# On the first $L^p$-cohomology of discrete groups

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**Abstract**

We exploit the isomorphism between the first $\ell^p$-cohomology $H^1_{(p)}(\Gamma)$ and the reduced 1-cohomology with coefficients in $\ell^p(\Gamma)$, to obtain vanishing results for $H^1_{(p)}(\Gamma)$: we treat e.g. groups acting on trees, groups with infinite center, wreath products, and lattices in product groups.

## 1 Introduction

$L^p$-cohomology for countable groups, in its simplicial version, was introduced by Gromov in Chapter 8 of [Gro93], as a useful invariant of countable groups.

Let $X$ be a simplicial complex which is a classifying space for $\Gamma$, and let $\tilde{X}$ be its universal cover. Denote by $\ell^p C^k$ the space of $p$-summable complex $k$-cochains on $\tilde{X}$, i.e. the $\ell^p$-functions on the set $C^k$ of $k$-simplices of $\tilde{X}$. The *$L^p$-cohomology of* $\Gamma$ is the reduced cohomology of the complex

$$d_k : \ell^p C^k \to \ell^p C^{k+1},$$

where $d_k$ is the simplicial coboundary operator; we denote it by

$$\overline{H}^k_{(p)}(\Gamma) = Ker\, d_k / \overline{Im\, d_{k-1}}.$$

As explained at the beginning of section 8 of [Gro93], this definition only depends on $\Gamma$. Of course, $L^2$-cohomology had been considered much earlier,

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the use of the von algebra of $\Gamma$ allowing to define, for $k \geq 0$, the $k$-th $L^2$-Betti number, i.e. the von Neumann dimension of $\overline{H}^k_{(2)}(\Gamma)$ (see [CG86]).

This paper is mainly devoted to vanishing results for the first $L^p$-cohomology of a finitely generated group. Our motivation for that is twofold. First, vanishing of the first $L^2$-Betti number has impact in geometric group theory and topology (see Eckmann's paper [Eck97]). Second, it was shown in [BMV05] that, whenever a non-amenable group $\Gamma$ acts properly isometrically on a proper $CAT(-1)$ space $X$, then for $p$ larger than the critical exponent $e(\Gamma)$ in $X$, the first $L^p$-cohomology of $\Gamma$ is NOT zero. If $e(\Gamma)$ is finite (which is the case when $Isom(X)$ acts co-compactly on $X$, by a result of Burger-Mozes [BM96]), we get a non-existence result for isometric actions: a non-amenable group $\Gamma$ with $\overline{H}^1_{(p)}(\Gamma) = 0$ for $p > 1$, cannot act properly isometrically on a $CAT(-1)$ space $X$ for which $Isom(X)$ is co-cocompact.

The following theorem is our main result (it subsumes Theorems 4.1, 4.2, 4.3, 4.6, 4.7, 4.8).

**Theorem** *Fix* $p \in ]1, +\infty[$.

*i) Let $\Gamma$ be a finitely generated group acting (without inversion) on a tree with non-amenable vertex stabilizers, and infinite edge stabilizers. If all vertex stabilizers have vanishing first $L^p$-cohomology, then so does $\Gamma$.*

*ii) Let $N$ be a normal, infinite, finitely generated subgroup of a finitely generated group $\Gamma$. Assume that $N$ is non-amenable, and that its centralizer $Z_\Gamma(N)$ is infinite. Then $\overline{H}^1_{(p)}(\Gamma) = 0$.*

*iii) Let $\Gamma$ be a finitely generated group. If the centre of $\Gamma$ is infinite, then $\overline{H}^1_{(p)}(\Gamma) = 0$.*

*iv) Let $H, \Gamma$ be (non trivial) finitely generated groups, and let $H \wr \Gamma$ be their wreath product. If $H$ is non-amenable, then $\overline{H}^1_{(p)}(H \wr \Gamma) = 0$.*

*v) Let $G = G_1 \times \ldots \times G_n$ be a direct product of non-compact, second countable locally compact groups ($n \geq 2$). Let $\Gamma$ be a finitely generated, cocompact lattice in $G$. If $\Gamma$ is non-amenable (equivalently, if some $G_i$ is non-amenable), then $\overline{H}^1_{(p)}(\Gamma) = 0$.*

*vi) Fix $n \geq 2$. For $i = 1, \ldots, n$, let $G_i$ be the group of $k_i$-rational points of some $k_i$-simple, $k_i$-isotropic linear algebraic group, for some local field $k_i$. Let $\Gamma$ be an irreducible lattice in $G_1 \times \ldots \times G_n$. Then $\overline{H}^1_{(p)}(\Gamma) = 0$.*

*Moreover, for $p = 2$, the results in (ii), (iv), (v) above hold without the non-amenability assumption.*[1]

Part (iii) of this Theorem extends a result of Gromov (Corollary on p. 221 of [Gro93]): if the center of $\Gamma$ contains an element of infinite order, then $\overline{H}^1_{(p)}(\Gamma) = 0$.

Part (vi) is a modest contribution to a conjecture of Gromov (question (?) on p. 253 in [Gro93]): if $\Gamma$ is a co-compact lattice of isometries of a Riemannian symmetric space (of non-compact type) or a Euclidean building $X$, then one should have $\overline{H}^k_{(p)}(\Gamma) = 0$ for $k < rank(X)$.

We now describe our approach to $\overline{H}^1_{(p)}$, which is to appeal to an identification between the first $L^p$-cohomology and the (reduced) first group cohomology with coefficients in $\ell^p(\Gamma)$. The relevant cohomological background is presented in section 2. Denote by $\lambda_\Gamma$ the left regular representation of $\Gamma$ on functions on $\Gamma$. For $1 \leq p < \infty$, denote by $\mathbf{D}_p(\Gamma)$ the space of functions $f$ on $\Gamma$ such that $\lambda_\Gamma(g)f - f \in \ell^p(\Gamma)$ for every $g \in \Gamma$: this is the space of *p-Dirichlet finite* functions on $\Gamma$. If $\Gamma$ is finitely generated, and $S$ is a finite, symmetric generating subset of $\Gamma$, we say (following [Pul]) that a function $f$ on $\Gamma$ is *p-harmonic* if

$$\sum_{s \in S} |f(s^{-1}x) - f(x)|^{p-2}(f(s^{-1}x) - f(x)) \;=\; 0$$

for every $x \in \Gamma$. We denote by $\mathbf{HD}_p(\Gamma)$ the set (not a linear space, if $p \neq 2$) of harmonic, $p$-Dirichlet finite functions on $\Gamma$. It was observed by by B. Bekka and the second author [BV97] for $p = 2$, and by M. Puls [Pul] in general, that for $\Gamma$ an infinite, finitely generated group, the following are equivalent:

i) The first $L^p$-cohomology $\overline{H}^1_{(p)}(\Gamma)$ is zero;

ii) $\mathbf{HD}_p(\Gamma) = \mathbb{C}$;

iii) $\ell^p(\Gamma)$ is dense in $\mathbf{D}_p(\Gamma)/\mathbb{C}$;

iv) $\overline{H^1}(\Gamma, \ell^p(\Gamma)) = 0$, where $\overline{H^1}(\Gamma, \ell^p(\Gamma))$ denotes the reduced 1-cohomology of $\Gamma$ with coefficients in the $\Gamma$-module $\ell^p(\Gamma)$.

[1] W. Lück informed us that, in the case $p = 2$, it is possible to prove part (i) of the Theorem without the non-amenability assumption, using his algebraic version of $L^2$-Betti numbers (see [Lue02]). The case of amalgamated products is treated in [Lue02], Theorem 7.2.(4).

In section 3 we add a fifth characterization to this list, giving much flexibility:

**Corollary 3.2** *For an infinite, finitely generated group* $\Gamma$*, the above properties are still equivalent to:* $\overline{H^1}(\Gamma, \ell^p(H)|_\Gamma) = 0$ *for every group* $H$ *containing* $\Gamma$ *as a subgroup.*

Section 4 contains our vanishing results for $\overline{H}^1_{(p)}$, while section 5 has a somewhat different flavor: using the Cheeger-Gromov vanishing result for $L^2$-cohomology of amenable groups [CG86], we obtain a new characterization of amenability for finitely generated groups:

**Proposition 5.3:** *Let* $\Gamma$ *be an infinite, finitely generated group. The following are equivalent:*

*i)* $\Gamma$ *is amenable;*

*ii)* $\ell^2(\Gamma)$ *is a dense, proper subspace of* $\mathbf{D}_2(\Gamma)/\mathbb{C}$.

