

# On nearly Kähler geometry \*

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December 18, 2001

## Abstract

We consider complete nearly Kähler manifolds with the canonical Hermitian connection. We prove some metric properties of strict nearly Kähler manifolds and give a sufficient condition for the reducibility of the canonical Hermitian connection. A holonomic condition for a nearly Kähler manifold to be a twistor space over a quaternionic Kähler manifold is given. This enables us to give classification results in 10-dimensions.

## 1 Introduction

Nearly Kähler (briefly NK) geometry is related to the concept of weak holonomy, introduced by A. Gray [7] in 1971. He proved that among those groups acting transitively on the sphere there are only 3 groups, namely

$U(n)$  in dimension  $2n$ ,  $G_2$  in dimension 7,  $Spin(9)$  in dimension 16

that can occur as weak holonomy groups and produce other geometries than the classical holonomy approach. Nearly Kähler geometry corresponds to weak holonomy  $U(n)$  and was intensively studied in the seventies by Gray [8, 9]. Also note that the class of NK-manifolds appears naturally as one of the sixteen classes of almost Hermitian manifolds described by the Gray-Hervella classification [10].

Recent interest for the study of such manifolds can be justified by the fact that in dimension 6 nearly Kähler manifolds are related to the existence of a Killing spinor (see [11]). Furthermore, nearly Kähler manifolds provide a natural example of almost Hermitian manifolds admitting a Hermitian connection with totally skew symmetric torsion. From this point of view they are of interest in string theory (see [5]).

The aim of this paper is to investigate a number of properties of NK-manifolds related to the reducibility of the canonical Hermitian connection. We begin by proving a decomposition result which allows us to restrict our attention to strict NK-manifolds (see section 1). Our first main result is the following.

**Theorem 1.1** *Let  $(M^{2n}, g, J)$  a complete, strict nearly Kähler manifold. Then the following hold :*

(i) *If  $g$  is not an Einstein metric then the canonical Hermitian connection has reduced holonomy.*

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\*MSC 2000 : 53C12, 53C24, 53C28, 53C29

Keywords : nearly Kähler manifolds, twistor spaces

(ii) The metric  $g$  has positive Ricci curvature, hence  $M$  is compact with finite fundamental group.

(iii) The scalar curvature of the metric  $g$  is a strictly positive constant.

The previous theorem is a synthesis of the results contained in section 2.

Let us recall now that one main class of examples of NK manifolds is formed by the so called 3-symmetric spaces [8]. Other examples are provided by total spaces of Riemannian submersions with totally geodesic fibers admitting a compatible Kähler structure. These manifolds admit a canonical NK structure such that the canonical Hermitian connection has reduced holonomy (see section 3). In particular twistor spaces over positive quaternion-Kähler manifolds (here positive means of positive scalar curvature) have canonical NK-structures, a result already proven in [1]. See also [14] for the case of twistor bundles over 4-manifolds.

In the second part of this paper we are concerned with the the study of the most simple case of reducible NK-geometry which is the following :

**Theorem 1.2** *Let  $(M^{2n}, g, J)$  be a complete, strict nearly Kähler manifold. If the holonomy group of the canonical Hermitian connection is contained in  $U(1) \times U(n-1)$  then  $M$  is the twistor space of a positive quaternionic-Kähler manifold endowed with its canonical NK-structure.*

In 6-dimensions, the theorem 1.2 was already proven by a different method in [2]. Our approach consists in showing that the torsion of the canonical Hermitian connection has to be of special algebraic type with respect to the holonomy decomposition. This will be done in section 4. Then, using standard arguments one can show that  $M$  carries a complex contact structure and a Kähler-Einstein metric. The conclusion follows by a theorem of LeBrun (see section 5).

As a corollary of theorem 1.2 we obtain a structure result in 10-dimensions. Note that in 8-dimensions it was already known by Gray [9] that there are no strict NK-manifolds.

