On the Dynamics of Isometries

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Abstract

We provide an analysis of the dynamics of isometries of metric spaces. Certain subsets of the boundary at infinity play a fundamental role and are identified completely for the standard boundaries of CAT(0)-spaces, Gromov hyperbolic spaces, Hilbert geometries, certain pseudoconvex domains, and partially for Thurston's boundary of Teichmüller spaces. A theory of groups of isometries is developed for any proper metric space. This extends the usual theory of hyperbolic groups and gives in particular several new results in the special case of CAT(0)-geometry, for example a metric Furstenberg's lemma.

1 Introduction

In order to understand the topology of a manifold which admits a metric of a certain kind, one is lead to analyze the action by isometry of the fundamental group on the universal covering space. For this and other reasons it is of evident interest to study the automorphism group of Riemannian manifolds, see [23] for an exposition of results known up until 1972. Hyperbolic geometries and their groups are of particular importance and have a rich theory ([28], [30], [12]).

In other contexts there are metrics which tightly connect with the structures under study (isomorphisms are isometric) and whose existence is remarkable. The most prominent instance is the category of complex spaces and holomorphic maps, where one has the Schwarz-Pick lemma, the Bergman metric, Kobayashi's pseudo-metric, etc. See [13, p. xvii] for other examples.

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Groups of matrices act by isometry on an associated symmetric space or building. Nonpositive curvature enters here (and elsewhere) predominantly and much work on general Cartan-Hadamard manifolds and CAT(0)-spaces has been carried out ([4], [2], [10], [7]). At this point one may insert two more specific facts: the representation theoretic property (T) can be characterized in terms of (affine) isometric actions on Hilbert spaces, and several ergodic theorems such as those of von Neumann, Birkhoff, and Oseledec can be deduced from a single dynamical result about isometries (see [20], [19]).

As a final example we mention that any group acts by isometry on the Cayley graph associated to some set of generators and that the subject of geometric group theory has greatly expanded in recent years ([12], [15]).

Nevertheless, in spite of the many and diverse examples there seems to be no well-developed general theory of isometries available. The purpose of the present paper is to make a contribution to such a theory.

This paper studies and exploits (generalized) halfspaces and their limits, *the stars at infinity*. These subsets are of fundamental importance for the dynamics of isometries. Even though halfspaces are classical in the definition of Dirichlet fundamental domains and appear particularly in the literature on Kleinian groups, it seems they have not been systematically considered previously. The stars relate well to standard concepts such as Tits geometry of CAT(0)-spaces, Thurston's boundary of Teichmüller space, hyperbolicity of metric spaces, strict pseudoconvexity, the face lattice of convex domains, rank 1 isometries, etc. See sections 1 and 4.

In the theory of word hyperbolic groups, the study of how the group acts on its boundary plays an important role. We extend part of this theory in sections 2 and 3 to isometries of any (proper) metric space and discuss among other things a generalization of Hopf's theorem on ends, free subgroups, an analog of Furstenberg's lemma, random walks and implications of discreteness.

In some ways we thus present elements of a unified and rather general theory. In addition, as is indicated in section 4, several of the results obtained are new even in the much studied case of CAT(0)-geometry. More theory, examples, and applications remain to be worked out.

From the point of view of metric geometry, the stars provide a framework for boundary estimates and is hence one way of organizing asymptotic geometrical information. Moreover, if Gromov's theory of δ-hyperbolic spaces was in part motivated by Mostow's proof of strong rigidity in rank 1, then similarily one could say that the concept of the stars and their incidence geometry (face lattice, curve complex, spherical building, etc.) gain motivation from Mostow's proof in the higher rank case.

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2 Halfspaces and stars at infinity

2.1 Definitions

Let X be a metric space. For a subset W of X we let

$$d(x, W) = \inf_{w \in W} d(x, w).$$

Fix a base point x_0. We define the *halfspace* defined by the subset W and the real number C to be

$$H(W, C) = H^{x_0}(W, C) := \{z : d(z, W) \leq d(z, x_0) + C\}.$$

We use the notation $H(W) := H(W, 0)$ and for two points x and y in X we let $H_y^x = \{z : d(z, y) \leq d(z, x)\}$, so $H_y^{x_0} = H(\{y\}, 0)$. Note that the latter sets define halfspaces in the more standard sense when X is a euclidean or real hyperbolic space.

Let X be a complete metric space. By a *bordification* of X we here mean a topological space $\overline{X}$ with X embedded as an open dense subset. The *boundary* is $\partial X = \overline{X} \backslash X$. If $\overline{X}$ is compact we refer to it as a *compactification*. We define $d(x, \xi) = \infty$ for any $x \in X$ and $\xi \in \partial X$ (which is consistent with the completeness of X) and extend the definition of $d(x, W)$ for $W \subset \overline{X}$ in the expected way. A metric space is *proper* if every closed ball is compact.