After hearing of a preliminary version of our results, Damien Gaboriau contributed a quite interesting complement, and we are grateful to him for allowing us to include this material in our paper. Thus the Appendix contains Gaboriau's proof of the fact that, if the first $L^2$-Betti number of $\Gamma$ is zero, then $\overline{H^1}(\Gamma, \ell^2(\Gamma)) = 0$, *without* assuming the group $\Gamma$ to be finitely generated (compare with [BV97], or Corollary 3.2). We conjecture that the converse should also hold in full generality.

This paper can be viewed a sequel to [BMV05], although it can be read independently.

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# 2 1-cohomology vs. reduced 1-cohomology

## 2.1 1-cohomology

Let $G$ be a topological group and let $V$ be a topological $G$-module, i.e. a real or complex topological vector space endowed with a continuous linear representation $\pi : G \times V \to V$; $(g, v) \mapsto \pi(g)v$. If $H$ is a closed subgroup we denote by $V|_H$ the space $V$ viewed as an $H$-module for the restricted action, and by $V^H$ the set of $H$-fixed points:

$$V^H = \{v \in V \mid \pi(h)v = v, \forall h \in H\}.$$

We say that $V$ is a *Banach $G$-module* if $V$ is a Banach space and $\pi$ is a representation of $G$ by isometries of $V$. A $G$-module is *unitary* if $V$ is a Hilbert space and $\pi$ a unitary representation.

We now introduce the space of 1-cocycles and 1-coboundaries on $G$, and the 1-cohomology with coefficients in $V$:

- $Z^1(G,V) = \{b : B \to V \text{ continuous} \mid b(gh) = b(g) + \pi(g)b(h), \forall g,h \in G\}$
- $B^1(G,V) = \{b \in Z^1(G,V) \mid \exists v \in V : b(g) = \pi(g)v - v, \forall g \in G\}$
- $H^1(G,V) = Z^1(G,V)/B^1(G,V)$

If $N$ is a closed normal subgroup of $G$, and $V$ is a $G$-module, there is a well-known action of $G$ on $H^1(N,V|_N)$. On $Z^1(N,V|_N)$, this action is given by:

$$(g.b)(n) = \pi(g)(b(g^{-1}ng)) \tag{1}$$

($b \in Z^1(N,V|_N)$, $g \in G$, $n \in N$). Clearly this action leaves $B^1(N,V|_N)$ invariant, so it defines an action of $G$ on $H^1(N,V|_N)$. We have for $m \in N$:

$$(m.b)(n) = b(n) + (\pi(n)b(m) - b(m)) \tag{2}$$

showing that the $N$-action on $H^1(N,V|_N)$ is trivial, hence the action of $G$ on $H^1(N,V|_N)$ factors through $G/N$. The following result is well-known (see e.g. Corollary 6.4 in [Bro82]) and usually proved using the Hochschild-Serre spectral sequence in group cohomology [2].

**Proposition 2.1** *1) There is an exact sequence*

$$0 \to H^1(G/N, V^N) \stackrel{i_*}{\to} H^1(G,V) \stackrel{Rest_G^N}{\to} H^1(N,V|_N)^{G/N} \to \ldots \tag{3}$$

*where $i : V^N \to V$ denotes the inclusion;*

[2] For a proof without spectral sequences, see 8.1 in Chapter 1 of [Gui80].

*2) If $V^N = 0$, then the restriction map*

$$Rest_G^N : H^1(G,V) \to H^1(N, V|_N)^{G/N}$$

*is an isomorphism.* $\square$

For a set $X$, we denote by $\mathcal{F}(X)$ the set of all functions $X \to \mathbb{C}$. If a group $\Gamma$ acts on $X$, we endow $\mathcal{F}(X)$ with the $\Gamma$-module structure given by $(\gamma f)(x) = f(\gamma^{-1}x)$ (for $f \in \mathcal{F}(X), \forall x \in X$). The following lemma is well known. We give the quick proof for completeness:

**Lemma 2.2** *Let $\Gamma$ be a (discrete) group and let $X$ be a set on which $\Gamma$ acts freely. Then $H^1(\Gamma, \mathcal{F}(X)) = 0$.*

**Proof:** Let $(s_i)_{i\in I}$ be a set of representatives for $\Gamma$-orbits in $X$. For $x \in X$, there exists a unique $i \in I$ and $\gamma \in \Gamma$ such that $x = \gamma s_i$. For $b \in Z^1(\Gamma, \mathcal{F}(X))$, define then $f(x) = (b(\gamma^{-1}))(s_i)$. It is readily verified that $\gamma f - f = b(\gamma)$ for every $\gamma \in \Gamma$. $\square$

## 2.2 Reduced 1-cohomology

Since $G$ is a topological group and $V$ is a topological $G$-module, we may endow $Z^1(G,V)$ with the topology of uniform convergence on compact subsets of $G$. We denote by $\overline{B^1(G,V)}$ the closure of $B^1(G,V)$ for this topology, and by

$$\overline{H^1}(G,V) = Z^1(G,V)/\overline{B^1(G,V)}$$

the quotient space, called the *reduced first cohomology* of $G$ with coefficients in $V$. We will use the abuse of notation $H^1(G,V) = \overline{H^1}(G,V)$ to mean "the canonical epimorphism $H^1(G,V) \to \overline{H^1}(G,V)$ is an isomorphism". We recall without proof the following result of Guichardet (Théorème 1 in [Gui72]):

**Proposition 2.3** *Let $G$ be a locally compact, second countable group and let $V$ be a Banach module such that $V^G = 0$. The following are equivalent:*

*i) $H^1(G,V) = \overline{H^1}(G,V)$;*

*ii) $V$ does not have almost invariant vectors ( this means that there exists a compact subset $K$ of $G$ and $\varepsilon > 0$ such that $\sup_K \|\pi(g)v - v\| \geq \varepsilon \|v\|$, for every $v \in V$).* $\square$

Let $\lambda_G$ denote the left regular representation of $G$ on $L^p(G)$ $(1 \leq p < \infty)$. Since $\lambda_G$ has almost invariant vectors if and only if $G$ is amenable (see [Eym72]), we immediately deduce (see Corollaire 1 in [Gui72]):

**Corollary 2.4** *Fix $1 \leq p < \infty$. Let $G$ be a locally compact, non compact, second countable group. The following are equivalent:*

*i)* $H^1(G, L^p(G)) = \overline{H^1}(G, L^p(G))$;

*ii)* *$G$ is not amenable.* □

Reduced 1-cohomology behaves well with respect to inductive limits:

**Lemma 2.5** *Let $G$ be a locally compact group which is the union of a directed system of open subgroups $(G_i)_{i\in I}$. Let $(V, \pi)$ be a Banach $G$-module, with $b \in Z^1(G, V)$. If $b|_{G_i} \in \overline{B^1}(G_i, V|_{G_i})$ for all $i \in I$, then $b \in \overline{B^1}(G, V)$. In particular, if $\overline{H^1}(G_i, V|_{G_i}) = 0$ for all $i \in I$, then $\overline{H^1}(G, V) = 0$.*

**Proof:** Let $K$ be a compact subset of $G$, and $\varepsilon > 0$. By compactness $K$ is covered by a finite union $G_{i_1} \cup ... \cup G_{i_n}$; with $i \geq i_1, ..., i_n$, we get $K \subset G_i$. Since $b|_{G_i} \in \overline{B^1}(G_i, V|_{G_i})$, we find a vector $v \in V$ such that $\sup_K \|b(g) - (\pi(g)v - v)\| < \varepsilon$, i.e. $b \in \overline{B^1(G, V)}$. □

Next result will be used to characterize vanishing of the first $L^p$-characterization in Corollary 3.2.

