite subset F of S such that $|\varphi(u)| \leq \varepsilon$ for all $u \in S \setminus F$;

(2) for every $x \in M$ and for every $\varepsilon > 0$, there exists a finite set $F \subset S$ such that $|\tau(ux)| \leq \varepsilon$ for all $u \in S \setminus F$;

(2') for any trace τ' on M, for every $x \in M$ and for every $\varepsilon > 0$, there exists a finite set $F \subset S$ such that $|\tau'(ux)| \leq \varepsilon$ for all $u \in S \setminus F$;

(3) for every τ-orthonormal finite set $\{x_1, \ldots, x_N\} \subset M$ and for every $\varepsilon > 0$ there exists a finite set $F \subset S$ such that

$$\sup\{|\tau(ux^*)| \ ; \ x \in \mathrm{span}\{x_1, \ldots, x_N\}, \ \|x\|_2 \leq 1\} \leq \varepsilon \quad \forall u \in S \setminus F;$$

(3') for every trace τ' on M, for every τ'-orthonormal finite set $\{x_1, \ldots, x_N\} \subset M$ and for every $\varepsilon > 0$ there exists a finite set $F \subset S$ such that

$$\sup\{|\tau'(ux^*)| \ ; \ x \in \text{span}\{x_1, \ldots, x_N\}, \ \|x\|_{2,\tau'} \leq 1\} \leq \varepsilon \quad \forall u \in S \setminus F;$$

In particular, if $S \subset U(M)$ satisfies the above conditions, if θ is a $*$-isomorphism of M onto some von Neumann algebra N, then $\theta(S) \subset U(N)$ satisfies the same conditions.

Proof. (1) $\Rightarrow$ (2′) $\Rightarrow$ (2) and (3′) $\Rightarrow$ (3) are trivial.
(2) $\Rightarrow$ (3′): If τ' is a trace on M, if $\{x_1, \ldots, x_N\} \subset M$ is τ'-orthonormal and if $\varepsilon > 0$ is fixed, there exists $h \in M$ such that

$$\|\tau' - \tau(h\cdot)\| \leq \frac{\varepsilon}{2\sqrt{N} \cdot \max \|x_j\|}.$$

Furthermore, one can find a finite set $F \subset S$ such that

$$|\tau(ux_j^*h)| \leq \frac{\varepsilon}{2\sqrt{N}} \quad \forall u \in S \setminus F \quad \text{and} \quad \forall j = 1, \ldots, N.$$

This implies that

$$|\tau'(ux_j^*)| \leq \frac{\varepsilon}{\sqrt{N}} \quad \forall u \in S \setminus F \quad \text{and} \quad \forall j = 1, \ldots, N.$$

Let $x \in \text{span}\{x_1, \ldots, x_N\}$, $\|x\|_{2,\tau'} \leq 1$. Let us write $x = \sum_{j=1}^N \xi_j x_j$, where $\xi_j = \tau'(xx_j^*)$, and $\sum_{j=1}^N |\xi_j|^2 = \|x\|_{2,\tau'}^2 \leq 1$ since the x_j's are τ'-orthonormal. Hence we get, for $u \in S \setminus F$:

$$|\tau'(ux^*)| = |\sum_{j=1}^N \overline{\xi_j} \tau'(ux_j^*)| \leq \left(\sum_{j=1}^N |\xi_j|^2\right)^{1/2} \left(\sum_{j=1}^N |\tau'(ux_j^*)|^2\right)^{1/2} \leq \varepsilon$$

uniformly on the set $\{x \in \text{span}\{x_1, \ldots, x_N\} \ ; \|x\|_{2,\tau'} \leq 1\}$.
(3) $\Rightarrow$ (1): Let $\varphi \in M_*$ and $\varepsilon > 0$. We choose $x \in M$ such that $\|\varphi - \tau(\cdot x)\| \leq \varepsilon/2$. Applying condition (3) to the singleton set $\{x/\|x\|_2\}$ as orthonormal set, we find a finite subset F of S such that $|\tau(ux)| \leq \varepsilon/2$ for every $u \in S \setminus F$. Hence we get $|\varphi(u)| \leq \varepsilon$ for all $u \in S \setminus F$.

The last statement follows readily from condition (1). □

Definition 2.5 *Let M be a finite von Neumann algebra gifted with some fixed finite trace τ. We say that an infinite subset $S \subset U(M)$ is* ***almost orthonormal*** *if it satisfies the equivalent conditions in Proposition 2.4.*

Remarks. (1) Since M has separable predual, an almost orthonormal subset S of $U(M)$ is necessarily countable. Indeed, let $\{x_n\ ;\ n \geq 1\}$ be a $\|\cdot\|_2$-dense countable subset of the unit ball of M with respect to the operator norm. For $n \geq 1$, put

$$S_n = \{u \in S\ ;\ \max_{1\leq j\leq n} |\tau(ux_j^*)| \geq \frac{1}{n}\}.$$

Then each S_n is finite, $S_n \subset S_{n+1}$ for every n and $S = \bigcup_n S_n$. Thus, if $S = (u_n)_{n\geq 1}$ is a sequence of unitary elements, then S is almost orthonormal if and only if u_n tends weakly to zero. In particular, every diffuse von Neumann algebra contains almost orthonormal sequences of unitaries.

(2) The reason why we choose the above definition comes from the fact that if S is almost orthonormal in M, then for every $u \in S$, for every $\varepsilon > 0$, there exists a finite set $F \subset S$ such that $|\tau(v^*u)| < \varepsilon$ for all $v \in S \setminus F$. A typical example of an almost orthonormal subset in a finite von Neumann algebra is a τ-orthonormal subset S of $U(M)$ for *some* trace τ on M: indeed, for every $x \in M$, the series $\sum_S |\tau(xu^*)|^2$ converges. For instance, let $v \in U(M)$ be such that $\tau(v^k) = 0$ and for all integers $k \in \mathbb{Z} \setminus \{0\}$. Then the subgroup generated by v is almost orthonormal. As another example, let Γ be a countable group and let Γ_1 be an infinite subgroup of Γ. Set $G = \lambda(\Gamma_1)$. Then G is almost orthonormal in $M = L(\Gamma)$. Indeed, if $x \in L(\Gamma)$, then $\tau(\lambda(g)x) = \tau(x\lambda(g)) = x(g^{-1})$ obviously tends to 0 as g tends to infinity of Γ_1.

