The Length of Harmonic Forms on a Compact Riemannian Manifold

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Abstract

We study n dimensional Riemanniann manifolds with harmonic forms of constant length and first Betti number equal to $n-1$ showing that they are 2-steps nilmanifolds with some special metrics. We also characterise, in terms of properties on the product of harmonic forms, the left invariant metrics among them. This allows us to clarify the case of equality in the stable isosytolic inequalities in that setting. We also discuss other values of the Betti number.

1 Introduction

Let (M^n, g) be a Riemannian manifold. In this note we focus on the case where all harmonic forms are of constant length. Recently these manifold have appeared in different settings where they play a singular part.

In dimension 4, recent work of Lebrun [Leb02] shows a strong interplay between the length of harmonic self-dual 2-forms of the manifold and the non-vanishing of Seiberg-Witten invariants, in particular the existence of a symplectic structure.

Geometrically Formal Manifolds, are closed Riemaniann manifolds having a metric such that the space of harmonic forms is a sub-algebra of the algebra of differential forms. If the manifold is also oriented, one easily sees that all harmonic forms have constant pointwise norm (see [Kot01]). In particular if the first Betti number is equal to the dimension (which actually bounds it), then the manifold is a flat torus (In fact it is also true if some other Betti number is maximal, provided the dimension is prime for example). From a remark made by Kotschik [Kot01] we also know that being geometrically formal and orientable implies a second obstruction on the first Betti number, if n is the dimension, the Betti number can't be $n-1$. However, n dimensional manifolds with one-harmonic forms of constant length, and first Betti number equal to $n-1$ exists.

The main result of this paper is the following theorem which says exactly what these manifolds are (see Definition 8 for the meaning of pseudo left invariant)

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* The second author was partially supported by european project ACR OFES number 00.0349 and grant of the FNRS 20-65060.01

Key-words : Harmonic forms, Spectrum of the Laplacian.

Classification : 53C20, 58J50

Theorem 1. *Let (N^{n+1}, g) be a compact orientable connected manifold such that all of its harmonic 1-forms are of constant length, and such that $b_1(N^{n+1}) = n$, then (N^{n+1}, g) is a 2-step nilmanifold whose kernel is one dimensional and g is a pseudo left invariant metric.*

The macroscopical spectrum of a nilmanifold, is given by the asymptotic behaviour of the eigenvalues of Laplace-Beltrami operator acting on the function on the metric balls of the universal covering of a nilmanifold, as the radius of the balls goes to infinity. In [Ver01] the second author showed that the first eigenvalue of this macroscopical spectrum satisfies an inequality, whose equality case is attained by the nilmanifolds having all harmonic 1-forms of constant norm. This shows that the nilmanifolds with left-invariant metrics are not the only one satisfying the equality case, as in the torus case. Hence theorem 8 gives the following corollary in that setting :

Corollary 2. *Let (M^n, g) be a nilpotent Riemannian manifold, with first Betti number $b_1 = n-1$. Let $B_g(\rho)$ be the ball of radius ρ induced by the lifted metrics on the universal covering of M^n. Let $\lambda_1\big(B_g(\rho)\big)$ be the first eigenvalue of the laplacian acting on functions over $B_g(\rho)$ for the Dirichlet problem. Then there are some functions $\lambda_1^\infty(g)$ and $\lambda_1^{al}(g)$ such that*

$$\lim_{\rho\to\infty} \rho^2 \lambda_1\big(B_g(\rho)\big) = \lambda_1^\infty(g) \leq \lambda_1^{al}(g)$$

with equality if and only if M^n is a 2-step nilmanifold with one dimensional center and g is pseudo left invariant.

In this paper we also study what other assumption, in term of product of harmonic forms (lead by the geometrically formal background), one could add to characterize the left invariant metrics in the case of the Betti number equal $n-1$, where n is the dimension (see theorem 9 to 11).

Isosystolic stable inequalities, studied among other by V. Bangert and M. Katz [BK] (see also references therein), give lower bounds on the volume of compact orientable manifold in terms of some short closed geodesics (systoles). The cases of equality is obtained by manifolds with one-forms of constant length. Hence Theorem 8 implies the following corollary (where stsys_1 is the stable systole)

Corollary 3. *Let (X, g) be a compact, oriented, n-dimensional Riemannian manifold with first Betti number $b = n-1$. Then there is a constant $c(n)$ such that*

$$\mathrm{stsys}_1(g)\mathrm{sys}_{n-1}(g) \leq c(n)\mathrm{vol}_g(X)$$

Equality occurs if and only if X is a 2-step nilmanifolds and there exist a dual-critical lattices L in $\mathbb{R}^{n-1}$ and a submersion of X into $\mathbb{R}^{n-1}/L$ (with minimal fibers).