**Proposition 2.6** *Fix $1 \leq p < \infty$. Let $H$ be a subgroup of the countable, discrete group $\Gamma$. Consider the following properties:*

*i)* $\overline{H^1}(H, \ell^p(H)) = 0$;

*ii)* $\overline{H^1}(H, \ell^p(\Gamma)|_H) = 0$;

*i')* $H^1(H, \ell^p(H)) = 0$;

*ii')* $H^1(H, \ell^p(\Gamma)|_H) = 0$.

*Then $i) \Leftrightarrow ii)$ and $i') \Leftrightarrow ii')$*

**Proof:** Choosing representatives $(s_n)_{n\geq 1}$ for the right cosets of $H$ in $\Gamma$, we may identify $\ell^p(\Gamma)|_H$, in an $H$-equivariant way, with the $\ell^p$-direct sum of $[\Gamma : H]$ copies of $\ell^p(H)$.

$ii) \Rightarrow i)$ and $ii') \Rightarrow i')$: The continuous map $Z^1(H, \ell^p(H)) \to Z^1(H, \ell^p(\Gamma)|_H)$, $b \mapsto (b, 0, 0, ...)$ induces inclusions $H^1(H, \ell^p(H)) \to H^1(H, \ell^p(\Gamma)|_H)$ and $\overline{H^1}(H, \ell^p(H)) \to \overline{H^1}(H, \ell^p(\Gamma)|_H)$.

$i) \Rightarrow ii)$: The result is obvious for $[\Gamma : H] < \infty$, so we assume $[\Gamma : H] = \infty$. For $b \in Z^1(H, \ell^p(\Gamma)|_H)$, let $b_n \in Z^1(H, \ell^p(H))$ be its projection on the $n$-th factor $\ell^p(Hs_n)$. So, for $h \in H$, one has $b(h) = \oplus b_n(h)$. Fix $K$ a finite subset of $H$, and $\varepsilon > 0$. Let $N > 0$ be such that $\sum_{n>N} \|b_n(h)\|^p < \frac{\varepsilon}{2}$ for every $h \in K$. For $i = 1, \ldots, N$, using the assumption we find a function $v_i \in \ell^p(H)$ such that $\|b_i(h) - (\lambda_H(h)v_i - v_i)\|^p < \frac{\varepsilon}{2N}$ for every $h \in K$. Set $v_n = 0$ for $n > N$, and define $v = \oplus v_n \in \ell^p(\Gamma)$. Then by construction $\|b(h) - [\lambda_\Gamma(h)(h)v - v]\|^p < \varepsilon$ for every $h \in K$, i.e. $b$ is a limit of 1-coboundaries.

$i') \Rightarrow ii')$: We consider two cases:

a) If $H$ is finite then $H^1(H, \ell^p(H)) = H^1(H, \ell^p(\Gamma)|_H) = 0$.

b) If $H$ is infinite then the assumption $H^1(H, \ell^p(H)) = 0$ implies, by Corollary 2.4, that $H$ is not amenable. By lemma 2 in [BMV05], this implies that $\ell^p(\Gamma)|_H$ does not almost have invariant vectors. By Proposition 2.3, we have $H^1(H, \ell^p(\Gamma)|_H) = \overline{H}^1(H, \ell^p(\Gamma)|_H)$, so that the result follows from the implication $i) \Rightarrow ii)$. ☐

**Remark:** Let $G$ be a locally compact second countable group and let $V$ be a Banach $G$-module with $V^G = 0$. Fix $p \in ]1, +\infty[$, and denote by $\infty_p V$ the $\ell^p$-direct sum of countably many copies of $V$. Consider the following properties:

i) $\overline{H^1}(G, V) = 0$;

ii) $\overline{H^1}(G, \infty_p V) = 0$;

i') $H^1(G, V) = 0$;

ii') $H^1(G, \infty_p V) = 0$.

Then the same proof as in Proposition 2.6 shows that $i) \Leftrightarrow ii)$ and $ii') \Rightarrow i')$. However, the implication $i') \Rightarrow ii')$ is not clear in general (as lemma 2 in [BMV05] is very special to $\ell^p$-spaces). A proof of that implication, using a different approach and assuming that $V$ is a uniformly convex Banach space, has been communicated to us by N. Monod.

# 3 First $L^p$-cohomology

## 3.1 $p$-Dirichlet finite functions

Let $\Gamma$ be a finitely generated group; fix a finite generating set $S$. Let $\Gamma$ act on a set $X$. Denote by $\lambda_X$ the permutation representation of $\Gamma$ on $\mathcal{F}(X)$. Fix $p \in [1, \infty[$.

The space of *$p$-Dirichlet finite functions* on $X$ (relative to the $\Gamma$-action) is

$$\begin{aligned} \mathbf{D}_p(X) &= \{f \in \mathcal{F}(X) \mid \|\lambda_X(g)f - f\|_p < \infty \, \forall g \in \Gamma\} \\ &= \{f \in \mathcal{F}(X) \mid \|\lambda_X(s)f - f\|_p < \infty \, \forall s \in S\}. \end{aligned}$$

Then $\mathbf{D}_p(X)^\Gamma$ is the space of functions on $X$ which are constant on $\Gamma$-orbits of $X$ (it does not depend on $p$). Define a semi-norm on $\mathbf{D}_p(X)$ by

$$\|f\|_{\mathbf{D}_p(X)} = \left[\sum_{s \in S} \|\lambda_X(s)f - f\|_p^p\right]^{\frac{1}{p}}.$$

The kernel of this semi-norm is precisely $\mathbf{D}_p(X)^\Gamma$, and the quotient $\mathcal{D}_p(X) = \mathbf{D}_p(X)/\mathbf{D}_p(X)^\Gamma$ is a Banach space (the norm on $\mathcal{D}_p(X)$ depends on the choice of $S$, but the underlying topology does not).

Define a linear map $\widetilde{\alpha} : \mathbf{D}_p(X) \to Z^1(\Gamma, \ell^p(X))$ by $\widetilde{\alpha}(f)(\gamma) = \lambda_X(\gamma)f - f$. The kernel of this map is $\mathbf{D}_p(X)^\Gamma$, so $\widetilde{\alpha}$ descends to a continuous injection $\alpha : \mathcal{D}_p(X) \to Z^1(\Gamma, \ell^p(X))$.

Let $\widetilde{i} : \ell^p(X) \to \mathbf{D}_p(X)$ be the canonical inclusion. Clearly $\ell^p(X)^\Gamma$ is the space of $\ell^p$-functions which are constant on $\Gamma$-orbits, and zero on infinite orbits. Set $l^p_\Gamma(X) = \ell^p(X)/\ell^p(X)^\Gamma$ (so that $l^p_\Gamma(X) = \ell^p(X)$ if all orbits are infinite). The map $\widetilde{i}$ induces a continuous inclusion $i : l^p_\Gamma(X) \to \mathcal{D}_p(X)$. Note that the image of $\alpha \circ i$ is exactly the space $B^1(\Gamma, \ell^p(X))$ of 1-coboundaries. This shows that:

- if $i$ is not onto, then $H^1(\Gamma, \ell^p(X)) \neq 0$;
- if the image of $i$ is not dense, then $\overline{H^1}(\Gamma, \ell^p(X)) \neq 0$

**Theorem 3.1** *Let $X$ be a free $\Gamma$-space. Then $\alpha : \mathcal{D}_p(X) \to Z^1(\Gamma, \ell^p(X))$ is a topological isomorphism, and consequently:*

$$H^1(\Gamma, l^p(X)) \simeq \mathcal{D}_p(X)/i(l^p_\Gamma(X));$$

$$\overline{H^1}(\Gamma, \ell^p(X)) \simeq \mathcal{D}_p(X)/\overline{i(l^p_\Gamma(X))}.$$

**Proof:** We already know that $\alpha$ is continuous and injective. Since the $\Gamma$-action on $X$ is free, we have $H^1(\Gamma, \mathcal{F}(X)) = 0$ by lemma 2.2. So for $b \in Z^1(\Gamma, \ell^p(X))$ there exists $f \in \mathcal{F}(X)$ such that $b(g) = \lambda_X(g)f - f$ for every $g \in \Gamma$. Clearly $f$ belongs to $\mathbf{D}_p(X)$, so that $\tilde{\alpha}(f) = b$, and $\alpha$ is onto. It is then clear that $\alpha^{-1}$ is continuous. □

When $\Gamma$ is infinite and $X = \Gamma$, we have $l^p_\Gamma(X) = \ell^p(\Gamma)$ and $\mathcal{D}_p(X) = \mathbf{D}_p(X)/\mathbb{C}$. It was already observed (see lemma 1 in [BMV05]; end of section 2 in [Pul]) that:

- $H^1(\Gamma, \ell^p(\Gamma))$ is isomorphic to $\mathbf{D}_p(\Gamma)/(\ell^p(\Gamma) + \mathbb{C})$;
- the first $\ell^p$-cohomology $\overline{H}^1_{(p)}(\Gamma)$ is isomorphic to $\mathbf{D}_p(\Gamma)/\overline{(\ell^p(\Gamma) + \mathbb{C})}$.