We come now to the main definition of our article.

Definition 2.6 *Let M and τ be as above and let A be an abelian, unital von Neumann subalgebra of M. We say that A is* ***strongly mixing*** *in M if, for every almost orthonormal infinite subgroup G of $U(A)$, the action of G by inner automorphisms on M is strongly mixing relative to A.*

Remark. As discussed previously, the above property is independent of the trace τ and it is a conjugacy invariant.

We present now our first main result; when A and M are group von Neumann algebras, the strong mixing property is characterised by a combinatorial property:

Theorem 2.7 *Let Γ be an infinite group and let Γ_0 be an infinite abelian subgroup of Γ. Let $M = L(\Gamma)$ and $A = L(\Gamma_0)$ be as above. Then the following conditions are equivalent:*

(1) the action of Γ_0 by inner automorphisms on M is strongly mixing relative to A;

(2) for all finite subsets $C, D \subset \Gamma \setminus \Gamma_0$ there exists a finite subset $E \subset \Gamma_0$ such that $gg_0h \notin \Gamma_0$ for all $g_0 \in \Gamma_0 \setminus E$, all $g \in C$ and all $h \in D$;

(3) for every almost orthonormal infinite subset $S \in U(A)$, for all $x, y \in M$ and for every $\varepsilon > 0$, there exists a finite subset $F \subset S$ such that

$$\|E_A(uxu^*y) - E_A(x)E_A(y)\|_2 < \varepsilon \quad \forall u \in S \setminus F;$$

(4) A is strongly mixing in M.

Proof. Trivially, (3) $\Rightarrow$ (4) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2): If C, D are as in (2), set

$$x = \sum_{g \in C} \lambda(g) \quad \text{and} \quad y = \sum_{h \in D} \lambda(h).$$

Thus $E_A(x) = E_A(y) = 0$, and there exists a finite subset $E \subset \Gamma_0$ such that

$$\|E_A(\lambda(g_0^{-1})x\lambda(g_0)y)\|_2 = \|E_A(x\lambda(g_0)y)\|_2 < 1 \quad \forall g_0 \in \Gamma_0 \setminus E.$$

But

$$E_A(x\lambda(g_0)y) = E_A\left(\sum_{g \in C, h \in D} \lambda(gg_0h)\right) = \sum_{g \in C, h \in D, gg_0h \in \Gamma_0} \lambda(gg_0h),$$

hence $\|E_A(x\lambda(g_0)y)\|_2^2 = |\{(g,h) \in C \times D \ ; \ gg_0h \in \Gamma_0\}| < 1$ for all $g_0 \notin E$, which implies that $gg_0h \notin \Gamma_0$ for $g_0 \notin E$ and for all $g \in C$ and $h \in D$.
(2) $\Rightarrow$ (3): Let S be an almost orthonormal infinite subset of $U(A)$, let $x, y \in M$ and fix $\varepsilon > 0$. A-bilinearity of E_A implies that we can assume that $E_A(x) = E_A(y) = 0$, and we have to prove that one can find a finite set $F \subset S$ such that $\|E_A(uxu^*y)\|_2 < \varepsilon$ for all $u \in S \setminus F$. To begin with, let us assume furthermore that x and y have finite support, and let us write $x = \sum_{g \in C} x(g)\lambda(g)$ and $y = \sum_{h \in D} y(h)\lambda(h)$ with $C, D \subset \Gamma \setminus \Gamma_0$ finite. Let $E \subset \Gamma_0$ be as in (2) with respect to the finite sets C and D of $\Gamma \setminus \Gamma_0$: $gg_0h \notin \Gamma_0$ for all $g \in C$, $h \in D$ and $g_0 \notin E$. We claim then that $E_A(x\lambda(g_0^{-1})y) = 0$ if $g_0 \in \Gamma_0 \setminus E^{-1}$. Indeed, if $g_0 \in \Gamma_0 \setminus E^{-1}$, we have:

$$\begin{aligned} E_A(x\lambda(g_0^{-1})y) &= E_A\left(\sum_{g \in C, h \in D} x(g)y(h)\lambda(gg_0^{-1}h)\right) \\ &= \sum_{g \in C, h \in D, gg_0^{-1}h \in \Gamma_0} x(g)y(h)\lambda(gg_0^{-1}h) = 0 \end{aligned}$$

because $g_0 \in \Gamma_0 \setminus E^{-1}$.
Choosing $\lambda(E) \subset A$ as a τ-orthonormal system, there exists a finite subset F of S such that, if $u \in S \setminus F$:

$$\sup\{|\tau(uz^*)| \ ; \ z \in \mathrm{span}\lambda(E), \ \|z\|_2 \leq 1\} < \frac{\varepsilon^2}{|E|\|x\|\|y\|}.$$

Thus, for fixed $u \in S \setminus F$, take $z = \sum_{g_0 \in E} u(g_0)\lambda(g_0)$, so that $z \in \text{span}\lambda(E)$, $\|z\|_2 \leq 1$ and

$$\sum_{g_0 \in E} |u(g_0)|^2 = \tau(uz^*) < \frac{\varepsilon^2}{|E|\|x\|\|y\|}.$$