**Corollary 1.1** *Let  $(M^{10}, g, J)$  be a complete NK-manifold. Then either the universal cover of  $M$  is a Riemannian product of a Kähler surface with a six dimensional NK-manifold, either  $M$  is the twistor space of a positive, 8-dimensional quaternionic Kähler manifold equipped with its canonical NK structure.*

Using results from [15] (see also [12]) we know that the only positive quaternionic-Kähler manifolds of 8-dimensions are the symmetric spaces  $P\mathbb{H}^2, Gr_2(\mathbb{C}^4), G_2/SO(4)$  with their canonical metrics. Hence their twistor spaces, which are described in [12], equipped with the canonical NK structure exhaust the list of complete, strict NK-manifolds of dimension 10.

## 2 Nearly Kähler geometry

A nearly Kähler manifold is an almost hermitian manifold  $(M^{2n}, g, J)$  such that

$$(\nabla_X J)X = 0$$

for every vector field  $X$  on  $M$  (here  $\nabla$  denotes the Levi-Civita connection associated to the metric  $g$ ). A NK manifold is called *strict* if  $\nabla_X J \neq 0$  for every  $X \in TM, X \neq 0$ .

Recall that the tensor  $\nabla J$  has a number of important algebraic properties that can be summarized as follows : the tensors  $A$  and  $B$  defined for  $X, Y, Z$  in  $TM$  by  $A(X, Y, Z) = \langle (\nabla_X J)Y, Z \rangle$  and  $B(X, Y, Z) = \langle (\nabla_X J)Y, JZ \rangle$  are skew-symmetric and have type  $(0, 3) + (3, 0)$  as real 3-forms. Denote by  $Ric$  the Ricci tensor of the metric  $g$  and by  $Ric^*$  its star version, that is the operator defined by

$$\langle Ric^*(X), Y \rangle = \frac{1}{2} \sum_{i=1}^{2n} R(X, JY, e_i, J e_i)$$

where  $R$  is the curvature tensor of  $(M, g)$  and  $\{e_1, \dots, e_{2n}\}$  a local frame field. The difference of these tensors, to be denoted by  $r$ , is given by the formula (see [9]) :

$$\langle rX, Y \rangle = \sum_{i=1}^{2n} \langle (\nabla_{e_i} J)X, (\nabla_{e_i} J)Y \rangle.$$

Obviously  $r$  is symmetric, positive and commutes with  $J$ . Another object of particular importance is the canonical hermitian connection defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{1}{2}(\nabla_X J)JY.$$

It is easy to see that  $\bar{\nabla}$  is the unique Hermitian connection on  $M$  with totally skew-symmetric torsion (see for example [5]). Note that the torsion of  $\bar{\nabla}$  given by  $T(X, Y) = (\nabla_X J)JY$  vanishes iff  $(M, g, J)$  is a Kähler manifold.

The tensor  $r$  has strong geometric properties. To begin, we have :

$$\mathbf{2.1} \quad \bar{\nabla} r = 0.$$

In fact, A. Gray proved in [9] that for all  $X, Y, Z$  in  $TM$  we have

$$2 \langle (\nabla_X r)Y, Z \rangle = \langle r(\nabla_X J)Y, JZ \rangle + \langle r(JY), (\nabla_X J)Z \rangle$$

. But this is nothing else that (2.1)!