Let $\mathcal{V} = \mathcal{V}_\xi$ denote the collection of open neighborhoods in $\overline{X}$ of a boundary point ξ. The *star based at* x_0 of a point $\xi \in \partial X$ is (the closures are taken in $\overline{X}$):

$$S^{x_0}(\xi) := \bigcap_{V \in \mathcal{V}_\xi} \overline{H(V)},$$

while the *star* of ξ is

$$S(\xi) := \overline{\bigcup_{C \geq 0} \bigcap_{V \in \mathcal{V}_\xi} \overline{H(V, C)}}.$$

The latter definition in particular removes an a priori dependence of x_0 as will be clear later on. Note also that because of the monotonicity built into the definition of H, we may restrict $\mathcal{V}$ to some fundamental system of neighborhoods of ξ.

We introduce the *star-distance*: Let s be the largest metric on ∂X taking values in $[0,\infty]$ such that $s(\xi,\eta)=0$ if $S(\xi)=S(\eta)$, and $s(\xi,\eta)=1$ if at least one of $\xi\in S(\eta)$ or $\eta\in S(\xi)$ holds. More explicitly, $s(\xi,\eta)$ equals the minimum number k such that there are points γ_i with $\gamma_0=\xi$, $\gamma_k=\eta$, and $s(\gamma_i,\gamma_{i+1})=1$ for all i.

It does not seem clear whether, or when, $\xi\in S(\eta)$ implies $\eta\in S(\xi)$. Let

$$S^{\vee}(\xi)=\{\eta:\xi\in S(\eta)\},$$

and we say that the bordification is *star-reflexive* when $S(\xi)=S^{\vee}(\xi)$ for all ξ. The examples below turn out to have this property.

The *face* of a subset A of ∂X is the intersection of all stars containing A. The face of the empty set is defined to be the empty set. By the notation $x_n\to S$, where x_n is a sequence of points and S a set, we mean that for any neighborhood U of S we have $x_n\in U$ for all sufficiently large n.

2.2 Some lemmas

Lemma 1 *For any $\xi\in\partial X$, the sets $\overline{H(V)}$ for $V\in\mathcal{V}_\xi$ contain V and $\xi\in S^{x_0}(\xi)\subset S(\xi)\subset\partial X$. If ∂X is compact, then for every neighborhood U of $S^{x_0}(\xi)$ there is a neigborhood V of ξ such that $\overline{H(V)}\subset U$.*

Proof. Note that $V\subset\overline{H(V)}$. Indeed, first observe that $V\cap X\subset H(V)$ because $d(v,V)=0$ for any $v\in V$. Secondly, note that for any $v\in V$ and any open neighborhood U of v, $U\cap V$ is again an open neighborhood and every open set in $\overline{X}$ has to intersect X. Finally, $S^{x_0}(\xi)$ is nonempty because ξ is contained in every V, and $S^{x_0}(\xi)\subset S(\xi)\subset\partial X$ since $d(V,x_0)$ is unbounded for $V\in\mathcal{V}_\xi$.

By compactness and the monotonicity built into $H(V)$, it follows from considering a fundamental system of neighborhoods that for every neighborhood U of $S^{x_0}(\xi)$ there is a neigborhood V of ξ such that $H(V)\subset U$ (otherwise one would have $\bigcap\overline{H(V)}\cap U^c\neq\emptyset$). ■

Note that if $z_n\to\xi$ and $d(z_n,y_n)<C$ then every limit point of y_n belongs to $S^{x_0}(\xi)$. A priori, $S^{x_0}(\xi)$ depends on x_0 although in the examples below this is not the case. On the other hand:

Lemma 2 *The sets $S(\xi)$ are independent of the base point x_0. If $z_n\to\xi\in\partial X$, $d(z_n,y_n)<C$ and $y_n\to\eta$, then $S(\xi)=S(\eta)$. Moreover, ξ and η belong to the same stars.*

Proof. The first statement follows from

$$H^{x_0}(W, C - d(x, x_0)) \subset H^x(W, C) \subset H^{x_0}(W, C + d(x, x_0)),$$

and because of the increasing union over $C \geq 0$ in the definition of $S(\xi)$. The other two claims hold for similar reasons. ∎