Then

$$\begin{aligned} \|E_A(uxu^*y)\|_2 &= \|uE_A(xu^*y)\|_2 = \|E_A(xu^*y)\|_2 \\ &\leq \sum_{g_0 \in E} |u(g_0)|\|E_A(x\lambda(g_0^{-1})y)\|_2 < \varepsilon, \end{aligned}$$

using Cauchy-Schwarz Inequality.
Finally, if $x, y \in M$ are such that $E_A(x) = E_A(y) = 0$, if $\varepsilon > 0$, let $x', y' \in L_f(\Gamma)$ be such that $\|x'\| \leq \|x\|$, $\|y'\| \leq \|y\|$, $E_A(x') = E_A(y') = 0$ and

$$\|x' - x\|_2, \|y' - y\|_2 < \frac{\varepsilon}{3 \cdot \max(\|x\|, \|y\|)}.$$

Take a finite subset $F \subset S$ such that $\|E_A(ux'u^*y')\|_2 < \varepsilon/3$ for every $u \in S \setminus F$. Then, if $u \in S \setminus F$,

$$\|E_A(uxu^*y)\|_2 < \|y\|\|x - x'\|_2 + \|x\|\|y - y'\|_2 + \frac{\varepsilon}{3} < \varepsilon.$$

This ends the proof of Theorem 2.7. □

We end this section by noticing that, using the same arguments as in the proofs of Lemma 2.2 and Corollary 2.4 of [3], we obtain:

Proposition 2.8 *Let M and Q be finite von Neumann algebras and let A be a strongly mixing MASA in M. Then A is also strongly mixing in the free product von Neumann algebra $M \star Q$.*

3 Examples

Except in Subsection 3.1 below, all our examples of pairs $A \subset M$ with A strongly mixing in M are based on Theorem 2.7: we exhibit pairs of groups (Γ_0, Γ), where Γ_0 is an abelian subgroup of Γ, that satisfy *condition (ST)*, which means that:

For all pairs of finite sets $C, D \subset \Gamma \setminus \Gamma_0$, there exists a finite subset E of Γ_0 such that $g\gamma h \notin \Gamma_0$ for all $g \in C$, all $h \in D$ and all $\gamma \in \Gamma_0 \setminus E$.

Observe that, taking finite unions of exceptional sets E, condition (ST) is equivalent to:

For all $g, h \in \Gamma \setminus \Gamma_0$, there exists a finite subset E of Γ_0 such that $g\gamma h \notin \Gamma_0$ for all $\gamma \in \Gamma_0 \setminus E$.

When Γ_0 is an infinite cyclic group generated by some element t, condition (ST) is still equivalent to:

For all $g, h \in \Gamma \setminus \Gamma_0$, there exists a positive integer N such that, for every $|k| > N$, one has $gt^k h \notin \Gamma_0$.

For future use in the present section, for every subset S of a group Γ we put $S^* = S \setminus \{1\}$.

3.1 Mixing actions and associated crossed products

Let Γ_0 be an infinite abelian group, let N be a finite von Neumann algebra gifted with some (finite, faithful, normal, normalised) trace τ and let $\alpha : \Gamma_0 \to \mathrm{Aut}(N)$ be a τ-preserving action of Γ_0 on N. We denote by M the associated crossed product von Neumann algebra $N \rtimes_\alpha \Gamma_0$. Recall that M is a von Neumann algebra acting on $L^2(N) \otimes \ell^2(\Gamma_0)$; it is generated by $N \cup \{\lambda(g) \ ; \ g \in \Gamma_0\}$, where $x \in N$ acts on $L^2(N, \tau) \otimes \ell^2(\Gamma_0)$ as follows:

$$(x \cdot \xi)(g) = \alpha_{g^{-1}}(x)\xi(g)$$

for all $\xi \in L^2(N, \tau) \otimes \ell^2(\Gamma_0)$, so that $\lambda(g)x\lambda(g^{-1}) = \alpha_g(x)$ for all g and $x \in N$. (By a slight abuse of notation, we also denote by λ the representation $1 \otimes \lambda$ on $L^2(N) \otimes \ell^2(\Gamma_0)$.) The trace τ extends to M, and every element $x \in M$ has a "Fourier expansion with coefficients in N", namely, the following series converges to x with respect to the $\| \cdot \|_2$-topology: $x = \sum_{\gamma \in \Gamma_0} x(\gamma)\lambda(\gamma)$ where $x(\gamma) = E_N(x\lambda(\gamma^{-1})) \in N$. In particular, $\|x\|_2^2 = \sum_\gamma \|x(\gamma)\|_2^2$.

Put $A = L(\Gamma_0) \subset M$. It is easy to check that the conditional expectation E_A satisfies:

$$E_A(x) = \sum_{\gamma \in \Gamma_0} \tau(x(\gamma))\lambda(\gamma)$$

if $x = \sum_\gamma x(\gamma)\lambda(\gamma) \in M$. Our next result strengthens Lemma 3.1 of [11] and motivates Definition 2.6.

Theorem 3.1 *Let Γ_0, N, α, $M = N \rtimes_\alpha \Gamma_0$ and $A = L(\Gamma_0)$ be as above. Then A is strongly mixing in M if and only if α is a strongly mixing action.*

Proof. If A is strongly mixing in M, let $a, b \in N$ and $\varepsilon > 0$. Since $\lambda(\Gamma_0)$ is (almost) orthonormal in $U(A)$, there exists a finite set $E \subset \Gamma_0$ such that

$$\|E_A(\lambda(g)a\lambda(g^{-1})b) - E_A(a)E_A(b)\|_2 < \varepsilon \quad \forall g \in \Gamma_0 \setminus E.$$

But $E_A(\lambda(g)a\lambda(g^{-1})b) = \tau(\alpha_g(a)b)$, $E_A(a) = \tau(a)$ and $E_A(b) = \tau(b)$. Thus $|\tau(\alpha_g(a)b) - \tau(a)\tau(b)| < \varepsilon$ for $g \in \Gamma_0 \setminus E$, and α is a strongly mixing action.

