The structure of $\mathcal{AK}_2$-manifolds. *

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January 21, 2003

Abstract

We study special almost Kähler manifolds whose curvature tensor satisfies the second curvature condition of Gray. It is shown that for such manifolds, the torsion of the first canonical Hermitian is parallel. This enables us to show that every $\mathcal{AK}_2$-manifold has parallel torsion. Some applications of this result, concerning the existence of orthogonal almost Kähler structures on spaces of constant curvature, are given.

Contents

1 Introduction

An almost Kähler manifold (shortly $\mathcal{AK}$) is a Riemannian manifold (M^{2n}, g), together with a compatible almost complex structure J, such that the Kähler form $\omega = g(J\cdot, \cdot)$ is closed. Hence, almost Kähler geometry is nothing else that symplectic geometry with a prefered metric and complex structure. Since symplectic manifolds often arise in this way, it is rather natural to ask under which conditions on the metric we get integrability of the almost complex J. In this direction, a famous conjecture of S. I. Golberg asserts that every compact, Einstein, almost Kähler manifold is, in fact, Kähler. At our present knowledge, this conjecture is still open. Nevertheless, they are a certain number of partial results, supporting this conjecture. First of all, K. Sekigawa proved [23] that the Goldberg conjecture is true when the scalar curvature is positive. We have to note that the Golberg conjecture is definitively not true with the compacity assumption removed. In fact, there are Hermitian symmetric spaces of

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non-compact type of any complex dimension $n \geq 3$ admitting almost Kähler structures commuting with the invariant Kähler one [4]. Not that, at the opposite, the real hyperbolic space of dimension at least 4 do not admits, even locally, orthogonal almost Kähler structures [22, 8]. In dimension 4, examples of local Ricci flat almost Kähler metrics are constructed in [3, 8, 18]. In the same paper, a potential source of compact almost Kähler, Einstein manifolds is considered, namely those compact Kähler manifolds whose Ricci tensor admits two distinct, constant eigenvalues; integrability is proven under certain positivity conditions. The rest of known results, most of them enforcing or replacing the Einstein condition with some other natural curvature assumption are mainly in dimension 4. To cite only a few of them, we mention the beautiful series of papers [5, 2, 6] giving a complete local and global classification of almost Kähler manifolds of 4 dimensions satisfying the second and third Gray condition on the Riemannian curvature tensor. Other recent results, again in 4-dimensions, are concerned with the study of local obstructions to the existence of Einstein metrics [8], $\star$-Einstein metrics [19], etc.

The aim of this paper is to close the circle of ideas from our previous paper [17] and to show coincidence between the class of $\mathcal{AK}_2$-manifolds- that is almost Kähler manifolds whose curvature tensor satisfies the second Gray's condition- and almost Kähler manifolds with parallel torsion. Here, both torsion and parallelism are with respect ot the first canonical Hermitian connection. More precisely, let us recall that it was proven in [17] that in an open dense set every $\mathcal{AK}_2$ locally splits a the Riemannian product of an almost Kähler manifolds with parallel torsion and a special $\mathcal{AK}_2$-manifold (see section 3 for the definition). Our main present result is to give a classification of special $\mathcal{AK}_2$-manifolds.

Theorem 1.1 *Let (M^{2n}, g, J) be a special $\mathcal{AK}_2$-manifold. Then M has parallel torsion and (M, g, J) is in fact a locally 3-symmetric space.*

Therefore we get the precise description, in terms of the torsion of almost Kähler manifolds whose curvature tensor satisfies the second Gray condition.

Theorem 1.2 *Let (M^{2n}, g, J) be an almost Kähler manifold in the class $\mathcal{AK}_2$. Then the torsion of (g, J) is parallel with respect to the first canonical Hermitian connection.*

As a corollary we obtain that there are no Einstein manifolds in the class $\mathcal{AK}_2$. This follows from [17] as a simple application of Sekigawa's formula combined with the paralleism of the torsion. Let us note that our actula classification of $\mathcal{AK}_2$-manifolds is of course not a complete one. What it remains to be done is to classify almost Kähler manifolds with parallel torsion with respect to the first canonical Hermitian connection.

Our present classification enables some applications relating to the non-existence of almost Kähler structures in hyperbolic geometry.

Theorem 1.3 *(i) Let (M, g) be a manifold of constant sectional curvature. If there exists an almost complex structure J such that (g, J) is almost Kähler, then g is a flat metric.*
(ii) Let (M^{2n}, g, J) be a Kähler manifold of constant negative holomorphic sectional curvature. If I is an almost complex structure commuting with I such that (g, I) is almost Kähler, then I has to be in fact a Kähler structure.