Hence in the three dimensional case, equality is only satisfied by the three dimensional Heisenberg group endowed with a metric such that there is a Riemannian submersion onto an equilateral torus.

Our result are also to be compared with the work of E. Aubry, B. Colbois, P. Ghanaat and E.A. Ruh [ACGR01], where it is showed that an n-dimensional oriented manifold M having $n-1$ small (compared to the diameter) eigenvalues (for the laplacian acting on one-forms) is diffeomorphic to a nilmanifold with an almost left invariant metric. However, in their paper the authors needed strong assumptions on the curvature. In this note, instead of a control of the curvature, we have a control of the length of harmonic forms. We would like to stress that this seems to be a hidden assumption on the curvature.

2 Implication of the existence of harmonic forms of constant length.

2.1 The Albanese map

Let (M^n, g) be a compact Riemannian manifold having all of its harmonic one forms of constant length.

Let us stress that we do not consider the case where there are harmonic forms of constant length, but that *all* harmonic forms are supposed to be of constant length. This implies for example that the pointwise scalar product of two harmonic forms is constant, which is not the case with just the existence assumption as the following example shows (which happens to answers question 7 of section 10 in [BK]:

Example 4. Let (M^n, g) be Riemannian manifold, consider $N^{n+2} = M^n \times \mathbb{T}^2$ endowed with the following Riemannian metric $h = g \oplus s$ where s is defined as follows. Take a function from M to $\mathbb{R}$ such that $f^2 < 1$ then we take the metric s on $\mathbb{T}^2$ such that $s(dx, dx) = 1 = s(dy, dy)$ and $s(dx, dy) = f$. We claim that $\alpha_1 = dx$ and $\alpha_2 = dy$ which are closed are also co-closed thus harmonic, but by construction their scalar product is not constant.

Proof. with respect to h in N^{n+2}. Let us take $(e_i)_{1\leq i\leq n+2}$ a local orthonormal basis of N^{n+2}, with $(e_k)_{1\leq k\leq n}$ spanning TM, and e_{n+1}, e_{n+2} spanning $T\mathbb{T}^2$. Consider also $(X_i)_{i=1,2}$ the dual vector field with respect to h of $(\alpha_i)_{i=1,2}$. Then remark that $X_i = x_i^1 \frac{\partial}{\partial x} + x_i^2 \frac{\partial}{\partial y}$ and $e_k = E_k^1 \frac{\partial}{\partial x} + E_k^2 \frac{\partial}{\partial y}$ where $(x_i^j)_{i,j=1,2}$ and (E_k^i) are functions from M to $\mathbb{R}$. Now write

$$-d^*\alpha_i = \sum_{1\leq k\leq n+2} h(\nabla_{e_k} X_i, e_k)$$

As the Levi-Civita connection is torsion free, we remark that for $i = 1, 2$ and $1 \leq k \leq n+2$

$$h(\nabla_{e_k} X_i, e_k) = h(\nabla_{X_i} e_k, e_k) + h([X_i, e_k], e_k).$$

However for $i = 1, 2$ and $1 \leq k \leq n+2$

$$h(\nabla_{X_i} e_k, e_k) = \frac{1}{2} X_i \cdot h(e_k, e_k) = 0$$

and noticing that for $1 \leq k \leq n$, $[\frac{\partial}{\partial x}, e_k] = [\frac{\partial}{\partial y}, e_k] = 0$ we obtain

$$\begin{aligned}[X_i, e_k] &= [x_i^1 \frac{\partial}{\partial x} + x_i^2 \frac{\partial}{\partial y}, e_k] \\ &= -(e_k \cdot x_i^1) \frac{\partial}{\partial x} - (e_k \cdot x_i^2) \frac{\partial}{\partial y}\end{aligned}$$

and has $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are in $T\mathbb{T}^2$ we get that for all $1 \leq k \leq n$ and $i = 1, 2$,

$$h(\nabla_{e_k} X_i, e_k) = 0$$

Now for $k = n+1, n+2$ we have

$$[X_i, e_k] = [x_i^1 \frac{\partial}{\partial x} + x_i^2 \frac{\partial}{\partial y}, E_k^1 \frac{\partial}{\partial x} + E_k^2 \frac{\partial}{\partial y}]$$

however, for any function f define on M, $T\mathbb{T}^2 \in \ker df$ and noticing that $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$ we finally get that α_i is indeed harmonic. ❑