Conversely, assume that α is a strongly mixing action, and let G be an almost orthonormal subgroup of $U(A)$. The $*$-algebra of all finite sums $\sum x(\gamma)\lambda(\gamma)$ with $x(\gamma) \in N$ being $\|\cdot\|_2$-dense in M, it suffices to prove that, for all $a, b \in N$, for all $g, h \in \Gamma_0$ and for every $\varepsilon > 0$, there exists a finite set $F \subset G$ such that

$$\|E_A(u^* a\lambda(g)ub\lambda(h)) - E_A(a\lambda(g))E_A(b\lambda(h))\|_2 < \varepsilon \quad \forall u \in G \setminus F.$$

Choose first a finite set $E \subset \Gamma_0$ such that

$$\sup_{\gamma \in \Gamma_0 \setminus E} |\tau(a\alpha_{\gamma g}(b)) - \tau(a)\tau(b)| < \frac{\varepsilon}{2}.$$

Next, choose a finite set $F \subset G$ such that

$$\sum_{\gamma \in E} |u(\gamma)|^2 = \sup\{|\tau(uz^*)| \ ; \ z \in \text{span}\lambda(E)\} < \left(\frac{\varepsilon}{4\|a\|_2\|b\|_2}\right)^2 \quad \forall u \in G \setminus F.$$

One has, for $u = \sum_\gamma u(\gamma)\lambda(\gamma) \in G$:

$$\begin{aligned} E_A(a\lambda(g)ub\lambda(h)) &= E_A(au\lambda(g)b\lambda(g^{-1})\lambda(gh)) \\ &= \sum_{\gamma \in \Gamma_0} u(\gamma)E_A(a\alpha_{\gamma g}(b)\lambda(\gamma gh)) \\ &= \sum_{\gamma \in \Gamma_0} u(\gamma)\tau(a\alpha_{\gamma g}(b))\lambda(\gamma gh), \end{aligned}$$

where we used the formula $E_A(x\lambda(\gamma)) = \tau(x)\lambda(\gamma)$ for all $x \in N$ and $\gamma \in \Gamma_0$. Similarly,

$$uE_A(a\lambda(g))E_A(b\lambda(h)) = \sum_{\gamma \in \Gamma_0} u(\gamma)\tau(a)\tau(b)\lambda(\gamma gh),$$

and both series converge in the $\|\cdot\|_2$-topology. As $E_A(u^*x) = u^*E_A(x)$ for $u \in U(A)$ and $x \in M$, we get for every $u \in G \setminus F$:

$$\begin{aligned} \|E_A(u^* a\lambda(g)ub\lambda(h)) - E_A(a\lambda(g))E_A(b\lambda(h))\|_2^2 &= \sum_{\gamma \in \Gamma_0} |u(\gamma)|^2 |\tau(a\alpha_{\gamma g}(b)) - \tau(a)\tau(b)|^2 \\ &\leq \sum_{\gamma \in E} |u(\gamma)|^2 |\tau(a\alpha_{\gamma g}(b)) - \tau(a)\tau(b)|^2 \\ &+ \sup_{\gamma \notin E} |\tau(a\alpha_{\gamma g}(b)) - \tau(a)\tau(b)|^2 \sum_{\gamma \notin E} |u(\gamma)|^2 \\ &< \varepsilon^2. \end{aligned}$$

□

Theorem 3.1 allows us to prove that there are singular MASA's that are not strongly mixing. It also relies on Theorem 4.2 of [12] which states the existence of free, weakly, but not strongly, mixing actions of infinite, discrete abelian groups on a Lebesgue probability space.

Corollary 3.2 *Let Γ_0 be an infinite abelian group and let α be a measure-preserving, free, weakly mixing but not strongly mixing action on some standard probability space $(X, \mathcal{B}, \mu)$. Set $N = L^\infty(X, \mathcal{B}, \mu)$ and let M be the corresponding crossed product II_1-factor. Then the abelian subalgebra $A = L(\Gamma_0)$ is a singular MASA in M, but it is not strongly mixing.*

Proof. Let $F \subset N$, $E \subset \Gamma_0$ be finite, and let $\varepsilon > 0$. As in the proofs of Theorem 3.1 and of Proposition 2.3, it suffices to prove that there exists $\gamma \in \Gamma_0$ such that

$$\|E_A(a\lambda(g)\lambda(\gamma^{-1})b\lambda(h)) - E_A(a\lambda(g))\lambda(\gamma^{-1})E_A(b\lambda(h))\|_2 < \varepsilon$$

for all $a, b \in F$ and all $g, h \in E$. As α is weakly mixing, by Corollary 1.6 of [1] and polarisation, one can find $\gamma \in \Gamma_0$ such that

$$|\tau(\alpha_\gamma(a)\alpha_g(b)) - \tau(a)\tau(\alpha_g(b))| = |\tau(\alpha_\gamma(a)\alpha_g(b)) - \tau(a)\tau(b)| < \varepsilon$$

for all $a, b \in F$ and all $g, h \in E$. As in the proof of Theorem 3.1, one has:

$$\begin{aligned} E_A(a\lambda(g)\lambda(\gamma^{-1})b\lambda(h)) - E_A(a\lambda(g))\lambda(\gamma^{-1})E_A(b\lambda(h)) &= \lambda(\gamma^{-1})[E_A(\alpha_\gamma(a)\alpha_g(b)) \\ &\quad -E_A(a)E_A(b)]\lambda(gh) \\ &= (\tau(\alpha_\gamma(a)\alpha_g(b)) \\ &\quad -\tau(a)\tau(\alpha_g(b)))\lambda(\gamma^{-1}gh). \end{aligned}$$

This proves that A is a singular MASA in M. However, since α is not strongly mixing, A is not a strongly mixing MASA in M, by Theorem 3.1. □

We give next a family of typical examples of strongly mixing actions on finite von Neumann algebras.