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The result in (i) is not new, with exception of its proof. In dimension beyond 8 it was proven in [22], and in dimension 4 and 6 in [8]. Note that in dimension 4 the result of (ii) was already obtained by J. Armstrong, as a consequence of his classification of almost Kähler, Einstein, 4-manifold of class $\mathcal{AK}_3$. In dimension 6 and beyond, the result is, at our knowledge, new. Also, note that, by results in [5], that they are Hermitian symmetric spaces of non-compact type, admitting a reversing strictly almost Kähler structure. Other symmetric spaces can also support strictly almost Kähler structures [21].

The paper is organised as follows. In section 2 we review some elementary facts from almost Kähler geometry and also a few results from our previous paper [17], to be used in the subsequent. Section 3 is devoted to the investigation of the curvature tensor of special $\mathcal{AK}_2$ manifolds. Some important technical tools are developed and at the end of the section it is proved that any special $\mathcal{AK}_2$-manifold with parallel torsion has to be locally 3-symmetric. The fourth section contains the proof of the theorem 1.1, whose basic ingredient is the observation that a partial holonomy reduction (with respect to the canonical Hermitian connection) extends in a canonical way to a global one.

2 Preliminaries

Let us consider an almost Hermitian manifold (M^{2n}, g, J), that is a Riemannian manifold endowed with a compatible complex structure. We denote by ∇ the Levi-Civita connection of the Riemannian metric g. Consider now the tensor ∇J, the first derivative of the almost complex structure and recall that for all X in TM we have that $\nabla_X J$ is a skew-symmetric (with respect to g) endomorphism of TM, which anticommutes with J. The tensor ∇J can be used to distinguish various classes of almost Hermitian manifolds. For example, (M^{2n}, g, J) is quasi-Kähler iff

$$\nabla_{JX} J = -J \nabla_X J$$

for all X in TM. If $\omega = g(J \cdot, \cdot)$ denotes the Kähler form of the almost Hermitian structure (g, J), we have an almost Kähler structure ($\mathcal{AK}$ for short), iff $d\omega = 0$. We also recall the well known fact that almost Kähler manifolds are always quasi-Kähler.

The almost complex structure J defines a Hermitian structure if it is integrable, that is the Nijenhuis tensor N_J defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]$$

for all vector fields X and Y on M identically vanishes. This is also equivalent to

$$\nabla_{JX} J = J \nabla_X J$$

whenever X is in TM. Therefore, an almost Kähler manifold which is also Hermitian must be Kähler.

In the rest of this section (M^{2n}, g, J) will be an almost Kähler manifold. We begin to recall some basic facts about the various notions of Ricci tensors.

Let Ric be the Ricci tensor of the Riemannian metric g. We denote by Ric' and Ric'' the J-invariant resp. the J-anti-invariant part of the tensor Ric. Then the Ricci form is defined by

$$\rho = <Ric'J\cdot, \cdot > .$$

We define the $\star$-Ricci form by

$$\rho^\star = \frac{1}{2} \sum_{i=1}^{2n} R(e_i, Je_i)$$

where $\{e_i, 1 \leq i \leq 2n\}$ is any local orthonormal basis in TM. Note that $\rho^\star$ is not, in general, J-invariant. The $\star$-Ricci form is related to the Ricci form by

2.1 $$\rho^\star - \rho = \frac{1}{2}\nabla^\star\nabla\omega.$$

The (classical) proof of this fact consists in using the Weitzenböck formula for the harmonic 2-form ω. Taking the scalar product with ω we obtain :

$$s^\star - s = \frac{1}{2}|\nabla J|^2$$

where the $\star$-scalar curvature is defined by $s^\star = 2 < R(\omega), \omega >$.

A basic object in almost Kähler geometry is the first canonical Hermitian connection, defined by :

$$\overline{\nabla}_X Y = \nabla_X Y + \eta_X Y$$

for all vector fields X and Y on M. Here, the tensor η is given by $\eta_X Y = \frac{1}{2}(\nabla_X J)JY$. Then $\overline{\nabla}$ is a metric Hermitian connection, that is it respects the metric and the almost complex structure. The torsion of $\overline{\nabla}$ is defined by $T_X Y = \eta_X Y - \eta_Y X$. Now, it is worthly to note that the almost Kähler condition (i.e. $d\omega = 0$) is equivalent to

$$< T_X Y, Z > = - < \eta_Z X, Y >$$

for all X, Y and Z in TM. In the subsequent we will refer simply to the tensor T as the torsion of the almost Kähler manifold (M^{2n}, g, J). Our main object of study in this paper is the class of almost Kähler manifolds introduced by the following definition.

Definition 2.1 *An almost Kähler manifold (M^{2n}, g, J) belongs to the class $\mathcal{AK}_2$ iff its Riemannian curvature tensor satisfies the identity :*

$$R(X, Y, Z, U) - R(JX, JY, Z, U) = R(JX, Y, JZ, U) + R(JX, Y, Z, JU)$$

for all X, Y, Z, U in TM.

Manifolds within this class have a simple caracterization in terms of the torsion of the canonical Hermitian connection.