**Proposition 2.1** *Let  $(M^{2n}, g, J)$  be a complete, simply connected, NK-manifold. Then  $M$  is a riemannian product  $M_1 \times M_2$  where  $M_1$  is a Kähler manifold and  $M_2$  a strict NK-manifold.*

**Proof :**

Set  $E_1 = Ker(r)$  and let  $E_2$  be the orthogonal complement of  $E_1$  in  $TM$ . By (2.1) both  $E_1$  and  $E_2$  are  $\bar{\nabla}$ -parallel. Since  $\nabla_X J$  vanishes whenever  $X$  is in  $E_1$  the distribution  $E_1$  is in fact  $\nabla$ -parallel. Now, if  $X$  is in  $TM$  and  $Y$  in  $E_2$  we have  $(\nabla_X J)Y \in E_1^\perp = E_2$ , hence  $E_2$  is  $\nabla$ -parallel. It is now easy to conclude by a theorem of de Rham ■

**Remark 2.1** *Proposition 2.1 was already proven in [9] under the assumption that the tensor  $r$  is  $\nabla$ -parallel.*

Therefore, we can restrict our attention to the class of strict NK-manifolds.

**Proposition 2.2** *Let  $(M^{2n}, g, J)$  a strict NK-manifold.*

(i) *Suppose that  $r$  has more than one eigenvalue. Then the canonical Hermitian connection has reduced holonomy.*

(ii) *If the tensor  $r$  has exactly one eigenvalue then  $M$  is a positive Einstein manifold. Furthermore, the first Chern class of  $(M, J)$  vanishes.*

**Proof :**

(i) If  $\lambda_i > 0, i = \overline{1, p}$  are the eigenvalues of  $r$  we have a  $\overline{\nabla}$ -parallel decomposition

$$2.2 \quad TM = \bigoplus_{i=1}^p E_i$$

where  $E_i$  is the eigenbundle corresponding to the eigenvalue  $\lambda_i$ . Hence, each factor is preserved by the holonomy group, which is thus reducible.

(ii) The proof can be found in [9], page 242. Let us give it for the sake of completeness. We recall the following formula :

$$2.3 \quad \sum_{i,j=1}^{2n} \langle r e_i, e_j \rangle (R(X, e_i, Y, e_j) - 5R(X, e_i, JX, J e_j)) = 0$$

(see [9]) where  $\{e_i\}_{i=\overline{1, 2n}}$  is a local orthonormal frame field and  $X, Y$  are in  $TM$ . If  $r = \lambda 1_{TM}, \lambda > 0$  this formula becomes  $Ric - 5Ric^* = 0$  hence,  $Ric = \frac{5\lambda}{4}$  as  $Ric - Ric^* = r$ . The second assertion follows by the description of the first Chern class of  $(M, J)$  given in [9] ■

The first part of the theorem 1.1 follows now from the previous proposition. We will now compute the Ricci tensor of a NK-manifold and show that it is completely determined by the spectral decomposition of the tensor  $r$ . This computation will be equally used in section 5.

**Lemma 2.1** *We have, by respect to the decomposition (2.2) :*

(i)  *$Ric(X, Y) = 0$  if  $X$  and  $Y$  are vector fields belonging to  $E_i$  and  $E_j$  respectively, and  $i \neq j$ .*

(ii) *If  $X, Y$  are vector fields in  $E_i$  :*

$$Ric(X, Y) = \frac{\lambda_i}{4} \langle X, Y \rangle + \frac{1}{\lambda_i} \sum_{s=1}^p \lambda_s \langle r^s(X), Y \rangle$$

where the tensors  $r^s : TM \rightarrow TM, 1 \leq s \leq p$  are defined by  $\langle r^s(X), Y \rangle = -Tr_{E_s}(\nabla_X J)(\nabla_Y J)$  whenever  $X, Y$  are in  $TM$ .

**Proof :**

(i) Let us denote by  $\overline{R}$  the curvature tensor of the connexion  $\overline{\nabla}$ . We have (see [9], page 237) :

$$2.4 \quad \begin{aligned} \overline{R}(X, Y, Z, T) &= R(X, Y, Z, T) - \frac{1}{2} \langle (\nabla_X J)Y, (\nabla_Z J)T \rangle + \\ &+ \frac{1}{4} \left[ \langle (\nabla_X J)Z, (\nabla_Y J)T \rangle - \langle (\nabla_X J)T, (\nabla_Y J)Z \rangle \right]. \end{aligned}$$