Let b_1 be the first Betti number of our manifold M. Using a basis of harmonic 1-forms we can define the Albanese map (or Jacobi map) π using integration, this gives a map onto a torus $\mathbb{T}^{b_1}$, on which we put the usual flat metric. Besides we have $b_1 \leq n$, because we can't have more than n linearly independant 1-forms. Hence in this case π is a Riemannian submersion. Moreover Albanese's map being harmonic (see A. Lichnerowicz [Lic69]), the fibers are minimal (see J. Eells and J.M. Sampson [ES64] and 9.34 [Bes87] page 243).

A theorem of R. Hermann (see theorem 9.3 in [Bes87] page 235) states that a Riemannian submersion whose total space is complete is a locally trivial fibre bundle. We sum this up in the following proposition.

Proposition 5. *Let (M^n, g) be a compact Riemannian manifold and b_1 be its first Betti number. Then all harmonic 1-forms are of constant length if and only if (M^n, g) is a locally trivial fiber bundle, with minimal fibers, over a b_1-dimensional flat torus, $b_1 \leq n$*

$$F^{n-b_1} \hookrightarrow M^n \xrightarrow{\pi} \mathbb{T}^{b_1}$$

Moreover if $b_1 = n$ then π is a Riemanian isometry, hence (M^n, g) is a flat torus.

Now let us look at some other consequences of the existence of harmonic forms of constant length. Thanks to the Albanese map we can lift the harmonic forms of the Albanese torus on the manifold. Let us call $\alpha_1, \ldots, \alpha_{b_1}$ an orthonormal family of lifted harmonic forms. Using the duality through the metric we can associates to these harmonic forms b_1 vector fields $X_1, \ldots, X_{b_1}$. These vector fields define a sub-bundle $\mathcal{H}$, which we will call the **Horizontal**, such that for all $x \in M$, $\mathcal{H}_x$ is generated by $X_1(x), \ldots, X_{b_1}(x)$.

If we let $\mathcal{V}$ be the orthogonal complement of $\mathcal{H}$ with respect to the metric, which we will refer to as the **Vertical**, then we have the following

Proposition 6. *Let (M^n, g) be a compact Riemannian manifold and b_1 be its first Betti number. If all harmonic 1-forms are of constant length, then there is distribution $\mathcal{H}$ given by an orthonormal frame of vector fields $X_1, \ldots, X_{b_1}$ dual to an orthonormal frame of harmonic 1-forms, such that the tangent bundle splits orthogonaly as follows :*

$$TM = \mathcal{V} \oplus \mathcal{H}.$$

Moreover the distribution $\mathcal{V}$ is integrable, and for any $1 \leq i, j \leq b_1$ and $U \in \mathcal{V}$, $[X_i, X_j] \in \mathcal{V}$ and $[X_i, U] \in \mathcal{V}$.

Proof. This comes from the fact that the forms $(\alpha_i)_{1 \leq i \leq b_1}$ are closed. Indeed for any closed 1-form α we have the following equality for any X, Y in TM :

$$\alpha([X, Y]) = X \cdot \alpha(Y) - Y \cdot \alpha(X)$$

thus for any i, j and k we have :

$$\alpha_i([X_j, X_k]) = X_j \cdot \alpha_i(X_k) - X_k \cdot \alpha_i(X_j) = X_j \cdot \delta_{ik} - X_k \cdot \delta_{ij} = 0$$

hence $[X_j, X_k]$ is orthogonal to any X_i. If U and V are vertical vector fields, then it is easily seen that for any $1 \leq i \leq b_1$

$$\alpha_i([X_j, U]) = 0, \quad \text{and} \quad \alpha_i([U, V]) = 0$$

❑

2.2 An useful decomposition

Let (M^n, g) be a compact Riemannian manifold with a unit vector field Z. Let $\mathcal{V}$ be the distribution generated by Z (which is sometimes called the **Vertical** distribution) and $\mathcal{H}$ its orthogonal complement with respect to g (wich we will call **Horizontal**). Then the tangent bundle splits as follows