Example. Consider a finite von Neumann algebra $B \neq \mathbb{C}$ gifted with some finite, faithful, normal, normalised trace τ_B, let Γ_0 be an infinite abelian group that acts properly on a countable set X in the sense that for every finite set $Y \subset X$, the set $\{g \in \Gamma_0 \,;\, g(Y) \cap Y \neq \emptyset\}$ is finite. Let $(N, \tau) = \bigotimes_{x \in X}(B, \tau_B)$ be the associated infinite tensor product. Then the corresponding Bernoulli shift action is the action σ of Γ_0 on N given by

$$\sigma_g(\otimes_{x \in X} b_x) = \otimes_{x \in X} b_{gx}$$

for every $\otimes_x b_x \in N$ such that $b_x = 1$ for all but finitely many x's. Then it is easy to see that properness of the action implies that σ is a strongly mixing action. The classical case corresponds to the simply transitive action by left translations on Γ_0.

Let now H be a discrete group, let Γ_0 be an infinite abelian group and let $\alpha : \Gamma_0 \to \mathrm{Aut}(H)$ be an action of Γ_0 on H. Then the semi-direct product $\Gamma = H \rtimes_\alpha \Gamma_0$ is the direct product set $H \times \Gamma_0$ gifted with the multiplication

$$(h, \gamma)(h', \gamma') = (h\alpha_\gamma(h'), \gamma\gamma') \quad \forall (h, \gamma), (h', \gamma') \in \Gamma.$$

The action α lifts from H to the von Neumann algebra $L(H)$, and $L(\Gamma) = L(H) \rtimes \Gamma_0$ is a crossed product von Neumann algebra in a natural way. In Theorem 2.2 of [11], the authors consider a sufficient condition on the action α on H which ensures that $L(\Gamma_0) \subset L(\Gamma)$ is a strongly singular MASA in $L(\Gamma)$ and that $L(\Gamma)$ is a type II_1 factor. In fact, it turns out that their condition implies that $L(\Gamma_0)$ is strongly mixing in $L(\Gamma)$, as we prove here:

Proposition 3.3 *Let H and Γ_0 be infinite discrete groups, Γ_0 being abelian, let α be an action of Γ_0 on H and let $\Gamma = H \rtimes_\alpha \Gamma_0$. If, for each $\gamma \in \Gamma_0^*$, the only fixed point of α_γ is 1_H, then:*

(1) Γ is an ICC group;

(2) the pair (Γ_0, Γ) satisfies condition (ST);

(3) the action of Γ_0 on $L(H)$ is strongly mixing.

In particular, $L(\Gamma)$ is a type II_1 factor and $L(\Gamma_0)$ is strongly mixing in $L(\Gamma)$.

Proof. Statement (1) is proved in [11]. Thus it remains to prove (2) and (3).

To do that, we claim first that if α is as above, then the triple (Γ_0, H, α) satisfies the following condition whose proof is inspired by that of Theorem 2.2 of [11]:

For every finite subset F of H^*, there exists a finite set E in Γ_0 such that $\alpha_\gamma(F) \cap F = \emptyset$ for all $\gamma \in \Gamma_0 \setminus E$.

Indeed, if F is fixed, set $I(F) = \{\gamma \in \Gamma_0 \ ; \ \alpha_\gamma(F) \cap F \neq \emptyset\}$. If $I(F)$ would be infinite for some F, set for $f \in F$:

$$S_f = \{\gamma \in \Gamma_0 \ ; \ \alpha_\gamma(f) \in F\},$$

so that $I(F) = \bigcup_{f \in F} S_f$, and S_f would be infinite for at least one $f \in F$. There would exist then distinct elements γ_1 and γ_2 of Γ_0 such that $\alpha_{\gamma_1}(f) = \alpha_{\gamma_2}(f)$, since F is finite. This is impossible because $\alpha_{\gamma_1^{-1}\gamma_2}$ cannot have any fixed point in $F \subset H^*$.

Let us prove (2). Fix some finite set $C \subset \Gamma \setminus \Gamma_0$. Without loss of generality, we assume that $C = C_1 \times C_2$ with $C_1 \subset H^*$ and $C_2 \subset \Gamma_0$ finite. Take $F = C_1 \cup C_1^{-1} \subset H^*$ in the condition above and let $E_1 \subset \Gamma_0$ be a finite set such that $\alpha_\gamma(F) \cap F = \emptyset$ for all $\gamma \in \Gamma_0 \setminus E_1$. Put $E = \bigcup_{\gamma \in C_2} \gamma^{-1} E_1$, which is finite. Then it is easy to check that for all $(h, \gamma), (h', \gamma') \in C$ and for every $g \in \Gamma_0 \setminus E$, one has

$$(h, \gamma)(1, g)(h', \gamma') = (h\alpha_{\gamma g}(h'), \gamma g \gamma') \notin \{1_H\} \times \Gamma_0.$$

Finally, in order to prove (3), it suffices to see that, if $a, b \in L_f(H)$, then there exists a finite subset E of Γ_0 such that

$$\tau(\alpha_\gamma(a)b) = \tau(a)\tau(b) \quad \forall \gamma \notin E.$$

Thus, let $S \subset \Gamma_0$ be a finite subset such that $a = \sum_{\gamma \in S} a(\gamma)\lambda(\gamma)$ and $b = \sum_{\gamma \in S} b(\gamma)\lambda(\gamma)$. Choose $E \subset \Gamma_0$ finite such that $\alpha_\gamma(S^* \cup (S^*)^{-1}) \cap (S^* \cup (S^*)^{-1}) = \emptyset$ for every $\gamma \in \Gamma_0 \setminus E$. Then, if $\gamma \notin E$, we have

$$\tau(\alpha_\gamma(a)b) = \sum_{h_1, h_2 \in S} a(h_1)b(h_2)\tau(\lambda(\alpha_\gamma(h_1)h_2)) = \tau(a)\tau(b)$$

since $\alpha_\gamma(h_1)h_2 \neq 1$ for $h_1, h_2 \in S^*$. $\square$

Example. Let $d \geq 2$ be an integer and let $g \in GL(d, \mathbb{Z})$. Then it defines a natural action of $\Gamma_0 = \mathbb{Z}$ on $H = \mathbb{Z}^d$ which has no non trivial fixed point if and only if the list of eigenvalues of g contains no root of unity. (See for instance Example 2.5 of [16].)