Let  $\{e_k\}_{k=1,2n}$  on orthonormal base of  $TM$  which gives orthonormal bases in  $E_s$  for  $1 \leq s \leq p$ . We get :

$$Ric(X, Y) = \sum_{s=1}^p \sum_{e_k \in E_s} R(X, e_k, Y, e_k).$$

If  $s \neq j$  we have  $\overline{R}(X, e_k, Y, e_k) = 0$  hence  $R(X, e_k, Y, e_k) = \frac{1}{4} \langle (\nabla_{e_k} J)X, (\nabla_{e_k} J)Y \rangle$  by (2.4). If  $s = j$  then  $s \neq i$  and as before we get

$$R(X, e_k, Y, e_k) = R(Y, e_k, X, e_k) = \frac{1}{4} \langle (\nabla_{e_k} J)X, (\nabla_{e_k} J)Y \rangle.$$

It follows that  $Ric(X, Y) = \frac{1}{4} \langle rX, Y \rangle = 0$ .

(ii) Using (2.3) we obtain :

$$\sum_{s=1}^p \lambda_s \left( \sum_{e_k \in E_s} R(X, e_k, Y, e_k) - 5R(X, e_k, JY, Je_k) \right) = 0.$$

Reasoning as in the proof of (i), we get for  $s \neq i$  that

$$R(X, e_k, JY, Je_k) = -3R(X, e_k, Y, e_k) = -\frac{3}{4} \langle (\nabla_{e_k} J)X, (\nabla_{e_k} J)Y \rangle.$$

It follows that

$$4 \sum_{\substack{s=1 \\ s \neq i}} \lambda_s \langle r^s X, Y \rangle + \lambda_i \left( \sum_{e_k \in E_s} R(X, e_k, Y, e_k) - 5R(X, e_k, JY, Je_k) \right) = 0$$

and further  $4 \sum_{\substack{s=1 \\ s \neq i}} (\lambda_s - \lambda_i) \langle r^s X, Y \rangle + \lambda_i \langle (Ric - 5Ric^*)X, Y \rangle = 0$ . We conclude

by using that  $Ric - Ric^* = r$  and  $\sum_{s=1}^p r^s = r$  ■

Note that by definition the tensors  $r^s, 1 \leq s \leq p$  are positive. Setting  $\lambda = \min\{\lambda_i : 1 \leq i \leq p\}$  the proposition 2.1 obviously implies that  $Ric \geq \lambda g$ . This, together with Myer's theorem proves the second assertion of theorem 1.1.

Another result we will use in the next section is :

**Lemma 2.2** *The tensors  $r^s, 1 \leq s \leq p$  are  $\overline{\nabla}$ -parallel.*

The proof is analogous to that of the  $\overline{\nabla}$ -parallelism of  $r$  so it will be left to the reader. Thus, using the lemma 2.1 we obtain that :

**Corollary 2.1** *The Ricci tensor and the Ricci  $\star$  tensor of a compact NK-manifold are  $\overline{\nabla}$ -parallel.*

It follows that the scalar curvature and more, the  $\star$ -scalar curvature of  $(M, g, J)$ , are strictly positive constants. The proof of the theorem 1.1 is now finished.

### 3 Examples of NK manifolds

Let us consider a Riemannian submersion with totally geodesic fibers

$$F \hookrightarrow (M, g) \rightarrow N$$

and let  $TM = \mathcal{V} \oplus H$  be the corresponding splitting of  $TM$ . We will suppose that  $M$  admits a complex structure  $J$  compatible with  $g$  and preserving  $\mathcal{V}$  and  $H$  such that  $(M, g, J)$  is a Kähler manifold. Consider now the Riemannian metric on  $M$  defined by

$$\hat{g}(X, Y) = \frac{1}{2}g(X, Y) \text{ if } X, Y \in \mathcal{V}, \hat{g}(X, Y) = g(X, Y) \text{ for } X, Y \text{ in } H.$$

The metric  $\hat{g}$  admits a compatible almost complex structure  $\hat{J}$  given by  $\hat{J}|_{\mathcal{V}} = -J$  and  $\hat{J}|_H = J$ . This almost complex structure was introduced in [4] for the case of twistor spaces over 4-manifolds.