Proposition 2.1 *[17] An almost Kähler manifold (M^{2n}, g, J) belongs to the class $\mathcal{AK}_2$ iff*

$$(\overline{\nabla}_X \eta)(Y, Z) = (\overline{\nabla}_Y \eta)(X, Z)$$

for all vector fields X, Y and Z on M.

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A number of usefull properties can be derived from the previous caracterization. At first let us define the Kähler nullity of the almost Kähler structure (g, J) to be $H = \{v : \eta_v = 0\}$. The orthogonal complement of H in TM will be denoted by $\mathcal{V}$. Since our study is purely local we can assume without loss of generality that H (and hence $\mathcal{V}$) have constant rank (this happens anyway in any connected component of a dense open subset of M). Then every closed property proved locally will extend to the whole manifold.

Proposition 2.2 *[17] We have :*
(i) both distributions $\mathcal{V}$ and H are integrable.
(ii) $\overline{\nabla}_V \eta = 0$ for all V in $\mathcal{V}$.
(iii) for any V, W in $\mathcal{V}$ and X in H we have that $\overline{\nabla}_V W$ and $\overline{\nabla}_V X$ belong to $\mathcal{V}$ and H respectively.

Let us denote now by $\overline{R}$ the curvature tensor of connection $\overline{\nabla}$. It has the following symmetry property :

$$\textbf{2.2} \qquad \overline{R}(X, Y, Z, U) - \overline{R}(Z, U, X, Y) = <[\eta_X, \eta_Y]Z, U> - <[\eta_Z, \eta_U]X, Y>$$

whever X, Y, Z, U are in TM. We end this section by recalling a technical consequence of proposition 2.1.

Lemma 2.1 *[17] Let V, W belong to $\mathcal{V}$ and X, Y be in H. We have :*
(i) $\overline{R}(V, W)\eta = 0$
(ii) $[\overline{R}(X, Y), \eta_V] = \eta_{\beta_V(X,Y)}$ where $\beta_V(X, Y) = \eta_{\eta_V Y} X - \eta_{\eta_V X} Y$.

3   Special $\mathcal{AK}_2$-manifolds

This section will be devoted to devellop a number of preliminary results to be used in the proof of theorem 1.1. We begin by recalling the definition of special $\mathcal{AK}_2$-manifolds which is of esentially algebraic nature.

Definition 3.1 *Let (M^{2n}, g, J) be in the class $\mathcal{AK}_2$. It is said to be special if and only if $\eta_\mathcal{V} \mathcal{V} = H$ where H is the Kähler nullity of (g, J) and $\mathcal{V}$ its orthogonal complement in TM.*

Algebraically speaking, the special condition ensures the vanishing of the torsion on $\mathcal{V}$, and also the symmetry of the restriction to $\mathcal{V}$ of the tensor η. We equally note that for a special $\mathcal{AK}_2$-manifold we also have :

$$(\nabla_\mathcal{V} J)H = \mathcal{V}.$$

From a geometric viewpoint, the special condition tells us that the integral manifolds of the integrable distribution $\mathcal{V}$ inherits from (M, g, J) the structure of Kähler manifolds.

In the rest of this section we will work on a given $\mathcal{AK}_2$-manifold (M^{2n}, g, J). The notations of the previous definition are to used without further comment. We are going to investigate the action of the curvature tensor $\overline{R}$ on the decomposition $TM = \mathcal{V} \oplus H$. Our starting point is the following intermediary result.

Lemma 3.1 *Let (M^{2m}, g, J) be a special $\mathcal{AK}_2$ manifold. Then the following holds :*

3.1
$$2\overline{R}(V_3, V_4, V_2, \eta_{V_1} X) = \overline{R}(V_3, V_4, X, \eta_{V_1} V_2)$$

for all $V_i, 1 \leq i \leq 4$ in $\mathcal{V}$ and X in H.

Proof :

We will make use of the following formula from [17], a consequence of the second Bianchi identity for the connection $\overline{\nabla}$:

3.2 $\quad \overline{R}(\eta_{V_2} X, V_1, V_2, V_3, V_4) - \overline{R}(\eta_{V_1} X, V_2, V_3, V_4) = - < [\eta_{V_3}, \eta_{V_4}] X, T_{V_1} V_2 >$

whenever $V_i, 1 \leq i \leq 4$ are in $\mathcal{V}$ and X is in H. First, we note that under the special condition the right hand side of (3.2) vanishes so that we have $\overline{R}(\eta_{V_2} X, V_1, V_2, V_3, V_4) = \overline{R}(\eta_{V_1} X, V_2, V_3, V_4)$. Now, using the symmetry property (2.2) we obtain :

$$\overline{R}(V_3, V_4, \eta_{V_2} X, V_1) + < [\eta_{\eta_{V_2} X}, \eta_{V_1}] V_3, V_4 > - < [\eta_{V_3}, \eta_{V_4}] \eta_{V_2} X, V_1 > =$$
$$\overline{R}(V_3, V_4, \eta_{V_1} X, V_2) + < [\eta_{\eta_{V_1} X}, \eta_{V_2}] V_3, V_4 > - < [\eta_{V_3}, \eta_{V_4}] \eta_{V_1} X, V_2 >$$