Lemma 3 *Assume that $\overline{X}$ is sequentially compact and that $S(\xi) = S^{x_0}(\xi)$ for every $\xi \in \partial X$. Let ξ_n and η_n be two sequences in ∂X converging to ξ and η, respectively. If $s(\xi_n, \eta_n) > 0$ for all n, then*

$$s(\xi, \eta) \leq \liminf_{n\to\infty} s(\xi_n, \eta_n).$$

Proof. By the assumption we can work with the S^{x_0}-stars. It is enough to consider $s(\xi_n, \eta_n) = 1$ for all n, because of the sequential compactness and the way s is defined. Moreover, we may suppose that $\xi_n \in S(\eta_n)$ for all n. Hence $\xi_n \in \overline{H(V)}$ for every neighborhood V of η_n. Given a neighborhood U of η, there is a N such that U is also a neighborhood of η_n for $n \geq N$. We therefore have that $\xi_n \in \overline{H(U)}$ for all $n \geq N$, and hence also $\xi \in \overline{H(U)}$. Because U was arbitrary, we have that $\xi \in S(\eta)$ and so $s(\xi, \eta) \leq 1$ as required. ∎

2.3 Examples

2.3.1 Hyperbolic bordifications

By a *hyperbolic bordification* we mean the definition to be found in [22]. Examples include the usual boundary of visibility spaces or δ-hyperbolic spaces, the end-compactification, and Floyd's boundary construction. Here is a relation between halfspaces and hyperbolicity:

$$(x|z) := \frac{1}{2}(d(x, x_0) + d(z, x_0) - d(x, z)) \geq \frac{1}{2}d(z, x_0)$$

if and only if $x \in H_z^{x_0}$.

Proposition 4 *Assume that $\overline{X}$ is a hyperbolic bordification. Then $S(\xi) = S^{x_0}(\xi) = \{\xi\}$ for every $\xi \in \partial X$.*

Proof. Given U a neighborhood of ξ in $\overline{X}$ and $C > 0$. By definition we may find R and $W \in \mathcal{W}$ such that $\{z : (z|W) > R - C/2\} \subset U$ (see [22]) and by making W smaller we can also arrange so that $R < d(W, x_0)/2$ (recall that $(z|W) := \sup(z|w)$ and $d(x_0, \xi) = \infty$). Now

$$\begin{aligned} H(W,C) &= \{z : d(z,W) \le d(z,x_0) + C\} \\ &= \{z : 0 \le \sup_w (d(z,x_0) - d(z,w)) + C\} \\ &\subset \{z : d(W,x_0) \le \sup_w (d(z,x_0) + d(w,x_0) - d(z,w)) + C\} \\ &= \{z : (z|W) > R - C/2\} \subset U, \end{aligned}$$

which proves the proposition, because $\mathcal{W}$ is a fundamental system of neighborhoods and C plays no role. ■

2.3.2 Hilbert's metric

Let X be a bounded convex domain in $\mathbb{R}^n$ and ∂X the usual boundary. Recall that in this context the *star* of a boundary point ξ, Star(ξ), is the intersection of ∂X with the union of all hyperplanes which are disjoint from X but contain ξ. We have:

Proposition 5 *Assume that X is a bounded convex domain equipped with Hilbert's metric and let $\overline{X}$ be the closure in $\mathbb{R}^n$. Then $S(\xi) = S^{x_0}(\xi) =$Star(ξ) for every $\xi \in \partial X$.*

Proof. The inclusion $S(\xi) \subset$Star(ξ) follows from the inclusion

$$H(W,C) \subset \{z : (z|W) \ge \frac{1}{2} d(W,x_0) + C'\}$$

proved in Proposition 4 using the same terminology, together with the proof of Theorem 5.2 in [21]. The other inclusion follows because given W and ζ it is simple to see that we can approximate ζ with a point arbitrary far from x_0 but staying on finite Hilbert distance to W (the Hilbert metric remains bounded near a line segment of the boundary in the direction parallel to this line segment). In particular, $\bigcap \overline{H(V,C)}$ is independent of C and equals $S^{x_0}(\xi)$. ■

2.3.3 Nonpositive curvature

Let X be a complete CAT(0)-space. Recall that the angular metric is $\angle(\xi,\xi') = \sup_{p\in X} \angle_p(\xi',\xi)$, where ξ,ξ' are points in the standard visual boundary ∂X of X. The following lemma and its proof can essentially be found in [4]:

Lemma 6 *Let c and c' be two geodesic rays emanating from x_0 and let $\xi = [c]$ and $\xi' = [c']$ be the corresponding boundary points. Let p_i denote the projection of $c'(i)$ onto c. If $\angle(\xi,\xi') > \pi/2$ then p_i stays bounded as $i \to \infty$. If $\angle(\xi,\xi') < \pi/2$, then p_i is unbounded. In the case $\angle(\xi,\xi') = \pi/2$ then $\{p_i\}$ is bounded if and only if x_0, c, and c' define a flat sector.*

Proof. First recall the basic angle property of projections [7, Prop. II.2.4]: $\angle_{p_i}(c'(i),\xi) \geq \pi/2$ and $\angle_{p_i}(c'(i),x_0)$.