**Proposition 3.1** *The manifold  $(M, \hat{g}, \hat{J})$  is nearly Kähler. The distributions  $\mathcal{V}$  and  $H$  are parallel with respect to the canonical Hermitian connection of  $(M, \hat{g}, \hat{J})$  which thus has reduced holonomy.*

**Proof :** Let  $A : TM \times TM \rightarrow TM$  be the O'Neill tensor of the Riemannian submersion  $(M, g)$ . As  $g$  is Kähler we must have  $A_X J = J A_X$  for all  $X$  in  $TM$ . Using the relations between the Levi-Civita connections of  $\hat{g}$  and  $g$  given in [3] we obtain after a standard computation :

$$\begin{aligned} (\hat{\nabla}_X \hat{J})V &= -(\hat{\nabla}_V \hat{J})X = -A_X(JV) \\ (\hat{\nabla}_V \hat{J})W &= 0, \quad (\hat{\nabla}_X \hat{J})Y = 2A_X(JY) \end{aligned}$$

for every  $X, Y$  in  $\mathcal{V}$  and  $V, W$  in  $H$ . It is now straightforward to conclude ■

**Corollary 3.1** *The twistor space of a positive quaternionic-Kähler manifold of dimension  $4k$  admits a canonical NK structure with reducible holonomy, contained in  $U(1) \times U(2k)$ .*

**Proof :**

We have only to recall [16] that such a twistor space is the total space of a Riemannian submersion with totally geodesic fibers of dimension 2 and that it admits a compatible Kähler structure ■

### 4 Reducible NK manifolds

In this section we consider strict NK-manifolds  $(M^{2n}, g, J)$  such that the holonomy of the canonical Hermitian connection is contained in  $U(1) \times U(n-1)$ . This leads to a  $\bar{\nabla}$ -parallel decomposition of  $TM$ , orthogonal with respect to  $g$  and stable by  $J$

$$TM = L \oplus E$$

with  $L$  of rank two. Note that the torsion of  $\bar{\nabla}$  vanishes on  $L$  and  $T(L, E) \subseteq E$ .

**Lemma 4.1** *We have :*

(i)  $\overline{R}(X, Y, V, JV) = -2 \langle (\nabla_V J)^2 X, JY \rangle$  for every vector fields  $X, Y$  on  $E$  and  $V$  on  $L$ .

(ii)  $\overline{R}(X, V, V, JV) = 0$  if  $X$  belongs to  $E$  and  $V$  to  $L$ .

**Proof :**

(i) Using (2.4) we get

$$\overline{R}(X, Y, V, JV) = R(X, Y, V, JV) - \frac{1}{2} \langle (\nabla_V J)^2 X, JY \rangle .$$

Now the first Bianchi identity gives  $R(X, Y, V, JV) = -R(Y, V, X, JV) + R(X, V, Y, JV)$ . As  $E$  is  $\overline{\nabla}$ -parallel we must have  $\overline{R}(Y, V, X, JV) = 0$  so we find by (2.4) that  $R(Y, V, X, JV) = \frac{3}{4} \langle (\nabla_V J)^2 X, JY \rangle$ . In the same way we have  $R(X, V, Y, JV) = -\frac{3}{4} \langle (\nabla_V J)^2 X, JY \rangle$  and the result follows easily.