Using the vanishing of the torsion T on $\mathcal{V}$, a standard verification leads to $< [\eta_{\eta_{V_2} X}, \eta_{V_1}] V_3, V_4 >$
$- < [\eta_{V_3}, \eta_{V_4}] \eta_{V_2} X, V_1 > = 0$ hence

3.3 $\quad\quad\quad\quad \overline{R}(V_3, V_4, \eta_{V_2} X, V_1) = \overline{R}(V_3, V_4, \eta_{V_1} X, V_2).$

But $(\overline{R}(V_3, V_4) . \eta)(V_2, X) = 0$ for $i = 1, 2$ (see lemma 2.1, (i)). Plugging this in the previous equation we obtain :

$$< \eta_{\overline{R}(V_3, V_4) V_2} X, V_1 > + < \eta_{V_2} \overline{R}(V_3, V_4) X, V_1 > = \overline{R}(V_3, V_4, \eta_{V_1} X, V_2).$$

Invoking proposition 2.2, (iii), one finds that the operator $\overline{R}(V_3, V_3)$ preserves $\mathcal{V}$ and H. Using again the vanishing of the torsion on $\mathcal{V}$ we have

$$< \eta_{\overline{R}(V_3, V_4) V_2} X, V_1 > = - < X, \eta_{\overline{R}(V_3, V_4) V_2} V_1 > =$$
$$- < X, \eta_{V_1} \overline{R}(V_3, V_4) V_2 > = \overline{R}(V_3, V_4, V_2, \eta_{V_1} X)$$

and the conclusion is now immediate ■

The relation in lemma 3.1 shows that the restriction of $\overline{R}$ to $\mathcal{V}$ is completely determined by the mixed curvature terms of type $\overline{R}(V, W, X, Y)$ with V, W in $\mathcal{V}$ and X, Y in H. To investigate these terms we introduce now the **configuration** tensor $A : H \times H \to \mathcal{V}$ by setting :

$$\overline{\nabla}_X Y = \tilde{\nabla}_X Y + A_X Y$$

for all X, Y in H. In a similar way, we define $B : H \times \mathcal{V} \to \mathcal{V}$ by

$$\overline{\nabla}_X V = \tilde{\nabla}_X V + B_X V.$$

Since H is integrable A is a symmetric tensor, that is $A_X Y = A_Y X$ for all X, Y in H. It is immediate to establish that $[A_X, J] = 0$ for all X in H. Now, the parallelism of η in the direction of $\mathcal{V}$ together with the caracterization in proposition 2.1 translates into additional algebraic properties of the tensor A, as follows.

Lemma 3.2 *Let X, Y be in H and V, W in $\mathcal{V}$ respectively. We have :*
(i) $B_X(\eta_V Y) = \eta_V(A_X Y).$
(ii) $A_X(\eta_V W) = \eta_V(B_X W).$

Proof :
Having in mind proposition 2.2, (iii) it suffices to project on H and $\mathcal{V}$ respectively the identities $(\overline{\nabla}_X \eta)(V, Y) = 0$ and $(\overline{\nabla}_X \eta)(V, W) = 0$. ∎

The configuration tensor A can be used to compute parts of the curvature tensor $\overline{R}$ in the following way.

Lemma 3.3 *Let* V, W *and* X, Y *be vector fields in* $\mathcal{V}$ *and* H *respectively. We have :*
(i)
$$\overline{R}(V, X, W, Y) = < W, (\overline{\nabla}_V A)(X, Y) > - < B_X V, B_Y W >$$
(ii)
$$< W, (\overline{\nabla}_V A)(X, Y) > = < V, (\overline{\nabla}_W A)(X, Y) > .$$
(iii)
$$\overline{R}(V, W, X, Y) = - < B_X V, B_Y W > + < B_X W, B_Y V >$$
for all V, W *in* $\mathcal{V}$ *and* X, Y *in* H.