$$TM = \mathcal{V} \oplus \mathcal{H}.$$

If $i_Z(\cdot) = Z \lrcorner \cdot$, is the interior product by Z then we can define the space of horizontal p-forms as follows :

$$\Lambda^p(\mathcal{H}) = \Lambda^p(M) \cap \ker(i_Z).$$

Let us introduce the 1-form $\vartheta = Z^\flat$ dual to Z with respect to g. Then it is an easy exercise to see that we have the following decomposition of p-forms :

$$\Lambda^p(M) = \Lambda^p(\mathcal{H}) \oplus \left[\Lambda^{p-1}(\mathcal{H}) \wedge \vartheta\right]. \tag{1}$$

Notice that following this decomposition we have

$$d\vartheta = b + \eta \wedge \vartheta$$

where b is a horizontal 2-form called the **curvature** of the horizontal distribution H and η is the horizontal one form associated to the horizontal vector field $-\nabla_Z Z$ with respect to g, a fact the reader may easily verify.

Let us introduce the horizontal exterior differential d_H, which associates to an horizontal differential form the horizontal part of its exterior differential.

We also introduce the multiplication operators

$$\begin{aligned} L &: \Lambda^q(\mathcal{H}) \to \Lambda^{q+2}(\mathcal{H}), \quad L := \cdot \wedge b \\ S &: \Lambda^q(\mathcal{H}) \to \Lambda^{q+1}(\mathcal{H}), \quad S := \eta \wedge \cdot \end{aligned} \tag{2}$$

Thanks to the decomposition (1), each p-form may be identified to a couple $(\alpha, \beta) \in \Lambda^p(\mathcal{H}) \times \Lambda^{p-1}(\mathcal{H})$ of horizontal forms. Thus we can see the exterior differential d acting on p-forms as an operator from $\Lambda^p(\mathcal{H}) \times \Lambda^{p-1}(\mathcal{H})$ to $\Lambda^{p+1}(\mathcal{H}) \times \Lambda^{p+2}(\mathcal{H})$. Then, we have the following proposition, where $\mathcal{L}_Z$ is the Lie derivative in the direction of Z, and where for any differential operator D, D^* is its formal adjoint with respect to g.

Proposition 7. **[Nag01].** *With respect to the decomposition (1) we have for the exterior differential acting on p-forms :*

$$d = \begin{pmatrix} d_H & (-1)^{p-1} L \\ (-1)^p \mathcal{L}_Z & d_H + S \end{pmatrix}$$

and for the codifferential :

$$d^* = \begin{pmatrix} d_H^* & (-1)^{p-1} \mathcal{L}_Z^* \\ (-1)^p L^* & d_H^* + S^* \end{pmatrix}$$

3 The case of the first Betti number being one less that the dimension

The aim of this section is to describe the n-dimensional manifolds having all of their harmonic one-forms of constant length, and with first Betti number equal to $n-1$. All manifolds considered here are of dimension greater or equal to 3 (in dimension 1 and 2 we have only the circle S^1 and the torus $\mathbb{T}^2$ which admit forms of constant length).

Let us recall that all 2-step nilmanifolds are principal torus bundles over a torus (see R.S. Palais and T.E. Stewart [PS61])

Definition 8. *Let N^{n+1} be a 2-step nilmanifold whose center is one dimensional and consider a submersion p from N^{n+1} onto a flat torus $\mathbb{T}^n$. Let $(\alpha_1, \dots, \alpha_n)$ be the lift of an orthonormal basis of harmonic 1-forms over the torus. Choose a principal connection form ϑ for this submersion. Let g_ϑ be the Riemanniann metric such that the dual basis of $(\alpha_1, \dots, \alpha_n, \vartheta)$ is orthonormal. Thus p becomes a Riemannian submersion. We will call such a metric* pseudo left invariant.

Remark that if the dual basis of $(\alpha_1, \dots, \alpha_n, \vartheta)$ in this definition is left invariant, then the metric g_ϑ is left invariant.