3.2 The case of HNN extensions

In Lemma 3.2 of [4], the first-named author proved that Thompson's group F satisfies condition (ST) with respect to its abelian subgroup Γ_0 generated by x_0. As is well-known, F is an HNN extension where x_0 is the stable letter. In this subsection, we extend the above result to some families of HNN extensions, and we refer to the monograph [5], Chapter IV, for basic definitions and properties of HNN extensions. Let $\Gamma = HNN(\Lambda, H, K, \phi)$ be an HNN-extension where H, K are subgroups of Λ and, as usual, where $\phi : H \to K$ is an isomorphism. Denote by t the stable letter such that $t^{-1}ht = \phi(h)$ for all $h \in H$ and by Γ_0 the subgroup generated by t. Recall that a sequence $g_0, t^{\varepsilon_1}, \ldots, t^{\varepsilon_n}, g_n$, $(n \geq 0)$ is *reduced* if $g_i \in \Lambda$ and $\varepsilon_i = \pm 1$ for every i, and if there is no subsequence t^{-1}, g_i, t with $g_i \in H$ or t, g_i, t^{-1} with $g_i \in K$. As is well known, if the sequence $g_0, t^{\varepsilon_1}, \ldots, t^{\varepsilon_n}, g_n$ is reduced and if $n \geq 1$ then the corresponding element $g = g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n \in \Gamma$ is non trivial (Britton's Lemma). We also say that such an element is in *reduced form.* Furthermore, if $g = g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n$ and $h = h_0 t^{\delta_1} \cdots t^{\delta_m} g_m$ are in reduced form and if $g = h$, then $n = m$ and $\varepsilon_i = \delta_i$ for every i. Hence the *length* $\ell(g)$ of $g = g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n$ (in reduced form) is the integer n. For future use, recall from [16] that, for every positive integer j, $\mathrm{Dom}(\phi^j)$ is defined by $\mathrm{Dom}(\phi) = H$ for $j = 1$ and, by induction, $\mathrm{Dom}(\phi^j) = \phi^{-1}(\mathrm{Dom}(\phi^{j-1}) \cap K) \subset H$ for $j \geq 2$.

The aim of the present subsection is to prove that the pair (Γ_0, Γ) satisfies condition (ST) under simple hypotheses.

Proposition 3.4 *Suppose that for each $\lambda \in \Lambda^*$, there exists $j > 0$ such that $\lambda \notin \mathrm{Dom}(\phi^j)$. Then the pair (Γ_0, Γ) satisfies condition (ST).*

We need two lemmas to achieve the proof.

Lemma 3.5 *For all $c, d \in \Gamma \setminus \Gamma_0$, there exists $N_1 > 0$ such that the elements ct^{N_1} and $t^{N_1}d$ have reduced forms $c_0 t^{\delta_1} c_1 \cdots t^{\delta_k} c_k t^p$ and $t^q d_0 t^{\varepsilon_1} d_1 \cdots t^{\varepsilon_l} d_l$ respectively with $p > k \geq 0$, $q > l \geq 0$, $c_k \neq 1$ and $d_0 \neq 1$.*

Lemma 3.6 *Let $g = g_0 t^{\delta_1} g_1 \cdots t^{\delta_k} g_k t^p$ with $p > k \geq 0$ and $g_k \neq 1$ (in reduced form). Under the assumptions of Proposition 3.4, there exists $N_2 \geq 0$ such that, for all $\alpha \in \mathbb{Z}$, the element $t^\alpha g t^{N_2}$ has a reduced form $t^\beta h_0 t^{\varepsilon_1} h_1 \cdots t^{\varepsilon_m} h_m t^r$ with $r > 0$, $m \geqslant 0$ and $h_m \neq 1$.*

Proof of Proposition 3.4. As discussed at the beginning of Section 3, it is sufficient to prove that for all $c, d \in \Gamma \setminus \Gamma_0$, there exists an integer $N > 0$ such that $ct^n d \notin \Gamma_0$ for all integers n satisfying $|n| \geq N$. We may even restrict to $n \geq N$ since one has $(ct^{-n}d)^{-1} = d^{-1}t^n c^{-1}$ where $d^{-1}, c^{-1} \notin \Gamma_0$ whenever $c, d \notin \Gamma_0$.

Let $c, d \in \Gamma \setminus \Gamma_0$. Take $N = 2N_1 + N_2$ where N_1, N_2 are given by Lemmas 3.5 and 3.6, and take $n \geq N$. We may write $ct^n d = ct^{N_1} t^{N_2} t^{n-2N_1-N_2} t^{N_1} d$. It follows from Lemma 3.5 that ct^{N_1} and $t^{N_1}d$ have reduced forms $c_0 t^{\delta_1} c_1 \cdots t^{\delta_k} c_k t^p$ and $t^q d_0 t^{\varepsilon_1} d_1 \cdots t^{\varepsilon_l} d_l$ with $p > k$, $q > l$, $c_k \neq 1$ and $d_0 \neq 1$. Next, by Lemma 3.6, we see that, for all $\alpha \in \mathbb{Z}$, the element $t^\alpha c t^{N_1} t^{N_2}$ has reduced form $t^\beta h_0 t^{\varepsilon_1} h_1 \cdots t^{\varepsilon_m} h_m t^r$ with $r > 0$ and $h_m \neq 1$. Hence, a reduced form of $t^\alpha c t^n d$ is given by $t^\beta h_0 t^{\varepsilon_1} h_1 \cdots t^{\varepsilon_m} h_m t^{r+(n-2N_1-N_2)+q} d_0 t^{\varepsilon_1} d_1 \cdots t^{\varepsilon_l} d_l$ (note that $n - 2N_1 - N_2 \geq 0$).