Proof :
The proof of (i) will be omitted since a standard computation using only the proposition 2.2, (iii). Now, (ii) comes from (i) by means of the symmetry property (2.2). To prove (iii) one uses the first Bianchi identity for $\overline{R}$ when noticing that the latter do not contains derivatives of the torsion, in virtue of proposition 2.1 and proposition 2.2, (ii). ∎

We will start now to compute parts of the curvature tensor $\overline{R}$. To begin with, let us define the symmetric, J-invariant, partial Ricci tensors $r_1 : \mathcal{V} \to \mathcal{V}$ and $r_2 : H \to H$ by setting :
$$\sum_{v_k \in \mathcal{V}} \overline{R}(v_k, J v_k) V = r_1(JV)$$
$$\sum_{v_k \in \mathcal{V}} \overline{R}(v_k, J v_k) X = r_2(JX)$$

Then we have :

Proposition 3.1 *The partial Ricci tensors* r_1, r_2 *can be computed by the following formulas :*
(i) $< r_1 V, W > = \sum\limits_{e_i \in H} < B_{e_i} V, B_{e_i} W >$
(ii) $< r_2 X, Y > = -2 \sum\limits_{v_k \in \mathcal{V}} < B_X v_k, B_Y v_k >$
where V, W *are in* $\mathcal{V}$ *and* X, Y *belong to* H *and* $\{v_k\}, \{e_i\}$ *are arbitrary orthonomal basis in* $\mathcal{V}$ *and* H *respectively.*

Proof :
(ii) follows by a simple computation involving lemma 3.3, (iii). Let us prove (i). Using (3.3), lemma 3.1 and (i) we obtain that
$$\sum_{v_k \in \mathcal{V}} \overline{R}(v_k, J v_k, V, \eta_W X) = \sum_{v_k \in \mathcal{V}} \overline{R}(v_k, J v_k, W, \eta_V X) = \tfrac{1}{2} \sum_{v_k \in \mathcal{V}} \overline{R}(v_k, J v_k, X, \eta_V W)$$
$$= - \sum_{v_k \in \mathcal{V}} < B_X v_k, B_{\eta_V W}(J v_k) > .$$

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Now, we have :

$$\sum_{v_k \in \mathcal{V}} < B_X v_k, B_{\eta_V W}(Jv_k) >= -\sum_{v_k \in \mathcal{V}} \sum_{e_i \in H} < e_i, B_{\eta_V W}(Jv_k) >< v_k, A_{e_i} X >$$
$$= \sum_{e_i \in H} < A_{e_i} \eta_V W, J A_{e_i} X > .$$

Or using in an apropriate way lemma 3.2 we have

$$< A_{e_i} \eta_V W, J A_{e_i} X >=< \eta_V B_{e_i} W, A_{e_i} J X >= - < B_{e_i} W, B_{e_i} \eta_V (JX) >=$$
$$< B_{e_i} W, B_{e_i} J \eta_V X >$$

and the conclusion is now straightforward. ∎

We are now able to give the main technical result of this section.

Proposition 3.2 *Let $(M^{2n}, g, J), n \geq 2$ be a special $\mathcal{AK}_2$-manifold. Then the following holds :*

3.4
$$\Delta^{\mathcal{V}} |A|^2 = -5|r_1|^2 - 2|\overline{\nabla}_{\mathcal{V}} A|^2.$$

Here, $\overline{\nabla}_{\mathcal{V}}$ denotes the restriction of $\overline{\nabla}$ to $\mathcal{V}$ and $\Delta^{\mathcal{V}}$ is the corresponding partial Laplacian, acting on functions.

Proof :
From lemma 3.3, (ii) we deduce that
3.5
$$(\overline{\nabla}_{JV} A)(JX, Y) = (\overline{\nabla}_V A)(X, Y)$$
for all V in $\mathcal{V}$ and X, Y in H respectively. We consider now the partial Laplacian $D^{\mathcal{V}}$, acting on A by :

$$(D^{\mathcal{V}} A)(X, Y) = -\sum_{v_k \in \mathcal{V}} (\overline{\nabla}^2_{v_k, v_k} A)(X, Y)$$

for all X, Y belonging to H, where $\{v_k\}$ is an arbitrary local orthonormal basis of $\mathcal{V}$. Derivating (3.5) it follows that

$$(D^{\mathcal{V}} A)(X, Y) = \frac{1}{2} J \sum_{v_k \in \mathcal{V}} (\overline{R}(v_k, Jv_k).A)(X, Y).$$

Using proposition 3.1, we obtain further :

$$(D^{\mathcal{V}} A)(X, Y) = -\frac{1}{2}(r_1 A_X Y + 2A_{r_2 X} Y + 2A_X r_2 Y).$$

Taking the scalar product with A gives now $< D^{\mathcal{V}} A, A >= -\frac{1}{2}(|r_1|^2 + 4|r_2|^2)$, or further $< D^{\mathcal{V}} A, A >= -\frac{5}{2}|r_1|^2$, after noticing that $|r_1|^2 = |r_2|^2$. Now , the standard Weitzenböck formula gives

$$\frac{1}{2}\Delta^{\mathcal{V}} |A|^2 =< D^{\mathcal{V}} A, A > -|\overline{\nabla}_{\mathcal{V}} A|^2$$

and the claimed formula follows now easily. ∎

Remark 3.1 *(i) Another way of proving formula (3.4) is the following. Define an almost complex structure I on M by setting*

$$I = J \text{ on } \mathcal{V} \text{ and } I = -J \text{ on } H.$$

Then it can be shown that (M^{2n}, g, I) is almost Kähler, and one can even show after some calculation that (g, I) belongs to the class $\mathcal{AK}_3$. Then the use of Sekigawa's formula gives exactly (3.4). Of course, one has to use (3.1) and furthermore compute all the remaining curvature terms. Since the calculations are of more length we prefered the direct approach.
(ii) If the manifold (M, g, I) belongs to the class $\mathcal{AK}_2$ then the function $|\nabla I|^2$ is known to be constant and by the previous proposition we get that (g, I) is in fact a Kähler structure.