Thanks to this definition we can recall our characterization

Theorem 8. *Let (N^{n+1}, g) be a compact orientable connected manifold such that all of its harmonic 1-forms are of constant length, and such that $b_1(N^{n+1}) = n$, then (N^{n+1}, g) is a 2-step nilmanifold whose kernel is one dimensional and g is a pseudo left invariant metric.*

Proof. We deduce from proposition 5 that we have the following fibration :

$$S^1 \hookrightarrow N^{n+1} \xrightarrow{\pi} \mathbb{T}^n$$

where π is the Albanese map.

We begin by taking a basis $(a_1, \dots, a_n)$ of harmonic forms over the Albanese torus and lift it to a basis of harmonic 1-forms $(\alpha_1, \dots, \alpha_n)$ over N^{n+1}. Let $X_1, \dots, X_n$ be their dual vector fields with respect to the metric, they span, by definition, $\mathcal{H}$.

Thanks to Proposition 6, for any i, j

$$[X_i, X_j] \in \mathcal{V} \tag{3}$$

(remark that we can also deduce from that fact that $[X_i, X_j] = 2\nabla_{X_i} X_j$). Now take Z the dual vector field to the 1-form $Z^\flat = *(\alpha_1 \wedge \dots \wedge \alpha_n)$ ($*$ is the Hodge operator, thus this form is co-closed), its length is constant by construction. Furthermore Z belongs to and spans $\mathcal{V}$.

For any co-closed one form α we have

$$\sum_k \big(\nabla_{X_k} \alpha\big) X_k + \big(\nabla_Z \alpha\big) Z = 0 \tag{4}$$

Taking for α each of the α_i in turn, since $[X_i, X_j] = 2\nabla_{X_i} X_j \in \mathcal{V}$, we deduce from equality (4) that for $i = 1, \dots, n$

$$g(\nabla_Z X_i, Z) = 0$$

however as the Levi-Civita is torsion free, for $i = 1, \dots, n$

$$g(\nabla_Z X_i, Z) = g(\nabla_{X_i} Z, Z) + g([Z, X_i], Z) = g([Z, X_i], Z)$$

but by proposition 6, $[Z, X_i] \in \mathcal{V}$ hence for any $i = 1, \dots, n$

$$[X_i, Z] = 0 \tag{5}$$

This also implies that Z is a Killing field.

From (3) we have the existence of functions f_{ij} such that

$$[X_i, X_j] = f_{ij} Z. \tag{6}$$

However we would like to have some structural constants instead of the functions f_{ij}.

Let us remark that (Z being a Killing field) we have

$$dZ^\flat(X, Y) = 2g(\nabla_X Z, Y) \tag{7}$$

thus if we decompose $dZ^\flat$ in the basis given by $\alpha_i \wedge \alpha_j$ and $Z^\flat \wedge \alpha_i$ for all i, j then thanks to (6) and (7) we get that

$$dZ^\flat = \sum_{i<j} f_{ij} \alpha_i \wedge \alpha_j \tag{8}$$

In other words $dZ^\flat$ is horizontal and because Z is a Killing field it is projectable, i.e. , there exists a unique 2-form β on the Albanese torus such that $dZ^\flat = \pi^* \beta$.

Remark that $d\beta = 0$, thus $\beta = \beta_0 + d\alpha$ by the Hodge-de Rham theorem, with $\alpha \in \Lambda^1(\mathbb{T}^{b_1})$ and β_0 harmonic. Hence if $\zeta_0 = Z^\flat - \pi^*\alpha$, then $d\zeta_0 = \pi^*\beta_0$. But now $\beta_0 = \sum_{ij} c_{ij} a_i \wedge a_j$ where the c_{ij} are constants. This implies that

$$d\zeta_0 = \sum_{ij} c_{ij} \alpha_i \wedge \alpha_j \tag{9}$$

Notice that ζ_0 is also a (principal) connection 1-form. We are now taking as a basis of vector fields the dual base $(X_1^0, \ldots, X_{n-1}^0, Z_0)$ of the base $(\alpha_1, \ldots, \alpha_{n-1}, \zeta_0)$ (i.e. $\alpha_i(X_j^0) = \delta_{ij}$, $\ker \zeta_0 = \langle X_1^0, \ldots, X_{n-1}^0 \rangle$ and $\zeta_0(Z_0) = 1$) then we have from (9)

$$[X_i^0, Z_0] = 0$$
$$[X_i^0, X_j^0] = c_{ij} Z_0$$

Thus we can built an homomorphism I_A between the Lie algebra $\mathfrak{a}$ defined by $A_i, \ldots, A_n, A_{n+1}$ and with brackets

$$[A_i, A_j] = c_{ij} A_{n+1}$$

all the other brackets being equal to zero, by taking $I_A(A_i) = X_i^0$ for $i = 1, \ldots, n$ and $I_A(A_{n+1}) = Z_0$.