So we get, using Proposition 2.6:

**Corollary 3.2** *Let $\Gamma$ be an infinite, finitely generated group. The following are equivalent:*

*i)* $\overline{H}^1_{(p)}(\Gamma) = 0$;

*ii)* $\ell^p(\Gamma)$ *is dense in* $\mathbf{D}_p(\Gamma)/\mathbb{C}$;

*iii)* $\overline{H^1}(\Gamma, \ell^p(\Gamma)) = 0$;

*iv)* $\overline{H^1}(\Gamma, \ell^p(H)|_\Gamma) = 0$ *for every group* $H$ *containing* $\Gamma$ *as a subgroup.* □

From this and Corollary 2.4, we get immediately:

**Corollary 3.3** *Let $\Gamma$ be an infinite, finitely generated group. The following are equivalent:*

*i)* $H^1(\Gamma, \ell^p(\Gamma)) = 0$;

*ii)* $\overline{H}^1_{(p)}(\Gamma) = 0$ *and* $\Gamma$ *is non-amenable.*

## 3.2 $p$-harmonic functions (after M. Puls)

This section is essentially borrowed from section 3 in Puls'paper [Pul]. We chose to include it mainly for the sake of completeness, but also to make sure that Puls' results hold for any $\Gamma$-action (not only for simply transitive ones). Our presentation, emphasizing the role of Gâteaux-differentials, is slightly different from the one in [Pul].

So we come back to the general setting of a finitely generated group $\Gamma$ (with a given, finite, symmetric, generating set $S$) acting on a countable set $X$. For $f \in \mathcal{F}(X)$ and $p > 1$, define

$$(\Delta_p f)(x) = \sum_{s \in S} |f(s^{-1}x) - f(x)|^{p-2}(f(s^{-1}x) - f(x))$$

with the convention, if $p < 2$, that $|f(s^{-1}x) - f(x)|^{p-2}(f(s^{-1}x) - f(x)) = 0$ if $f(s^{-1}x) = f(x)$. Say that $f$ is *$p$-harmonic* if $\Delta_p f = 0$, and denote by $\mathbf{HD}_p(X)$ the set of $p$-harmonic functions in $\mathbf{D}_p(X)$. For $p \neq 2$, the set $\mathbf{HD}_p(X)$ is not necessarily a linear subspace in $\mathbf{D}_p(X)$. Note however that it contains the linear subspace $\mathbf{D}_p(X)^\Gamma$.

For $f \in \mathbf{D}_p(\Gamma)$, define a linear form on $\mathcal{D}_p(X)$ by

$$d_f(g) = \sum_{x \in X} \sum_{s \in S} |f(s^{-1}x) - f(x)|^{p-2}(f(s^{-1}x) - f(x))(g(s^{-1}x) - g(x))$$

(where $g \in \mathbf{D}_p(X)$; but clearly $d_f(g)$ only depends on the image of $g$ in $\mathcal{D}_p(\Gamma)$, and $d_f$ only depends on the image of $f$ in $\mathcal{D}_p(\Gamma)$). Let $q$ be the conjugate exponent of $p$ (so that $\frac{1}{p} + \frac{1}{q} = 1$); by Hölder's inequality, we get

$$|d_f(g)| \leq \left[\sum_{x \in X}^{s \in S} |f(s^{-1}x) - f(x)|^{(p-1)q}\right]^{\frac{1}{q}} \left[\sum_{x \in X} \sum_{s \in S} |g(s^{-1}x) - g(x)|^p\right]^{\frac{1}{p}}$$

$$\leq \|f\|_{\mathbf{D}_p(X)}^{p-1} \|g\|_{\mathbf{D}_p(X)},$$

proving continuity of $d_f$ as a linear form on $\mathbf{D}_p(X)$.

This linear form $d_f$ can be understood as follows. Let us identify a function $f \in \mathbf{D}_p(X)$ with its image in $\mathcal{D}_p(X)$. Consider the strictly convex, continuous, non-linear functional on $\mathcal{D}_p(X)$ given by:

$$F(f) = \|f\|_{\mathcal{D}_p(X)}^p.$$

The Gâteaux-differential of $F$ at $f \in \mathcal{D}_p(X)$ (see [ET74], Def. 5.2 in Chapter I) is given by

$$F'_f(g) = \lim_{t \to 0^+} \frac{F(f+tg) - F(f)}{t}$$

$(g \in \mathcal{D}_p(X))$. An easy computation shows that:

$$F'_f = p\, d_f.$$

The following lemma extends lemma 3.1 in [Pul].

**Lemma 3.4** *For $f_1, f_2 \in D_p(X)$, the following are equivalent:*

*i)* $f_1 - f_2 \in \mathbf{D}_p(X)^\Gamma$;

*ii)* $d_{f_1}(f_1 - f_2) = d_{f_2}(f_1 - f_2)$.

**Proof:** If $f_1 - f_2 \in \mathbf{D}_p(X)^\Gamma$, then $d_f(f_1 - f_2) = 0$ for every $f \in \mathbf{D}_p(X)$, in particular $d_{f_1}(f_1 - f_2) = 0 = d_{f_2}(f_1 - f_2)$.

Conversely, if $f_1 - f_2 \notin \mathbf{D}_p(X)^\Gamma$, then $f_1, f_2$ define distinct elements in $\mathcal{D}_p(X)$. As $F$ is strictly convex on $\mathcal{D}_p(X)$, by Proposition 5.4 in Chapter I of [ET74], we have

$$F(f_1) > F(f_2) + F'_{f_2}(f_1 - f_2) = F(f_2) + p\, d_{f_2}(f_1 - f_2).$$

Similarly:

$$F(f_2) > F(f_1) + F'_{f_1}(f_2 - f_1) = F(f_1) - p\, d_{f_1}(f_1 - f_2).$$

So $d_{f_1}(f_1 - f_2) > d_{f_2}(f_1 - f_2)$. $\square$.

Next lemma generalizes lemma 3.2 and Proposition 3.4 in [Pul].

**Lemma 3.5** *For $f \in \mathbf{D}_p(X)$, the following are equivalent:*

1. *$f$ is $p$-harmonic;*
2. *$d_f(\delta_y) = 0$ for every $y \in X$;*
3. *$d_f(g) = 0$ for every $g \in \overline{i(l^p_\Gamma(X))}$ (where the closure is in $\mathcal{D}_p(X)$).*

**Proof:** $(i) \Leftrightarrow (ii)$ We compute:

$$\begin{aligned} d_f(\delta_y) &= \sum_{s\in S} |f(y)-f(sy)|^{p-2}(f(y)-f(sy)) \\ &\quad - \sum_{s\in S} |f(s^{-1}y)-f(y)|^{p-2}(f(s^{-1}y)-f(y)) \\ &= -2(\Delta_p f)(y) \end{aligned}$$

as $S$ is symmetric.

$(ii) \Leftrightarrow (iii)$ The linear span of the $\delta_y$'s ($y \in X$) is dense in $\ell^p(X)$. By continuity of $i : l^p_\Gamma(X) \to \mathcal{D}_p(X)$, the linear span of the $\delta_y$'s is dense in $\overline{i(l^p_\Gamma(X))}$. This shows the desired equivalence. □

The following result extends Theorem 3.5 in [Pul].