(ii) Using (2.4) twice we get

$$\overline{R}(X, V, V, JV) = R(X, V, V, JV) = R(V, JV, X, V) = \overline{R}(V, JV, X, V)$$

and we conclude by the fact that  $E$  is  $\overline{\nabla}$ -parallel ■

Let us denote by  $\Omega$  the curvature form of the line bundle  $L$ . Then we have

$$\overline{R}(X, Y)V = \Omega(X, Y)JV$$

for  $X, Y$  in  $TM$  and  $V$  in  $L$ . We denote by  $\omega^L$  the restriction of the Kähler form  $\omega$  to  $L$ . Let  $F$  be the endomorphism of  $TM$  defined by  $\langle FX, Y \rangle = -\frac{1}{2} \text{Tr}_L(\nabla_X J)(\nabla_Y J)$  whenever  $X, Y$  are in  $TM$ .

**Remark 4.1** *If  $V$  is a local vector field on  $L$  of norm 1 we have  $F = -(\nabla_V J)^2$ . Hence  $F$  is symmetric and positive, with  $[F, J] = 0$ . By lemma 2.2  $F$  is  $\overline{\nabla}$ -parallel and it follows easily that  $\nabla_V F = 0$  for every vector field  $V$  in  $L$ .*

If  $q^E$  is the 2-form on  $E$  defined by  $q^E(X, Y) = \langle FX, JY \rangle$  for  $X, Y$  in  $E$  we obtain by lemma 4.1 that :

$$\Omega = f\omega^L + 2q^E$$

where  $f$  is a smooth function on  $M$ .

**Lemma 4.2** *We have :*

(i)  $d\omega^L(X, V, JV) = dq^E(X, V, JV) = 0$  if  $V$  is in  $L$  and  $X$  in  $E$ .

(ii)

$$\begin{aligned} d\omega^L(V, X, Y) &= - \langle (\nabla_V J)X, Y \rangle \\ dq^E(V, X, Y) &= -2 \langle F(\nabla_V J)X, Y \rangle \end{aligned}$$

where  $V, X, Y$  are vector fields belonging to  $L$  resp.  $E$ .

**Proof :**

The proof of (i) is straightforward. We leave it to the reader and concentrate on (ii). We have

$$d\omega^L(V, X, Y) = \nabla_V \omega^L(X, Y) - \nabla_X \omega^L(V, Y) + \nabla_Y \omega^L(V, X).$$

The fact that  $\omega^L$  vanishes as soon as we take a direction in  $E$  gives us that  $\nabla_V \omega^L(X, Y) = 0$ ,  $\nabla_X \omega^L(V, Y) = -\omega^L(V, \nabla_X Y)$  and  $\nabla_Y \omega^L(V, X) = -\omega^L(V, \nabla_Y X)$ . The claimed formula for  $d\omega^L(V, X, Y)$  follows using the fact that  $\overline{\nabla}_X Y$  and  $\overline{\nabla}_Y X$  belong to  $E$ . Next, we have

$$dq^E(V, X, Y) = (\nabla_V q^E)(X, Y) - (\nabla_X q^E)(V, Y) + (\nabla_Y q^E)(V, X).$$

The vanishing of  $q^E$  on  $L \times E$  implies that

$$(\nabla_V q^E)(X, Y) = \langle (\nabla_V F)X, JY \rangle + \langle FX, (\nabla_V J)Y \rangle = \langle FX, (\nabla_V J)Y \rangle$$

(see the remark 4.1) and

$$(\nabla_X q^E)(V, Y) = \frac{1}{2} \langle F(\nabla_V J)X, Y \rangle, \quad (\nabla_Y q^E)(V, X) = \frac{1}{2} \langle F(\nabla_V J)Y, X \rangle.$$

We conclude by using the fact that  $F$  commutes with  $\nabla_V J$  ■

Let  $\omega^E$  be the restriction of the form  $\omega$  to  $E$ . We can now have a complete description of the curvature form of our line bundle  $L$  as follows.