4   Proof of theorem 1.1

In this section we will give the proof of the theorem 1.1. This will be done by showing that it is always possible to restrict, at least locally and in dimension at least 6, the study of special $\mathcal{AK}_2$- manifolds, to the case when the function $|A|^2$ is constant. This, together with proposition 3.2 of the previous section, will enable us to prove theorem 1.1.

As in the section 3 let (M^{2n}, g, J) be a special $\mathcal{AK}_2$-manifold, with Kähler nullity H and let $\mathcal{V}$ be the distribution orthogonal to H. We will first study the integral manifolds of H. For every X in H define a linear map :

$$\gamma_X : \mathcal{V} \to \mathcal{V} \text{ by } \gamma_X V = \eta_V X.$$

The vanishing of the torsion on $\mathcal{V}$ implies that γ_X is symmetric for all X in H.

Let us denote by $\tilde{R}$ the curvature tensor of the connection $\tilde{\nabla}$, where we recall that $\tilde{\nabla}$ is the orthogonal projection of $\overline{\nabla}$ onto the decomposition $TM = \mathcal{V} \oplus H$ (see section 3). The maps γ_X are in relation with the curvature of H (with respect to the connection $\tilde{\nabla}$), as the following lemma shows.

Lemma 4.1 *Let X, Y, Z be in H and V, W in $\mathcal{V}$. We have :*

$$\tilde{R}(X, Y, \eta_V W, Z) = <[[\gamma_X, \gamma_Y], \gamma_Z]V, W>.$$

Proof :
Using the definition of $\tilde{\nabla}$ we obtain after a short computation that

4.1 $\overline{R}(X, Y, Z', Z) = \tilde{R}(X, Y, Z', Z) + <A_Y Z', A_X Z> - <A_X Z', A_Y Z>$

for all X, Y, Z, Z' in H. Now, by lemma 2.1, (ii) one obtains

4.2 $\overline{R}(X, Y, \eta_V W, Z) + \overline{R}(X, Y, W, \eta_V Z) = <\eta_{\beta_V(X,Y)} W, Z>.$

But the symmetry property (2.2) yields to

$$\overline{R}(X, Y, W, \eta_V Z) - \overline{R}(W, \eta_V Z, X, Y) = -<[\eta_W, \eta_{\eta_V Z}]X, Y>$$

since H is the Kähler nullity of (g, J). The use of lemma 3.3, (iii) gives then

$$
\begin{aligned}
\overline{R}(W, \eta_V Z, X, Y) &= -<B_X W, B_Y(\eta_V Z)> + <B_X(\eta_V Z), B_Y W> \\
&= -<B_X W, \eta_V(A_Y Z)> + <\eta_V(A_X Z), B_Y W> \\
&= <A_X(\eta_V W), A_Y Z> - <A_X Z, A_Y(\eta_V W)>.
\end{aligned}
$$

where we used succesivelly lemma 3.2, (i) and (ii). It follows that

$$
\overline{R}(X, Y, W, \eta_V Z) = <A_X(\eta_V W), A_Y Z> - <A_X Z, A_Y(\eta_V W)> - <[\eta_W, \eta_{\eta_V z}]X, Y>.
$$

Using this in (4.2) and taking $Z' = \eta_V W$ in (4.1) we get

$$
\tilde{R}(X, Y, \eta_V W, Z) = <\eta_{\beta_V(X,Y)} W, Z> + <[\eta_W, \eta_{\eta_V z}]X, Y>.
$$

It remains to take into account, in the last equation, the definition of the maps γ_U, U in H and our lemma follows routineously. ∎

Our basic tool in the study of special $\mathcal{AK}_2$-manifolds will be the following intermediary result showing that a partial holonomy reduction of H extends in a canonical way to a holonomy reduction over TM.