**Theorem 3.6** *Every $f \in \mathbf{D}_p(X)$ can be decomposed as $f = g + h$, where $g \in \overline{i(\ell^p(X))}$ and $h \in \mathbf{HD}_p(X)$. This decomposition is unique, up to an element of $\mathbf{D}_p(X)^\Gamma$.*

**Proof:** We start with uniqueness. So assume that $f = g_1+h_1 = g_2+h_2$. Then $d_{h_1}(h_1-h_2) = d_{h_1}(g_2-g_1) = 0$ by appealing to lemma 3.5 (since $h_1$ is $p$-harmonic). Similarly $d_{h_2}(h_1-h_2) = 0$. By lemma 3.4, it follows that $h_1 - h_2$ is in $\mathbf{D}_p(X)^\Gamma$.

To prove existence, we denote by $g$ the projection of $f$ on the closed convex subset $\overline{i(l^p(X))}$ in $\mathcal{D}_p(X)$ (this projection is well-defined by uniform convexity and reflexivity of $\mathcal{D}_p(X)$, see Theorem 2.8 in [BL00] or lemma 6.2 in [BFGM]). Setting $h = f - g$, we must show that $h$ is $p$-harmonic. For every $j \in i(l^p(X))$, consider the smooth function

$$G_j : \mathbb{R} \to \mathbb{R} : t \mapsto \|f - g + tj\|^p_{\mathcal{D}_p(X)}.$$

Since $g$ minimizes the distance between $f$ and $\overline{i(l^p(X))}$, the function $G_j(t)$ assumes its minimal value at $t = 0$, hence $G_j'(0) = 0$. The same computation as for the Gâteaux-differential of $F$, shows that $G_j'(0) = p\, d_h(j) = 0$. Since this holds for every $j \in i(l^p(X))$, we conclude that $h$ is harmonic, by lemma 3.5. □

Comparing Theorem 3.6 with Theorem 3.1, we immediately get:

**Corollary 3.7** *Let $X$ be a free $\Gamma$-space. Then $\overline{H^1}(\Gamma, \ell^p(X))$ identifies with $\mathbf{HD}_p(X)/\mathbf{D}_p(X)^\Gamma$ (where two functions in $\mathbf{HD}_p(X)$ are identified if and only if they differ by an element in $\mathbf{D}_p(X)^\Gamma$).* □

# 4 Vanishing of first $L^p$-cohomology

## 4.1 Groups acting on trees

**Theorem 4.1** *Fix $p \in [1, +\infty[$. Let $G$ be a finitely generated group acting (without inversion) on a tree with non-amenable vertex stabilizers, and infinite edge stabilizers. If all vertex stabilizers have vanishing first $L^p$-cohomology, then so does $G$.*

**Proof:** By Bass-Serre theory (see [Ser77]), $G$ is the fundamental group of a graph of groups $(\mathcal{G}, Y)$. So $Y$ is a graph and $\mathcal{G}$ is a system of groups attached to edges and vertices of $Y$, in such a way that the edge groups are infinite and the vertex groups are non-amenable and have vanishing first $L^p$-cohomology. Consider the following cases:

1) If $Y$ is a segment, then $G$ is an amalgamated product $G = \Gamma_1 \star_A \Gamma_2$ with $A$ infinite, and $\Gamma_1$, $\Gamma_2$ non-amenable. The first cohomology of a $G$-module $V$ is computed by means of the Mayer-Vietoris sequence (see [Bro82], formula (9.1)):

$$0 \to V^G \to V^{\Gamma_1} \oplus V^{\Gamma_2} \to V^A \to$$

$$H^1(G, V) \to H^1(\Gamma_1, V|_{\Gamma_1}) \oplus H^1(\Gamma_2, V|_{\Gamma_2}) \stackrel{Rest^A_{\Gamma_1} - Rest^A_{\Gamma_2}}{\longrightarrow} H^1(A, V|_A) \longrightarrow ... \quad (4)$$

We apply this to $V = \ell^p(G)$. By Corollary 3.3, we have $H^1(G, \ell^p(G)) = 0$. Therefore $\overline{H}^1_{(p)}(G) = 0$.

2) If $Y$ is a loop, then $G$ is a $HNN$-extension $G = HNN(\Gamma, A, \theta)$, with $A$ infinite and $\Gamma$ non-amenable. The first cohomology of a $G$-module $V$ is computed by means of the Mayer-Vietoris sequence (see [Bro82], formula (9.2)):

$$0 \to V^G \to V^\Gamma \to V^A \to H^1(G, V) \to H^1(\Gamma, V|_\Gamma) \to H^1(A, V|_A) \longrightarrow ...$$

We apply this to $V = \ell^p(G)$. By Corollary 3.3, we have $H^1(G, \ell^p(G)) = 0$, so again $\overline{H}^1_{(p)}(G) = 0$.

3) If $Y$ is finite, we can argue by induction on the number $n$ of edges. If $n = 1$, the result follows from the first two cases. For arbitrary $n$, we choose an edge and contract it. If this edge is a segment with vertex groups $\Gamma_1$, $\Gamma_2$ and edge group $A$, we replace it by a vertex whose group is $\Gamma_1 *_A \Gamma_2$; if the edge is a loop with vertex group $\Gamma$, and edge group $A$, we replace it by a vertex whose group is $HNN(\Gamma, A, \theta)$. This operation does not change the fundamental group and we obtain a graph with $n - 1$ edges, so the induction assumption applies.

4) In the general case, $Y$ is the increasing union of finite subgraphs, so we may apply lemma 2.5. □

The converse of Theorem 4.1 fails. We give two examples for $p = 2$, one for amalgamated products, one for $HNN$-extensions.

**Example 1** *Let $q$ be a prime, and consider $\Gamma = SL_2(\mathbb{Z}[\frac{1}{q}])$. It follows from example 4 below that $\overline{H}^1_{(2)}(\Gamma) = 0$. But (see [Ser77]):*

$$\Gamma = SL_2(\mathbb{Z}) \star_A SL_2(\mathbb{Z})$$

*(with $A = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \bmod q\}$); and $\overline{H}^1_{(2)}(SL_2(\mathbb{Z})) \neq 0$.*

**Example 2** *Let $M$ be a closed, hyperbolic 3-manifold fibering over $S^1$; the fiber is a hyperbolic surface $\Sigma_g$. Then $\Gamma = \pi_1(M)$ is a semi-direct product (hence a particular case of an $HNN$-extension):*

$$\Gamma = \pi_1(\Sigma_g) \rtimes \mathbb{Z};$$

*then $\overline{H}^1_{(2)}(\Gamma) = 0$ but $\overline{H}^1_{(2)}(\pi_1(\Sigma_g)) \neq 0$.*

## 4.2 Normal subgroups with large commutant

**Theorem 4.2** *Let $N$ be a normal, infinite, finitely generated subgroup of a finitely generated group $\Gamma$. Assume that $N$ is non-amenable, and that its centralizer $Z_\Gamma(N)$ is infinite. Then $\overline{H}^1_{(p)}(\Gamma) = 0$, for $1 < p < +\infty$. If $p = 2$, this holds without the non-amenability assumption on $N$.*

**Proof:** We consider $\Gamma$ as a free $N$-space. Let $\mathbf{D}_p(\Gamma)$ be the space of $p$-Dirichlet finite functions *with respect to* $N$ on $\Gamma$, and $\mathcal{D}_p(\Gamma) = \mathbf{D}_p(\Gamma)/\mathbf{D}_p(\Gamma)^N$

as in the previous section. It is clear that $\mathcal{D}_p(\Gamma)$ is a Banach $Z_\Gamma(N)$-module, where $Z_\Gamma(N)$ acts by left translations.