**Proposition 4.1** (i) *There exists a constant  $k > 0$  such that  $F|_E = \frac{k}{4}1_E$ . Moreover, the curvature form of the line bundle  $L$  is*

$$\frac{k}{2}(-2\omega^L + \omega^E).$$

(ii) *We have that  $(\nabla_X J)Y$  belongs to  $L$  whenever  $X, Y$  are in  $E$ .*

**Proof :**

(i) As  $\Omega$  is closed we get  $f d\omega^L + df \wedge \omega^L = -2dq^E$ . If  $X$  resp.  $V$  are vector fields in  $E$  resp.  $L$  it follows by lemma 4.2, (i) that  $X.f = 0$ , hence  $df|_E = 0$ . This implies that  $[X, Y].f = 0$  whenever  $X, Y$  are vector fields in  $E$  and further that  $(\nabla_X J)Y.f = 0$  (here we used that  $E$  is  $\overline{\nabla}$ -parallel and  $\overline{\nabla}_X Y - \overline{\nabla}_Y X = [X, Y] + (\nabla_X J)JY$ ). But the map  $u : E \times E \rightarrow L$  defined at  $(v, w) \in E \times E$  as the orthogonal projection of  $(\nabla_v J)w$  on  $L$  is surjective by the injectivity of  $F|_{E_m}$ . Hence  $df$  vanishes on  $L$  and thus  $df = 0$ , that is  $f$  is constant, equal to  $c$ .

Let now  $X, Y$  resp.  $V$  be vector fields in  $E$  resp.  $L$ . As  $d\Omega(V, X, Y) = 0$  we get by lemma 4.2, (ii)

$$-c \langle (\nabla_V J)X, Y \rangle - 4 \langle F(\nabla_V J)X, Y \rangle = 0.$$

We deduce that  $(\nabla_V J)(4F + c) = 0$  and further  $F(4F + c) = 0$  on  $E$ . As the restriction of  $F$  to  $E$  is injective it follows that  $F = \frac{-c}{4}id$  on  $E$ . We set  $k = -c$ .

(ii) Let  $X, Y, Z$  be vector fields on  $E$ . As we obviously have  $d\omega^L(X, Y, Z) = 0$  it follows by (i) that  $d\omega^E(X, Y, Z) = 0$ . A straightforward computation gives  $(\nabla_X \omega^E)(Y, Z) = -\langle (\nabla_X J)Y, Z \rangle$  from which we deduce that  $d\omega^E(X, Y, Z) = -\langle (\nabla_X J)Y, Z \rangle$  ■

**Corollary 4.1** (i) *The tensor  $r$  has exactly two eigenvalues :  $\frac{k(n-1)}{2}$  resp.  $k$  with eigenbundles  $L$  resp.  $E$ .*

(ii) *The Ricci tensor of  $(M, g)$  has exactly two eigenvalues :  $\frac{k(n+7)}{8}$  and  $\frac{k(n+2)}{4}$  with eigenbundles  $L$  resp.  $E$ .*

**Proof :**

- (i) The fact that  $r|_L = \frac{k(n-1)}{2}$  follows easily by the fact that  $F$  is constant on  $E$ . If  $x$  is in  $E$  let  $v$  in  $L$  be unitary, and  $\{e_i\}_{1 \leq i \leq 2(n-1)}$  an orthogonal basis of  $E$ . Then we have  $\langle rx, x \rangle = 2\|(\nabla_v J)x\|^2 + \sum_{i=1}^{2(n-1)} \|(\nabla_{e_i} J)x\|^2$ . As  $(\nabla_{e_i} J)x$  belongs to  $L$ , the last sum equals  $2\|(\nabla_v J)x\|^2$  and we use  $F|_E = \frac{k}{4}$ .
- (ii) Follows from lemma 2.1 and (i) ■

## 5 The twistor structure

Let us define a new Riemannian metric on  $M$ , called  $\bar{g}$ , as follows :

$$\bar{g}(X, Y) = g(X, Y) \text{ if } X, Y \in E, \quad \bar{g}(X, Y) = 2g(X, Y) \text{ for } X, Y \text{ in } L.$$

The *reversing* almost complex structure defined by  $\bar{J}|_L = -J$  and  $\bar{J}|_E = J$  is in fact integrable, the proof being identical to that given in six dimensions in [2]. The Kähler form of  $(M, \bar{g}, \bar{J})$  is exactly  $-2\omega^L + \omega^E$  and hence it is closed by proposition 4.1, (i). Thus,  $(M, \bar{g}, \bar{J})$  is a Kähler manifold.