Finally, we obtain $t^\alpha c t^n d \neq 1$ for all $\alpha \in \mathbb{Z}$, by Britton's lemma. This implies $ct^n d \notin \Gamma_0$, as desired. □

Proof of Lemma 3.5. Let $c, d \in \Gamma \setminus \Gamma_0$. We set $N_1 = \max\{\ell(c), \ell(d)\} + 1$. Let us now write reduced forms of c and d (with $\ell(c) = i$ and $\ell(d) = j$):

$$c = g_0 t^{\delta_1} g_1 \cdots t^{\delta_i} g_i \ ; \ d = h_0 t^{\varepsilon_1} h_1 \cdots t^{\varepsilon_j} h_j \ .$$

Multiplying by t^{N_1} on the right (respectively left) and reducing, we obtain the announced reduced forms:

- we obtain $p > k$ since $N_1 > \ell(c)$ and since each reduction will decrease the exponent of the last t by 1 and the length of the prefix by 1;
- in a similar way, we get $q > l$;
- the inequalities $c_k \neq 1$ and $d_0 \neq 1$ hold because $c, d \notin \Gamma_0$.

The proof is complete. □

Proof of Lemma 3.6. Let $g = g_0 t^{\delta_1} g_1 \cdots t^{\delta_k} g_k t^p$ with $p > k \geqslant 0$ and $g_k \neq 1$ (reduced form) and let $\alpha \in \mathbb{Z}$.

If $\alpha \geq -k$, it is clear that, reducing $t^\alpha g t^n$ (for any $n \geq 0$), we obtain the desired reduced form. We thus suppose $\alpha < -k$. We distinguish three cases:

- If $\delta_i = -1$ for some i, the reduction of $t^\alpha g t^n$ for any $n \geqslant 0$ gives the desired reduced form because the reduction cannot go through this δ_i.
- If $\delta_i = 1$ for all i and if the reduction of $t^\alpha g$ gives $t^\beta h_0 t^{\varepsilon_1} h_1 \cdots t^{\varepsilon_m} h_m t^p$ with $m \geqslant 1$, then $t^\alpha g t^n$ has the desired reduced form for any $n \geqslant 0$.
- If $\delta_i = 1$ for all i and if $t^\alpha g = t^{\alpha'} \lambda t^p$ with $\lambda \in \Lambda^*$ we set $N_2 = \max\{j \in \mathbb{N} : \lambda \in \mathrm{Dom}(\phi^j)\}$ (note that λ does not depend on α). Then $t^\alpha g t^{N_2}$ will have reduced form $\phi^{-\alpha'}(\lambda) t^{p+N_2+\alpha'}$ with $N_2 > \alpha'$, or $t^{\alpha'+N_2}\phi^{N_2}(\lambda)t^p$ with $N_2 \leq -\alpha'$.

In all cases, $t^\alpha g t^{N_2}$ has the desired reduced form. □

Examples. (1) Consider F given by its usual presentation

$$F = \langle x_0, x_1, \ldots \mid x_i^{-1} x_n x_i = x_{n+1}\ 0 \leq i < n \rangle.$$

For every integer $k \geq 1$, denote by F_k the subgroup of F generated by $x_k, x_{k+1}, \ldots$, and denote by σ the "shift map" defined by $\sigma(x_n) = x_{n+1}$, for $n \geq 0$. Its restriction to F_k is an isomorphism onto F_{k+1}, and in particular, the inverse map $\phi : F_2 \to F_1$ is an isomorphism which satisfies $\phi(x) = x_0 x x_0^{-1}$ for every $x \in F_2$. As in Proposition 1.7 of [2], it is evident that F is the HNN extension $HNN(F_1, F_2, F_1, \phi)$ with $t = x_0^{-1}$ as stable letter. With these choices, F satisfies the hypotheses of Proposition 3.3, and this proves that the pair (Γ_0, F) satisfies condition (ST).
(2) Let m and n be non-zero integers. The associated *Baumslag-Solitar group* is the group which has the following presentation:

$$BS(m, n) = \langle a, b \mid ab^m a^{-1} b^{-n} \rangle.$$

Since $a^{-1} b^n a = b^m$, $BS(m, n)$ is an HNN extension $HNN(\mathbb{Z}, n\mathbb{Z}, m\mathbb{Z}, \phi)$ where $\phi(nk) = mk$ for every integer k. Assume first that $|n| > |m|$ and denote by Γ_0 the subgroup generated by a. Then it is easy to check that the pair $(\Gamma_0, BS(m, n))$ satisfies the condition in Proposition 3.4. Thus, it satisfies also condition (ST). If $|m| > |n|$, replacing a by a^{-1} (which does not change Γ_0), one gets the same conclusion. Thus, all Baumslag-Solitar groups $BS(m, n)$ with $|m| \neq |n|$ satisfy condition (ST) with respect to the subgroup generated by a. Observe that the latter class is precisely the class of Baumslag-Solitar groups that are ICC ([16]).

3.3 Some free products examples

It was noted in [4] that if Γ is the free group F_N of rank $N \geq 2$ on free generators $a_1, \ldots, a_N$ and if Γ_0 is the subgroup generated by some fixed a_i, then the pair (Γ_0, F_N) satisfies condition (ST). See also Corollary 3.4 of [13].

The present subsection contains three observations that extend the above case. Proofs make use of uniqueness of normal forms in free products with amalgamation and are omitted.