Claim: $\mathcal{D}_p(\Gamma)^{Z_\Gamma(N)} = 0$

Indeed, let a class $[f] \in \mathcal{D}_p(\Gamma)^{Z_\Gamma(N)}$ be represented by the function $f \in \mathbf{D}_p(\Gamma)$; then $\lambda_\Gamma(g)f - f \in \mathbf{D}_p(\Gamma)^N$ for every $g \in Z_\Gamma(N)$. This means that, for every $n \in N$:

$$\lambda_\Gamma(n)(\lambda_\Gamma(g)f - f) = \lambda_\Gamma(g)f - f;$$

or else (using $gn = ng$):

$$\lambda_\Gamma(g)(\lambda_\Gamma(n)f - f) = \lambda_\Gamma(n)f - f.$$

Since $f \in \mathbf{D}_p(\Gamma)$, we have $\lambda_\Gamma(n)f - f \in \ell^p(\Gamma)$, hence

$$\lambda_\Gamma(n)f - f \in \ell^p(\Gamma)^{Z_\Gamma(N)}.$$

As $Z_\Gamma(N)$ is infinite, this shows that $\lambda_\Gamma(n)f - f$ vanishes identically, so that $f \in \mathbf{D}_p(\Gamma)^N$, hence $[f] = 0$. This proves the Claim.

Consider the map $\alpha : \mathcal{D}_p(\Gamma) \to Z^1(N, \ell^p(\Gamma)|_N)$ from Theorem 3.1. Let $\Gamma$ act by translations on $\mathcal{D}_p(\Gamma)$, and let it act on $Z^1(N, \ell^p(\Gamma)|_N)$ by the action of formula (1) in section 2.1. A simple computation shows that $\alpha$ is $\Gamma$-equivariant. In view of Corollary 3.7, $\alpha$ induces a $Z_\Gamma(N)$-equivariant bijection between $\mathbf{HD}_p(\Gamma)/\mathbf{D}_p(\Gamma)^N$ and $\overline{H^1}(N, \ell^p(\Gamma)|_N)$. We now separate two cases:

i) $N \cap Z_\Gamma(N)$ is infinite. Since we know that the action of $N$ on $\overline{H^1}(N, \ell^p(\Gamma)|_N)$ is trivial (by equation (2)), every element of $\overline{H^1}(N, \ell^p(\Gamma)|_N)$ is $(N \cap Z_\Gamma(N))$-fixed. So every element of $\mathbf{HD}_p(\Gamma)/\mathbf{D}_p(\Gamma)^N$ is $(N \cap Z_\Gamma(N))$-fixed. Now by the Claim (noticing that we may replace there $Z_\Gamma(N)$ by $N \cap Z_\Gamma(N)$, since the latter is infinite), the only $(N \cap Z_\Gamma(N))$-fixed point in $\mathcal{D}_p(\Gamma)$ is 0. So $\mathbf{HD}_p(\Gamma)/\mathbf{D}_p(\Gamma)^N = \{0\}$, hence $\overline{H^1}(N, \ell^p(\Gamma)|_N) = 0$. By Corollary 3.2, we get $\overline{H}^1_{(p)}(N) = 0$. By Theorem 1 in [BMV05] (which uses non-amenability of $N$) we conclude that $\overline{H}^1_{(p)}(\Gamma) = 0$.

ii) $N \cap Z_\Gamma(N)$ is finite. Since $N$ is non-amenable, we have $\overline{H^1}(N, \ell^p(\Gamma)|_N) = H^1(N, \ell^p(\Gamma)|_N)$. By the Claim, there is no fixed point in $\mathbf{HD}_p(\Gamma)/\mathbf{D}_p(\Gamma)^N$ under $Z_\Gamma(N)/(N \cap Z_\Gamma(N))$. So we have $H^1(N, \ell^p(\Gamma)|_N)^{Z_\Gamma(N)/(N \cap Z_\Gamma(N))} = 0$. In particular $H^1(N, \ell^p(\Gamma)|_N)^{\Gamma/N} = 0$. By equation (3), we have $H^1(\Gamma, \ell^p(\Gamma)) = 0$, so $\overline{H}^1_{(p)}(\Gamma) = 0$ by Corollary 3.2.

If $p = 2$, we may assume that $N$ is amenable. Then $\Gamma$ contains an infinite, amenable, normal subgroup, so by the Cheeger-Gromov vanishing theorem [CG86], all the $L^2$-cohomology of $\Gamma$ does vanish. □

**Remarks:**

a) It was observed by Marc Bourdon that part (ii) in the above proof can be obtained differently in case $Z_\Gamma(N)$ contains an element $z$ of *infinite* order. Indeed let $H$ be the subgroup generated by $N \cup \{z\}$. Then $z$ is central in $H$, so $\overline{H}^1_{(p)}(H) = 0$ by the Corollary on p. 221 in [Gro93]. The result then follows from Theorem 1 in [BMV05].

b) Among the recent vanishing results for $L^2$-cohomology, the most striking is probably Gaboriau's (Théorème 6.8 in [Gab02]): assume that $\Gamma$ contains a normal subgroup $N$ which is infinite, has infinite index, and is finitely generated (as a group): then $\overline{H}^1_{(2)}(\Gamma) = 0$. To prove this, Gaboriau needs a substantial part of his theory of $L^2$-Betti numbers for measured equivalence relations and group actions. It is very tempting to try to get a simpler proof of that result, and this is what motivated this section. More precisely, one possible line of attack for Gaboriau's result is the following: if the normal subgroup $N$ is amenable, then all of the $L^2$-cohomology of $\Gamma$ does vanish, by the Cheeger-Gromov vanishing theorem [CG86]. So we may assume that $N$, hence also $\Gamma$, is non-amenable. We must then prove that $H^1(\Gamma, \ell^2(\Gamma)) = 0$; by the exact sequence (3), this is still equivalent to $H^1(N, \ell^2(\Gamma)|_N)^{\Gamma/N} = 0$.

   Although a proof of Gaboriau's result along these lines remains elusive so far, this line of attack opened the possibility of replacing $L^2$-cohomology by $L^p$-cohomology, which resulted in Theorem 4.2 above.

**Theorem 4.3** *Let $\Gamma$ be a finitely generated group. If the centre of $\Gamma$ is infinite, then $\overline{H}^1_{(p)}(\Gamma) = 0$ for $1 < p < \infty$.*

**Proof:** We apply the first case of the proof of Theorem 4.2, with $N = \Gamma$. It yields $\overline{H}^1_{(p)}(\Gamma) = 0$, in full generality (i.e. without appealing to non-amenabilility). □

The following example, kindly provided by M. Bourdon, shows that the previous Theorem 4.3 does *not* hold for $p = 1$.

**Example 3** *One has* $\overline{H}^1_{(1)}(\mathbb{Z}) \neq 0$. *To see it, first observe that every function* $f \in \mathbf{D}_1(\mathbb{Z})$ *admits a limit at* $+\infty$ *and* $-\infty$. *Indeed, the sequence* $(f(n))_{n\geq 1}$ *is Cauchy, since for* $n > m$:

$$|f(n) - f(m)| \;=\; |\sum_{k=m}^{n-1}(f(k+1) - f(k))| \;\leq\; \sum_{k=m}^{n-1} |(f(k+1) - f(k))|$$

*and the RHS goes to zero for* $m, n \to +\infty$ *as* $f \in \mathbf{D}_1(\mathbb{Z})$. *Similarly, the sequence* $(f(-n))_{n\geq 1}$ *is Cauchy.*

*Next consider the linear form* $\tau$ *on* $\mathbf{D}_1(\mathbb{Z})$ *defined by*

$$\tau(f) = (\lim_{n\to+\infty} f(n)) - (\lim_{n\to-\infty} f(n))$$

($f \in \mathbf{D}_1(\mathbb{Z})$)*. The form* $\tau$ *is continuous on* $\mathbf{D}_1(\mathbb{Z})$ *because*

$$|\tau(f)| = |\lim_{n\to+\infty}(f(n) - f(-n))| = |\lim_{n\to+\infty}\sum_{k=-n}^{n-1}(f(k+1) - f(k))| \leq \|f\|_{\mathbf{D}_1(\mathbb{Z})}.$$

*Since* $\tau$ *is a continuous non-zero linear form on* $\mathbf{D}_1(\mathbb{Z})$ *which vanishes on* $\mathbb{C} + \ell^1(\mathbb{Z})$, *we conclude that* $\overline{H}^1_{(1)}(\mathbb{Z}) \neq 0$.