**Lemma 5.1**  $(M, \bar{g})$  is an Einstein manifold, with Einstein constant  $\frac{n+1}{4}k$ .

**Proof :**

This is a computation very similar to that of [3], page 232, where the Ricci tensor of the canonical variation of a Riemannian submersion is computed. Let  $\tilde{\nabla}$  be the Levi-Civita connection of the metric  $\bar{g}$ . If  $V$  resp.  $X, Y$  are vector fields in  $L$  resp.  $E$  we have :  $\tilde{\nabla}_V X = \bar{\nabla}_V X$ ,  $\tilde{\nabla}_X V = \bar{\nabla}_X V - (\nabla_X J)JV$  and  $\tilde{\nabla}_X Y = \nabla_X Y$ . Moreover,  $\tilde{\nabla}_V W = \nabla_V W$  whenever  $V, W$  are in  $L$ . This follows from the definition of the Levi-Civita connection and by the fact that the  $\bar{\nabla}$ -parallelism of  $L$  and  $E$  allows us to identify the projections on  $L$  resp.  $E$  of brackets of the type  $[V, X]$  and  $[X, Y]$ .

Let  $\tilde{R}$  be the curvature tensor of  $\tilde{\nabla}$ . Using the above formulas we get, after a standard computation :

$$\begin{aligned} \tilde{R}(V, X, V, X) &= \langle FX, X \rangle \|V\|^2 = \frac{k}{4}\|V\|^2\|X\|^2 \\ \tilde{R}(X, Y, X, Y) &= \bar{R}(X, Y, X, Y) - \frac{1}{2}\|(\nabla_X J)Y\|^2 = R(X, Y, X, Y) - \frac{3}{4}\|(\nabla_X J)Y\|^2 \end{aligned}$$

by (2.4). The result follows now by corollary 4.1 ■

Thus,  $(M, \bar{g}, \bar{J})$  is a Kähler-Einstein manifold, which is also Fano. Moreover, the distribution  $E$  defines a complex contact structure on the complex manifold  $(M, \bar{J})$  as it is  $\bar{J}$ -holomorphic and the map  $(X, Y) \in E \times E \rightarrow (\nabla_X J)Y$  which gives the Frobenius obstruction is everywhere non-degenerate. By a result of LeBrun (see [13])  $(M, \bar{g})$  is the twistor space of a positive quaternionic-Kähler manifold. Moreover, from the construction of the metric  $\bar{g}$  we deduce that  $(M, g)$  is the twistor space of a positive quaternionic-Kähler manifold endowed with its canonical NK structure. This proves theorem 1.2.

**Remark 5.1** *If  $M$  is of dimension 6, it has constant type and proposition 4.1 is automatically satisfied. Corollary 4.1 follows by the fact that every 6-dimensional NK manifold is Einstein [9]. Thus all we need to prove the theorem 1.1 in this case is lemma 5.1.*

Let us prove now the corollary 1.1. It is well known (see [9]) that in 10-dimensions the eigenvalues of  $r$  are  $4(\alpha^2 + \beta^2)$  with multiplicity 2,  $4\alpha^2$  and  $4\beta^2$  each of multiplicity 4, where  $\alpha \geq \beta \geq 0$ . If  $\beta = 0$  then it follows by [9] that the universal cover of  $M$  is a Riemannian product as stated. If  $\beta > 0$  then  $M$  is strict and we apply theorem 1.1.

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