Proposition 3.7 *Let* $\Gamma = \Gamma_0 \star_Z \Gamma_1$ *be an amalgamated product where* Γ_0 *is an infinite abelian group,* Z *is a finite subgroup of* Γ_0 *and of* Γ_1*. Then the pair* (Γ_0, Γ) *satisfies condition (ST).*

We also have:

Proposition 3.8 *Let* $\Gamma = A \star B$ *be the free product of two subgroups such that* $|A| \geq 2$ *and that* B *contains an element* b *of order at least 3. Let* $a \neq 1$ *be an element of* A*. Set* $t = ab$ *and let* Γ_0 *be the subgroup generated by* t*. Then the pair* (Γ_0, Γ) *satisfies condition (ST).*

Finally, we get from Theorem 2.7 and Proposition 2.8:

Proposition 3.9 *Let* Γ *be an infinite group, let* Γ_0 *be an infinite abelian subgroup of* Γ *such that the pair* (Γ_0, Γ) *satisfies condition (ST) and let* G *be any countable group. Then the pair* $(\Gamma_0, \Gamma \star G)$ *satisfies also condition (ST).*

3.4 Final examples and remarks

So far, in almost all our examples of pair (Γ_0, Γ) that satisfy condition (ST), Γ_0 is an infinite cyclic subgroup of Γ. It turns out that examples that appear in Section 5 of [14] give pairs (Γ_0, Γ) with Γ_0 non-cyclic, some of them satisfying condition (ST), and some others not:

As in Example 5.1 of [14], let $\mathbb{Q}$ be the additive group of rational numbers and denote by $\mathbb{Q}^\times$ the multiplicative group of nonzero rational numbers. For each positive integer n, set

$$F_n = \{\frac{p}{q} \cdot 2^{kn} \ ; \ p, q \in \mathbb{Z}_{\text{odd}}, \ k \in \mathbb{Z}\} \subset \mathbb{Q}^\times$$

and

$$F_\infty = \{\frac{p}{q} \ ; \ p, q \in \mathbb{Z}_{\text{odd}}\} \subset \mathbb{Q}^\times.$$

Next, for $n \in \mathbb{N} \cup \{\infty\}$, set

$$\Gamma(n) = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} \ ; \ f \in F_n, \ x \in \mathbb{Q} \right\}$$

and let $\Gamma_0(n)$ be the subgroup of diagonal elements of $\Gamma(n)$. As is shown in [14], $\Gamma(n)$ is an ICC, amenable group. We claim that the pair $(\Gamma_0(n), \Gamma(n))$ satisfies condition (ST). Indeed, if $g = \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} b & y \\ 0 & 1 \end{pmatrix}$ belong to $\Gamma(n) \setminus \Gamma_0(n)$ and if $\gamma = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(n)$ then $g\gamma h = \begin{pmatrix} abz & ayz + x \\ 0 & 1 \end{pmatrix}$ belongs to $\Gamma_0(n)$ for at most one value of z since x and y are both nonzero.

However, if we consider larger matrices, the corresponding pairs of groups do not satisfy condition (ST). As in Example 5.2 from [14], let us fix two positive integers m and n, and set

$$\Gamma(m,n) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & f_1 & 0 \\ 0 & 0 & f_2 \end{pmatrix} \ ; \ f_1 \in F_m,\ f_2 \in F_n,\ x,y \in \mathbb{Q} \right\}$$

and let $\Gamma_0(m,n)$ be the corresponding diagonal subgroup. Then we have:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

which belongs to Γ_0 for all $f \in F_m$. Thus the pair $(\Gamma_0(m,n), \Gamma(m,n))$ does not satisfy condition (ST), though it satisfies condition (SS) and thus $A = L(\Gamma_0(m,n))$ is a singular MASA in $M = L(\Gamma(m,n))$, by Example 5.2 of [14]. This shows once again that singular MASA's are not necessarily strongly mixing.

Finally, let Γ be an infinite group and let Γ_0 be an infinite abelian subgroup of Γ. In [8], the following condition was used to obtain orthogonal pairs of von Neumann subalgebras; it was used later in [13] to get asymptotic homomorphism conditional expectations:

($\star$) *For every $g \in \Gamma \setminus \Gamma_0$, one has $g\Gamma_0 g^{-1} \cap \Gamma_0 = \{1\}$.*

Recall that a subgroup Γ_0 of Γ that satisfies ($\star$) is called **malnormal**. Then condition ($\star$) implies condition (ST). Indeed, for $g, h \in \Gamma$, set $E(g,h) = \{\gamma \in \Gamma_0 \ ; \ g\gamma h \in \Gamma_0\} = g^{-1}\Gamma_0 h^{-1} \cap \Gamma_0$. Then, if (Γ_0, Γ) satisfies ($\star$), and if $g, h \in \Gamma \setminus \Gamma_0$, $E(g,h)$ contains at most one element (see the proof of Lemma 3.1 of [13]). In turn, condition (ST) means exactly that $E(g,h)$ is finite for all $g, h \in \Gamma \setminus \Gamma_0$.

If Γ_0 is torsion free, then conditions ($\star$) and (ST) are equivalent because, in this case, if $g \in \Gamma \setminus \Gamma_0$, the finite set $E(g^{-1}, g) = g\Gamma_0 g^{-1} \cap \Gamma_0$ is a finite subgroup of Γ_0, hence is trivial. For instance, all pairs coming from HNN extensions as in Proposition 3.4 satisfy condition ($\star$).

However, condition ($\star$) is strictly stronger than condition (ST) in general: let $\Gamma = \Gamma_0 \star_Z \Gamma_1$ be as in Proposition 3.7 above, and assume further that $Z \neq \{1\}$ and that there exists $g \in \Gamma_1 \setminus Z$ such that $zg = gz$ for every $z \in Z$. Then $g\Gamma_0 g^{-1} \cap \Gamma_0 \supset Z$, and (Γ_0, Γ) does not satisfy ($\star$). Observe that, in this case, Γ is not necessarily an ICC group. However, replacing it by a non trivial free product group $\Gamma \star G$, we get a pair $(\Gamma_0, \Gamma \star G)$ satisfying condition (ST) by Proposition 3.9 but not ($\star$), and $\Gamma \star G$ is an ICC group.

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Université de Neuchâtel,
Institut de Mathémathiques,
Emile-Argand 11
Case postale 158
CH-2009 Neuchâtel, Switzerland
paul.jolissaint@unine.ch
yves.stalder@unine.ch