## 4.3 Wreath products

**Lemma 4.4** *Let* $G_1$, $G_2$ *be non-compact, locally compact groups. Let* $N \lhd G_1 \times G_2$ *be a closed normal subgroup such that* $N \cap (G_1 \times \{1\})$ *(resp.* $N \cap (\{1\} \times G_2)$*) is not co-compact in* $G_1 \times \{1\}$ *(resp.* $\{1\} \times G_2$*). Set* $G = (G_1 \times G_2)/N$. *Then:*

1) $\overline{H^1}(G, L^2(G)) = 0$;

2) *If moreover* $G$ *is non-amenable, then* $H^1(G, L^p(G)) = 0$ *for* $1 < p < \infty$.

**Proof:**

1) We appeal to a result of Shalom ([Sha00a], Theorem 3.1): if $(V, \pi)$ is a unitary $(G_1 \times G_2)$-module, and $b \in Z^1(G_1 \times G_2, V)$, then $b$ is cohomologous in $\overline{H^1}(G_1 \times G_2, V)$ to a sum $b_0 + b_1 + b_2$, where $b_0$ takes values in $V^{G_1\times G_2}$ and, for $i = 1, 2$, $b_i$ factors through $G_i$ and takes values in a $(G_1 \times G_2)$-invariant subspace on which $\pi$ factors through $G_i$. This implies the following alternative: either $\overline{H^1}(G_1 \times G_2, V) = 0$, or there exists in $V$ a non-zero vector fixed by some restriction $\pi|_{G_i}$.

We apply this to the regular representation $\lambda_G$, viewed as a representation of $G_1 \times G_2$. Our assumption ensures that the restriction $\lambda_G|_{G_i}$ ($i = 1, 2$) does not have non-zero invariant vectors. So $\overline{H^1}(G_1 \times G_2, L^2(G)) = 0$, hence also $\overline{H^1}(G, L^2(G)) = 0$.

2) We replace Shalom's result by a recent result of Bader-Furman-Gelander-Monod ([BFGM], Theorem 7.1): let $(V, \pi)$ be a Banach $(G_1 \times G_2)$-module, with $V$ uniformly convex, such that $\pi$ does not almost have invariant vectors, and $H^1(G_1 \times G_2, V) \neq 0$; then for some $i \in \{1, 2\}$, there exists a non-zero $\pi(G_i)$-fixed vector.

   We apply this to the regular representation $\lambda_G$ on $L^p(G)$, viewed as a representation of $G_1 \times G_2$. Our assumptions ensures that the restriction $\lambda_G|_{G_i}$ ($i = 1, 2$) does not have non-zero invariant vectors, and that $\lambda_G$ does not almost have invariant vectors. So $H^1(G_1 \times G_2, L^p(G)) = 0$, hence also $H^1(G, L^p(G)) = 0$. □

**Lemma 4.5** *Fix $n \geq 2$. Let $G_1, \ldots, G_n$ be non-compact, locally compact groups. Assume that at least one $G_i$ is non-amenable. Set $G = G_1 \times \ldots \times G_n$. Then $H^1(G, L^p(G)) = 0$ for $1 < p < \infty$.*

**Proof:** Re-numbering the groups if necessary, we may assume that $G_1$ is non-amenable. The result then follows from lemma 4.4, by induction over $n$ (the case $n = 2$ being lemma 4.4, with $N = \{1\}$). □

As an application, we show the vanishing of the first $L^p$-cohomology for wreath products. For $p = 2$, that fact can also be deduced from Theorem 7.2.(2) in [Lue02].

**Theorem 4.6** *Let $H$, $\Gamma$ be (non trivial) finitely generated groups. Then*

*1) $\overline{H}^1_{(2)}(H \wr \Gamma) = 0$;*

*2) If $H$ is non-amenable, then $\overline{H}^1_{(p)}(H \wr \Gamma) = 0$ for $1 < p < \infty$.*

**Proof:** Let $N = \bigoplus_\Gamma H$. Note that $N$ is amenable exactly when $H$ is. We separate two cases:

i) Proof of (1) when $N$ is amenable. If $N$ is finite, then so are $H$, $\Gamma$, $H \wr \Gamma$ and the result is clear. If $N$ is infinite, then the result follows from the Cheeger-Gromov vanishing theorem [CG86].

ii) Proof of (1) + (2) when $N$ is non-amenable. Then $N$ can be written as the direct product of two infinite groups. By lemma 4.5, we have $H^1(N, \ell^p(N)) = 0$, hence also $H^1(N, \ell^p(H \wr \Gamma)|_N) = 0$ by Proposition 2.6. The result then follows from equation (3). □

## 4.4 Lattices in products

**Theorem 4.7** *Let $G = G_1 \times \ldots \times G_n$ be a direct product of non-compact, second countable locally compact groups ($n \geq 2$). Let $\Gamma$ be a finitely generated, cocompact lattice in $G$. Then:*

*i)* $\overline{H}^1_{(2)}(\Gamma) = 0$*;*

*ii) if $\Gamma$ is non-amenable (equivalently, if some $G_i$ is non-amenable), then* $\overline{H}^1_{(p)}(\Gamma) = 0$ *for $1 < p < \infty$.*

**Proof:** By the version of Shapiro's lemma proved in Proposition 4.5 of [Gui80], since $\Gamma$ is cocompact, there exists a topological isomorphism $H^1(\Gamma, \ell^p(\Gamma)) \simeq H^1(G, {}_pInd_\Gamma^G \ell^p(\Gamma))$, where ${}_pInd_\Gamma^G V$ denotes the induced module in the $L^p$-sense, i.e. ${}_pInd_\Gamma^G V = (L^p(G, V))^\Gamma$. But ${}_pInd_\Gamma^G \ell^p(\Gamma)$ is $G$-isomorphic to $L^p(G)$, so we get $\overline{H^1}(\Gamma, \ell^p(\Gamma)) \simeq \overline{H^1}(G, L^p(G)) = 0$ by lemma 4.5. □

**Theorem 4.8** *Fix $n \geq 2$. For $i = 1, \ldots, n$, let $G_i$ be the group of $k_i$-rational points of some $k_i$-simple, $k_i$-isotropic linear algebraic group, for some local field $k_i$. Let $\Gamma$ be an irreducible lattice in $G_1 \times \ldots \times G_n$. Then $\overline{H}^1_{(p)}(\Gamma) = 0$ for $1 < p < \infty$.*

**Proof:** We need some terminology: a lattice $\Lambda$ in a locally compact group $G$ is *p-integrable* if either it is cocompact, or it is finitely generated and for some finite generating set $S \subset \Lambda$, there is a Borel fundamental domain $D \subset G$ such that

$$\int_D |\chi(g^{-1}h)|_S^p \, dh < \infty$$

for every $g \in G$; here $|\cdot|_S$ denotes word length, and $\chi : G \to \Gamma$ is defined by $\chi(\gamma g) = \gamma$ for $\gamma \in \Gamma$, $g \in G$.