Proposition 4.1 *Let (M, g, J) be a special $\mathcal{AK}_2$-manifold. Assume that we have an orthogonal, J-invariant decomposition $H_1 \oplus H_2$ which is also $\tilde{\nabla}$-parallel (inside H). Then :*
(i) If we put $\mathcal{V}_i = \eta_\mathcal{V} H_i$ for $i = 1, 2$ then we have an orthogonal and J-invariant decomposition

$$
\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{V}.
$$

Moreover we have $\eta_{\mathcal{V}_1} \mathcal{V}_2 = 0$ and $\eta_{\mathcal{V}_i} \mathcal{V}_i = H_i, i = 1, 2$.
(ii) The decomposition

$$
TM = (\mathcal{V}_1 \oplus H_1) \oplus (\mathcal{V}_2 \oplus H_2)
$$

defines a local splitting of M into the Riemannian product of two special $\mathcal{AK}_2$-manifolds, with corresponding Kähler nullities H_1 and H_2.

Proof :
(i) Let X_1 and X_2 be in H_1 and H_2 respectively. Then the partial parallelism of H_1, together with the symmetry property of $\tilde{R}$ (a consequence of (4.1) and (2.2)) ensures that $\tilde{R}(X_1, X_2, \eta_V W, Z) = $ for all V, W in $\mathcal{V}$ and Z in H. Then, by the previous lemma we obtain

$$
[[\gamma_{X_1}, \gamma_{X_2}], \gamma_Z] = 0
$$

for all Z in H. Taking $Z = X_1$ we find that

$$
\gamma_{X_1}^2 \gamma_{X_2} + \gamma_{X_2} \gamma_{X_1}^2 = 2\gamma_{X_1} \gamma_{X_2} \gamma_{X_1}.
$$

We change now X_2 in JX_2 in the previous equation and take into account that $\gamma_{JX} = \gamma_X J = -J\gamma_X$. It follows that

$$
\gamma_{X_1}^2 \gamma_{X_2} + \gamma_{X_2} \gamma_{X_1}^2 = -2\gamma_{X_1} \gamma_{X_2} \gamma_{X_1}
$$

hence we must have

$$
\gamma_{X_1}^2 \gamma_{X_2} + \gamma_{X_2} \gamma_{X_1}^2 = \gamma_{X_1} \gamma_{X_2} \gamma_{X_1} = 0.
$$

This yields to $\gamma_{X_1}^3 \gamma_{X_2} = 0$ and since γ_X is a symmetric operator for all X in H we get that $\gamma_{X_1} \gamma_{X_2} = 0$. But this fact is easily seen to be equivalent to the orthogonality of the spaces $\mathcal{V}_1 = \eta_{\mathcal{V}} H_1$ and $\mathcal{V}_2 = \eta_{\mathcal{V}} H_2$. The remaining afirmations of (i) are direct consequences of this fact.

(ii) We are going to prove first that the distribution $\mathcal{V}_1 \oplus H_1$ is $\overline{\nabla}$-parallel. Let U be in TM and V, W in $\mathcal{V}_1$. As $(\overline{\nabla}_U \eta)(V, W) = (\overline{\nabla}_V \eta)(U, W) = 0$ we get that

$$\overline{\nabla}_U(\eta_V W) = \eta_{\overline{\nabla}_U V} W + \eta_V(\overline{\nabla}_U W)$$

belongs to $\eta_{TM} \mathcal{V}_1 + \eta_{\mathcal{V}_1} TM = \mathcal{V}_1 \oplus H_1$. Since $H_1 = \eta_{\mathcal{V}_1} \mathcal{V}_1$ we conclude that $\overline{\nabla}_U X$ belongs to $\mathcal{V}_1 \oplus H_1$ for all X in H_1. Take now V in $\mathcal{V}_1$ and X in H_1. As before, we have :

$$\overline{\nabla}_U(\eta_V X) = \eta_{\overline{\nabla}_U V} X + \eta_V(\overline{\nabla}_U X)$$

belongs to $\eta_{TM} H_1 + \eta_{\mathcal{V}_1} TM = \mathcal{V}_1 \oplus H_1$ and using the fact that $\eta_{\mathcal{V}_1} H_1 = \mathcal{V}_1$ we conclude that $\mathcal{V}_1 \oplus H_1$ is $\overline{\nabla}$-parallel. In the same way it can be proven that $\mathcal{V}_2 \oplus H_2$ is $\overline{\nabla}$-parallel. Now, if $E_i = \mathcal{V}_i \oplus H_i$ for $i = 1, 2$ it follows from (i) that $\eta_{E_i} E_i \subseteq E_i$ and $\eta_{E_i} E_j = 0$ if $i \neq j$. This shows that E_1 and E_2 are in fact ∇-parallel and the proof is finished. ∎

We will now study properties of the tensor $r_2 : H \to H$ defined in the previous section.

Lemma 4.2 *Let (M^{2n}, g, J) be a special $\mathcal{AK}_2$-manifold. Then :*

$$(\tilde{\nabla}_X r_2) Y = 0$$

for all X, Y in H.