Now if $\mathcal{A}$ is the simply connected Lie group associated to $\mathfrak{a}$, thanks to the compactness of N^{n+1} we can integrate the homomorphisme I_A to obtain an action of $\mathcal{A}$ on N^{n+1} (see Corollary 3 and 4 of theorem 2.9 page 113 in [Oni93]) .

Since each orbit is open and N^{n+1} is connected, this action is transitive. From this we deduce (see F.W. Warner [War83] theorem 3.62) that N^n is a Lie group. And thanks to the constant of structures (c_{ij}) we deduce that it is a two step nilpotent lie group (see M. Spivak [Spi79] volume I theorem 17 in chapter 10 for example).

Concerning g, it is pseudo left invariant by the above discussion.

❏

It is worth noticing that if the manifold is not orientable this is no more true. Indeed, in the three dimensional case there are quotients of flat tori by a finite group whith first Betti number equal to 2.

Note also that theorem 8 implies that if M^n is a geometrically formal closed oriented manifold then $b_1(M^n) \neq n-1$ as already showed by D. Kotschik [Kot01].

A natural question arising is whether one can characterize the left invariant metrics among pseudo left invariant metric ones, in term of properties of harmonic forms. We first look at the 3-dimensional case.

Theorem 9. *Let (N^3, g) be a compact orientable connected manifold such that all harmonic 1-forms are of constant length, $b_1(N^3) = 2$, and such that the wedge product of two harmonic 1-forms is an eigenform of the laplacian, then (N^3, g) is a compact quotient of the 3-dimensional Heisenberg group and g is a left invariant metric.*

Proof. From the added assumption we get that $Z^\flat$ of the previous proof is an eigenform of the laplacian. That is to say that there is some constant λ such that

$$\Delta Z^\flat = \lambda Z^\flat$$

However, using the useful decomposition, knowing that $Z^\flat$ is coclosed and $dZ^b = b = \pi^*(\beta)$ (following the decomposition (1), b corresponds to $(b, 0)$ and $Z^\flat$ corresponds to $(0, 1)$) we must also have that

$$\Delta Z^\flat = d^* dZ^b = d^* b = (L^* b) Z^\flat = (L^* L \cdot 1) Z^\flat = |\beta|^2 Z^\flat$$

which means that β is of constant length. But β is a 2-form on a 2-dimensional Torus, which also means that β is proportional to the volume form, in other words the function (which is unique in that case) in the equality (8) is a constant. Remark that this also tells us that the eigenvalue is $\lambda = |\beta|^2 = f_{12}^2$. ❏

Now for the higher dimensional case.

Theorem 10. *For $n > 3$, let (N^n, g) be a compact orientable connected manifold such that all harmonic 1-forms are of constant length, $b_1(N^n) = n - 1$ and the wedge product of any $n - 2$ harmonic 1-forms is an eigenform of the Laplacian, then (N^n, g) is a 2-step nilmanifold whose kernel is one dimensional and g is a left invariant metric.*

Proof. We use the same notations as in the proof of theorem 8. From the new assumption we get thanks to the star Hodge operator that for any harmonic 1-form α, $\alpha \wedge Z^\flat$ is a co-closed eigenform. Using the decomposition (1) we associate to $\alpha \wedge Z^\flat$ the pair $(0, \alpha)$. As $\alpha \wedge Z^\flat$ is coclosed $\Delta(\alpha \wedge Z^\flat) = d^* d(\alpha \wedge Z^\flat)$. Let us use proposition 7 without forgetting that α is harmonic, and noticing that as $dZ^\flat$ is horizontal by the proof of theorem 8, $S = 0$

$$d(\alpha \wedge Z^\flat) \equiv \begin{pmatrix} d_H & -L \\ \mathcal{L}_Z & d_H \end{pmatrix} \begin{pmatrix} 0 \\ \alpha \end{pmatrix} = \begin{pmatrix} -L\alpha \\ d_H \alpha \end{pmatrix} = \begin{pmatrix} -L(\alpha) \\ 0 \end{pmatrix}$$