We then appeal to a result of Bader-Furman-Gelander-Monod (see section 8.2 in [BFGM], especially the few lines preceding Proposition 8.7): if $\Lambda$ is a $p$-integrable lattice in $G$, and $V$ is a Banach $\Lambda$-module, then there is a topological isomorphism

$$H^1(\Lambda, V) \simeq H^1(G, {}_p Ind_\Lambda^G V).$$

In our case, set $G = G_1 \times \ldots \times G_n$. Then $\Gamma$ is $p$-integrable for every $p \geq 1$, by a result of Shalom (section 2 in [Sha00a]). With $V = \ell^p(\Gamma)$, we get ${}_p Ind_\Gamma^G V \simeq L^p(G)$, so the result follows using lemma 4.5. □

**Example 4** *Let $q$ be a prime; $\Gamma = SL_2(\mathbb{Z}[\frac{1}{q}])$ is an irreducible non-uniform lattice in $SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_q)$. Theorem 4.8 applies to give $\overline{H}^1_{(p)}(\Gamma) = 0$.*

# 5 Application to amenable groups

**Proposition 5.1** *Let $\Gamma$ be a finitely generated group. If $\Gamma$ has an infinite amenable normal subgroup (in particular if $\Gamma$ is infinite amenable), then $\overline{H^1}(\Gamma, \ell^2(\Gamma)) = 0$.*

**Proof:** By the Cheeger-Gromov vanishing result [CG86], the assumptions imply $\overline{H}^1_{(2)}(\Gamma) = 0$. So the result follows from Corollary 3.2. □

When $\Gamma$ is itself amenable, the finite generation assumption can be removed:

**Corollary 5.2** *Let $\Gamma$ be an amenable discrete group. Then $\overline{H^1}(\Gamma, \ell^2(\Gamma)) = 0$.*

**Proof:** Let $(\Gamma_i)_{i \in I}$ be the directed system of finitely generated subgroups of $\Gamma$ (so that $\Gamma = \bigcup_{i \in I} \Gamma_i$). By Proposition 5.1, we have $\overline{H^1}(\Gamma_i, \ell^2(\Gamma_i)) = 0$, for every $i \in I$. By Proposition 2.6, this implies $\overline{H^1}(\Gamma_i, \ell^2(\Gamma)|_{\Gamma_i}) = 0$ for every $i \in I$. The conclusion then follows from lemma 2.5. □

We also get a new characterization of amenability for finitely generated, infinite groups:

**Proposition 5.3** *Let $\Gamma$ be an infinite, finitely generated group. The following are equivalent:*

*i)* $\Gamma$ *is amenable;*

*ii)* $\ell^2(\Gamma)$ *is a dense, proper subspace of* $\mathbf{D}_2(\Gamma)/\mathbb{C}$.

**Proof:** In view of Theorem 3.1, $\ell^2(\Gamma)$ is a dense, proper subspace of $\mathbf{D}_2(\Gamma)/\mathbb{C}$ if and only if $H^1(\Gamma, \ell^2(\Gamma)) \neq 0$ and $\overline{H^1}(\Gamma, \ell^2(\Gamma)) = 0$. If this happens, then $\Gamma$ is amenable by Corollary 3.3. Conversely, if $\Gamma$ is amenable, then $H^1(\Gamma, \ell^2(\Gamma)) \neq \overline{H^1}(\Gamma, \ell^2(\Gamma))$ by the converse of Corollary 3.3, and the latter space is zero by Proposition 5.1. □

# 6 Appendix: A result of D. Gaboriau, and a conjecture

Recall that Cheeger and Gromov defined $L^2$-Betti numbers $b^i_{(2)}(\Gamma)$ (taking values in $[0, \infty]$) for any countable group $\Gamma$ (see formula (2.8) in [CG86]). The definition of the first $L^2$-Betti number $b^1_{(2)}(\Gamma)$ will be recalled in the course of the proof of Proposition 6.1 below.

**Conjecture 1** *Let $\Gamma$ be a countable group. A necessary and sufficient condition for $b^1_{(2)}(\Gamma) = 0$ is $\overline{H^1}(\Gamma, \ell^2(\Gamma)) = 0$.*

For finitely generated groups, this is Corollary 3.2. The next result shows that the necessity is true in general.

**Proposition 6.1** *(D. Gaboriau) Let $\Gamma$ be a countable group. If $b^1_{(2)}(\Gamma) = 0$, then $\overline{H^1}(\Gamma, \ell^2(\Gamma)) = 0$.*

**Proof:** Write $\Gamma$ as the union $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ of an increasing family of finitely generated subgroups. Let $S_n$ be a finite generating subset of $\Gamma_n$, with $S_n \subset S_{n+1}$. Let $\Gamma_n = \langle S_n | R_n \rangle$ be a presentation of $\Gamma_n$; clearly we may arrange so that $R_n \subset R_{n+1}$. Let $K_n$ be the presentation 2-complex associated with this presentation of $\Gamma_n$.

Set then $S_\infty = \bigcup_{n=1}^{\infty} S_n$ and $R_\infty = \bigcup_{n=1}^{\infty} R_n$, so that $\langle S_\infty | R_\infty \rangle$ is a presentation of $\Gamma$. Let $K$ be the presentation 2-complex associated with this presentation, and let $p : \tilde{K} \to K$ be its universal cover. By construction $K_n$ is a subcomplex of $K$. Set $L_n = p^{-1}(K_n)$: this is a (non-connected) $\Gamma$-invariant subcomplex of $\tilde{K}$.

We now recall how $b^1_{(2)}(\Gamma)$ is defined. For each $n$, we have the first $L^2$-cohomology $\overline{H}^1_{(2)}(L_n)$, which is a unitary $\Gamma$-module. For $m < n$, the inclusion $L_m \subset L_n$ induces, by contravariance, a $\Gamma$-module map:

$$j_{m,n} : \overline{H}^1_{(2)}(L_n) \to \overline{H}^1_{(2)}(L_m).$$

Since $Im(j_{m,n+1}) \subset Im(j_{m,n})$, the $\Gamma$-dimension $\dim_\Gamma \overline{Im(j_{m,n})}$ is decreasing as a function of $n$; moreover $\lim_{n\to\infty} \dim_\Gamma \overline{Im(j_{m,n})}$ is now increasing as a function of $m$, and we define

$$b^1_{(2)}(\Gamma) = \lim_{m\to\infty} \lim_{n\to\infty, n>m} \dim_\Gamma \overline{Im(j_{m,n})}$$

$$= \sup_{m\in\mathbb{N}} \inf_{n>m} \dim_\Gamma \overline{Im(j_{m,n})}.$$

It is checked in [CG86] that $b^1_{(2)}(\Gamma)$ does not depend on the choice of the exhaustive sequence of $\Gamma$-co-compact subcomplexes $L_n$ in $\tilde{K}$.

We are going to use the following fact (explained e.g. in section 1 of [BMV05]): if $H$ is a finitely generated group, and $\tilde{X}$ is the universal cover of a presentation complex for $H$, then there is a natural isomorphism between $\overline{H^1}(H, \ell^2(H))$ and the first $\ell^2$-cohomology $\overline{H}^1_{(2)}(\tilde{X})$. In particular, if $L^0_m$ is any connected component of $L_m$, we have a natural isomorphism between $\overline{H^1}(\Gamma_m, \ell^2(\Gamma_m))$ and $\overline{H}^1_{(2)}(L^0_m)$. As a consequence, we have a natural isomorphism

$$\alpha_m : \overline{H^1}(\Gamma_m, \ell^2(\Gamma)) \to \overline{H}^1_{(2)}(L_m).$$

With this we can really start the proof of the Proposition. If $b^1_{(2)}(\Gamma) = 0$, from the second definition of $b^1_{(2)}(\Gamma)$ above, we get

$$\inf_{n>m} \dim_\Gamma \overline{Im(j_{m,n})} = 0$$

for every $m$, i.e.

$$\bigcap_{n>m} \overline{Im(j_{m,n})} = 0$$

which implies

$$\bigcap_{n>m} Im(j_{m,n}) = 0$$

for every $m$.

Take now $b \in Z^1(\Gamma, \ell^2(\Gamma))$ and consider the class $[b_m]$ of $b|_{\Gamma_m}$ in $\overline{H^1}(\Gamma_m, \ell^2(\Gamma))$. By naturality, for $n > m$, we have $j_{m,n}(\alpha_n[b_n]) = \alpha_m[b_m]$. So, for fixed $m$, we have $\alpha_m[b_m] \in \bigcap_{n>m} Im(j_{m,n})$, i.e $\alpha_m[b_m] = 0$.

Since $\alpha_m$ is an isomorphism, we have $[b_m] = 0$, i.e. $b|_{\Gamma_m} \in \overline{B^1}(\Gamma_m, \ell^2(\Gamma))$. By lemma 2.5, we deduce $b \in \overline{B^1}(\Gamma, \ell^2(\Gamma))$; since $b$ was arbitrary, this means that $\overline{H^1}(\Gamma, \ell^2(\Gamma)) = 0$. $\square$

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