Proof :

We will make use of the following formula which has been proven in [17] :

4.3 $\qquad\qquad\qquad (\overline{\nabla}_X \overline{R})(V_1, V_2, V_3, V_4) = 0$

for all X in H and V_i in $\mathcal{V}, 1 \leq i \leq 4$. It follows easily that $(\tilde{\nabla}_X \overline{R})(V_1, V_2, V_3, V_4) = 0$. Now, we recall that lemma 3.1 states that

$$2\overline{R}(V_3, V_4, V_2, \eta_{V_1} Y) = \overline{R}(V_3, V_4, Y, \eta_{V_1} V_2)$$

whenever Y belongs to H and V_i in $\mathcal{V}, 1 \leq i \leq 4$. To conclude, it suffices to derive the last equation in the direction of X in H, with respect to the connection $\tilde{\nabla}$, and next take (4.3) into account. ∎

We will give now the proof of the theorem 1.1 stated in the introduction.

Proof of theorem 1.1

Let U the open dense set of M where we have an orthogonal splitting :

$$H = H_1 \oplus \ldots H_p$$

where $H_i, 1 \leq i \leq p$ are the eigenbundles of r_2 with corresponding eigenfunctions $\lambda_i, 1 \leq i \leq p$. Note that by lemma 4.2 we have in the standard way $X.\lambda_i = 0$ for all

X in H and $1 \leq i \leq p$ hence the distributions H_i are $\tilde{\nabla}$-parallel, inside H. We set $\mathcal{V}_i = \eta_{\mathcal{V}} H_i, 1 \leq i \leq p$ and use proposition 4.1 to obtain an orthogonal, J-invariant and ∇-parallel decomposition :

$$4.4 \qquad\qquad TM = \bigoplus_{i=1}^{p} (\mathcal{V}_i \oplus H_i).$$

Of course, each factor corresponds to a special $\mathcal{AK}_2$-manifold such the corresponding tensor r_2 has the Kähler nullity as eigenspace. Therefore it remains us to consider this situation.

Suppose that (M^{2n}, g, J) is special $\mathcal{AK}_2$-manifold, with decomposition $TM = \mathcal{V} \oplus H$ and such that $r_2 = \lambda \cdot 1_H$. Then using (3.1) and the fact that $\eta_{\mathcal{V}} H = \mathcal{V}$ we obtain that

$$\sum_{v_k \in \mathcal{V}} \overline{R}(v_k, Jv_k, V, W) = -\frac{\lambda}{2} < JV, W >$$

for all V, W in $\mathcal{V}$. If $dim_{\mathbb{R}} \mathcal{V} = 2$ the special condition implies that $dim_{\mathbb{R}} H = 2$ and we know by the work in [5] that the torsion has to be parallel. If the dimension is greater, using the second Bianchi identity for $\overline{\nabla}$ on $\mathcal{V}$, exactly in the way one shows that a manifold of dimension greater than 3, with Ricci tensor proportional to the metric tensor is Einstein, one obtains that $V.\lambda = 0$ for all V in $\mathcal{V}$. But using proposition 3.2 it follows immediately that the configuration tensor A vanishes, ensuring the $\overline{\nabla}$-parallelism of the decomposition $TM = \mathcal{V} \oplus H$. Moreover, by propositions 2.1 and 2.2, (ii) we obtain that the torsion is also $\overline{\nabla}$-parallel.

Now it clear that each factor of the decomposition (4.4) has parallel torsion and it follows that the torsion of M is parallel over U and by continuity over M, hence the first part of theorem 1.1 is proved. It remains us to show that (M^{2n}, g, J) is locally 3-symmetric. The vanishing of A implies by lemmas 3.1 and 3.3 that curvature terms of the form $\overline{R}(V_1, V_2, V_3, V_4), \overline{R}(V_1, V_2, X, Y)$ and $\overline{R}(V_1, X, V_2, Y)$ where $V_i, 1 \leq i \leq 4$ are in $\mathcal{V}$ and X, Y in H must be all equal to 0. Furthermore, the restriction of $\overline{R}$ to H is computed by lemma 4.1 and using the parallelism of the torsion it is an easy exercise to see that $\overline{\nabla R} = 0$, in other words $\overline{\nabla}$ is an Ambrose-Singer connection. We conclude now by [15]. ∎

It is a good place now to introduce, in view of future use, the following definition.

Definition 4.1 *A locally 3-symmetric space of type I is a special $\mathcal{AK}_2$ with parallel torsion.*

Remark 4.1 *From the proof of theorem 1.1 we get the explicit dependence on the torsion of the curvature tensor $\overline{R}$ of a space of type I. This can be used to get algebraic caracterizations as a homogeoneous space of such a manifold. Since this discussion is beyond the scope of the present paper it will be omitted.*

Proof of theorem 1.3 :
It is easy to see that an almost Kähler structure satisfying the conditions in (i) or (ii) has to satisfy the second Gray condition and therefore must have parallel torsion. We conclude by recalling (cf. [17]) that any Einstein manifold supporting a compatible almost Kähler structure with parallel torsion is Kähler.

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