and

$$d^* d(\alpha \wedge Z^\flat) \equiv \begin{pmatrix} d_H^* & \mathcal{L}_Z^* \\ -L^* & d_H^* \end{pmatrix} \begin{pmatrix} -L(\alpha) \\ 0 \end{pmatrix} = \begin{pmatrix} d_H^*(-L(\alpha)) \\ L^* L(\alpha) \end{pmatrix}$$

but the last term is also equal to $(0, \lambda\alpha)$ by the eigenform assumption, hence

$$d_H^*(\alpha \wedge dZ^\flat) = 0$$

But α and $dZ^\flat$ are horizontal and projectable, thus in fact there are one 1-form a and one 2-form β such that we can write

$$0 = d_H^*(\alpha \wedge dZ^\flat) = \pi^*\big(d^*(a \wedge \beta)\big).$$

In other words, for any 1-harmonic form a on the torus we have

$$d^*(a \wedge \beta) = 0. \tag{10}$$

Let us take an orthonormal base (e_i) of parallel vector fields, then for any form ω

$$-d^*\omega = \sum_i e_i \lrcorner \nabla_{e_i}\omega$$

We want to apply that last equality to $a \wedge \beta$. First notice that (a is parallel)

$$\nabla_{e_i}(a \wedge \beta) = a \wedge \nabla_{e_i}\beta.$$

Contracting by e_i one obtains

$$e_i \lrcorner \big(\nabla_{e_i}(a \wedge \beta)\big) = a(e_i)\nabla_{e_i}\beta - \alpha \wedge (e_i \lrcorner \nabla_{e_i}\beta)$$

Sum over i and use the coclosed condition (i.e. (10)) and you get

$$\nabla_{a^\#}\beta + a \wedge d^*\beta = 0 \tag{11}$$

Now we take the interior product with e_i of (11), with $a = e_i^\flat$ which gives

$$e_i \lrcorner \nabla_{e_i}\beta + e_i \lrcorner (e_i^\flat \wedge d^*\beta) = e_i \lrcorner \nabla_{e_i}\beta + d^*\beta - d^*\beta(e_i)e_i^\flat = 0$$

we sum over i one more time

$$(n-3)d^*\beta = 0$$

and by assumption $n > 3$ thus $d^*\beta = 0$, that is to say that β is harmonic over the torus. Hence it follows that β is parallel and all the functions f_{ij} of (8) are constants, which allows us to conclude. ❏

There is another case where things can be precised.

Theorem 11. *Let $(N^{2m+1}, \omega, g_\omega)$ be a compact contact manifold with a contact form ω and an adapted Riemannian metric g_ω such that all harmonic 1-forms are of constant length, then (N^{2m+1}, g) is a compact quotient of an Heisenberg group and g is a left invariant metric.*

Proof. From theorem 8 we get that N^{2m+1} is a two step nilmanifold and g_ω is pseudo left-invariant. We also get that $g_\omega = \vartheta^2 + \pi^*(h)$ where ϑ is a one form, such that $d\vartheta = \pi^*(\beta)$ for some closed 2-form β over the Albanese torus. Now theorem 3.2 and theorem 3.4 of H.P. Pak and T. Takahashi in [PT01] implies that for all harmonic 1-forms α, if T is the Reeb vector field attached to ω,

$$T \lrcorner \alpha = \alpha(T) = 0$$

wich means that $T = fZ$ for some function f, and as T and Z are of constant unit length for the metric g_ω it means that $\omega = \vartheta$. Now, thanks to theorem 8 we know that $T = Z$ is a Killing field. This implies that the almost complex structure J on $\ker \omega$ lives on the flat torus given by the Albanese submersion. Hence we are in front of an almost-Kähler flat torus, but following [OLS78] and [Arm02] it has to be Kähler, thus β is parallel. ❏

4 Some remarks on the general case

The aim of this section is to point out the main differences between the case $b_1 = n-1$ and $b_1 \leq n-2$ for n dimensional manifolds admitting one-harmonic forms of constant length. We want to give some hint on the failure of our approach.

Our first remark is that one should restrict oneself to the study of locally irreducible orientable Riemannian manifolds to avoid the following cases : the direct product of a sphere of dimension $p > 1$ and a flat torus of dimension $n-p$, gives a manifold whose first Betti number is $n-p$, whose dimension is n and with $n-p$ harmonic 1-forms of constant length.

The second remark is in the following lemma, which shows the limitation of our method. Indeed to apply the same ideas one needs far stronger assumptions.

Lemma 12. *Let (N^n, g) be a compact locally irreducible manifold such that all harmonic 1-forms are of constant length, $b_1(N^n) = n-p$ and possessing a pointwise orthonormal base (ϑ_i) of the orthogonal complement of the harmonic 1-forms. Moreover assume that $(d\vartheta_i)$ are lifts of closed 2-forms on the Albanese torus. Then (N^n, g) is a two step nilpotent nilmanifolds whose kernel is $n-p$ dimensional.*

Proof. $TM = \mathcal{V} + \mathcal{H}$ where $\mathcal{H}$ is spanned by $X_1, \dots, X_{n-p}$ the dual vector fields to $\alpha_1, \dots, \alpha_{n-p}$, which are lifts of harmonic 1-forms on the Albanese torus, and $\mathcal{V}$ is the orthogonal complement. We associate, thanks to the metric, the dual vector fields (Z_k) to the 1-forms (ϑ_k). As the α_i are closed we get that for $1 \leq i, j \leq n-p$,

$$[X_i, X_j] \in \mathcal{V}.$$

and our assumptions imply that

$$d\vartheta_k \in \Lambda^2\mathcal{H}$$

where $\Lambda^2\mathcal{H} = \Lambda^2 M \cap \bigcap_k \ker i_{Z_k}$ and that

$$d\vartheta_k = \pi^*(\beta_k)$$

where β_k is a 2-form on Albanese's torus. However, $d\beta_k = 0$, hence for some harmonic 2-form β_k^0 and some 1-form a_k over the Albanese torus we have

$$\beta_k = \beta_k^0 + da_k$$

Notice, that β_k^0 is non zero otherwise ϑ_k would be horizontal, which is not the case by assumption. Hence if we consider the independent forms (as one can easily verify)

$$\theta_k = \vartheta_k - \pi^*(a_k)$$

then

$$d\theta_k = \pi^*(\beta_k^0).$$

hence we can conclude as in the proof of Theorem 8. ❑

As the last section involved nilmanifold, we could decide to focus on a family of compact manifolds close to them, say solvmanifold. But even that assumption is not enough to clarify the situation, as the next lemma with the following remark point out.

Lemma 13. *Let* (M^n, g) *be a solvmanifold all of whose* 1*-forms are of constant length and whose first Betti number is* $n-2$*, then we have the following fibration with minimal fibers :*

$$\mathbb{T}^2 \hookrightarrow M^n \xrightarrow{\pi} \mathbb{T}^{n-2}$$

Proof. This comes from the fact that the fibration in Proposition 5

$$F^2 \hookrightarrow M^n \xrightarrow{\pi} \mathbb{T}^{n-2}$$

gives a long exact sequence on the homotopy groups :

$$\ldots \to \pi_n(F^2) \to \pi_n(M^n) \to \pi_n(\mathbb{T}^{n-2}) \to \pi_{n-1}(F^2) \to \pi_{n-1}(M^n) \to \ldots$$

Now, $\mathbb{T}^{n-2}$ and M^n have only their fundamental group which is not trivial, hence we have the following exact sequence on the fundamental groups

$$0 \to \pi_1(F^2) \to \pi_1(M^n) \to \pi_1(\mathbb{T}^{n-2}) \to 0$$

and $\pi_k(F^2)$ is trivial if $k > 1$. Which means that $\pi_1(F^2)$ can be seen as a subgroup of the solvable group $\pi_1(M^n)$, hence it is also solvable. Thus the fiber is compact and with solvable fundamental group. However in dimension 2 the only compact oriented manifold with solvable fundamental groups are the sphere and the torus, but here the sphere is excluded because $\pi_2(S^2) \neq 0$. ❑

Without further assumptions we can't expect a more precise results. Indeed the example of the Sol geometry in dimension 3, or of any 2-step nilmanifold with a 2 dimensional center will satisfy Lemma 13 with many different metrics, however built in the same way : following the construction 8.1 in [BK].

As a conclusion, the rigidity of the cases $b_1 = n$ and $b_1 = n-1$ do not propagates to lower values of the first Betti number

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