If p_i is bounded we may assume $p_i \to p$ (along some subsequence), because the points p_i are restricted to a compact subset of c. Then by the upper semicontinuity of angles ([7, Prop. II.9.2]) we have:

$$\angle(\xi',\xi) \geq \angle_p(\xi',\xi) \geq \limsup_i \angle_{p_i}(c'(i),\xi) \geq \pi/2.$$

If p_i is unbounded, then in view of [7, Prop. II.9.8] we have

$$\begin{aligned} \angle(\xi,\xi') &= \lim_{i\to\infty} (\pi - \angle_{p_i}(c'(i),x_0) - \angle_{c'(i)}(p_i,x_0)) \\ &\leq \pi/2 - \lim_{i\to\infty} \angle_{c'(i)}(p_i,x_0) \leq \pi/2. \end{aligned}$$

It remains to analyze the case $\angle(\xi',\xi) = \pi/2$. If p_i is a bounded sequence then as above

$$\pi/2 \geq \angle_p(\xi',\xi) \geq \limsup_i \angle_{p_i}(c'(i),\xi) \geq \pi/2$$

and then [7, Cor. II.9.9] shows that x_0, c, and c' define a flat sector. The converse is trivial: $p_i = x_0$. ■

Proposition 7 *Assume X is a complete CAT(0)-space and $\overline{X}$ is the visual bordification. Then $S(\xi) = S^{x_0}(\xi) = \{\eta : \angle(\eta,\xi) \leq \pi/2\}$ for every $\xi \in \partial X$.*

Proof. Consider two rays c_1 and c_2 from x_0 representing ξ and η respectively. Assume that the projections of $c_2(i)$ onto c_1 are unbounded. Since by definition projections realize the shortest distance, we then have that for any neighborhood V of ξ and for every large enough i (so that $p_i \in V$) that $d(c_2(i), V) \leq d(c_2(i), p_i) \leq d(c_2(i), x_0)$. In the case $\angle(\eta, \xi) = \pi/2$ and c_1, c_2, and x_0 define a flat sector, then by euclidean geometry V contains a point ξ' with $\angle(\xi', \eta) < \pi/2$. In view of Lemma 6 we hence have $\{\eta : \angle(\eta, \xi) \leq \pi/2\} \subset S^{x_0}(\xi)$.

Assume $\angle(\xi, \eta) > \pi/2$ and given $C > 0$. By definition there is a point y such that $\angle_y(\xi, \eta) > \pi/2$. By continuity ([7, Prop. II.9.2.(1)]) we can find neigborhoods V of ξ and U of η in $\overline{X}$ such that $\angle_y(z, w) \geq \pi/2 + \theta$ for every $z \in U$, $w \in V$ and some $\theta > 0$. Further we make V smaller (if necessary) so that $d(y, V)|\cos(\pi/2 + \theta)| \geq d(x_0, y) + C'$ for some $C' > C$. For any $w \in V \cap X$, $z \in U \cap X$ we have by the cosine inequality (i.e. comparison with the euclidean cosine law):

$$\begin{aligned} d(z,w)^2 &\geq d(y,z)^2 + d(y,w)^2 - 2d(y,z)d(y,w)\cos\angle_y(z,w) \\ &\geq d(y,z)^2 + d(y,w)^2 + 2d(y,z)d(y,w)|\cos(\pi/2+\theta)| \\ &\geq d(y,z)^2 + (d(x_0,y)+C')^2 + 2d(y,z)(d(x_0,y)+C') \\ &= (d(x_0,y)+C'+d(y,z))^2 \end{aligned}$$

which implies that $d(z, w) > d(z, x_0) + C'$ by the triangle inequality. Therefore $d(z, V) > d(z, x_0) + C$ for all $z \in U \cap X$ and it follows that $\eta \notin S(\xi)$ as desired. ■

2.3.4 Kobayashi's metric on bounded domains

Theorem 8 *Let X be a bounded domain in $\mathbb{C}^n$ with C^2-smooth boundary equipped with Kobayashi's metric. If ξ_1 and ξ_2 are two distinct boundary points at which X is strictly pseudoconvex, then $s(\xi_1, \xi_2) \geq 2$.*

Proof. Combining [24, Thm. 4.5.8] with an estimate due to Forstneric-Rosay, cf. [24, Cor. 4.5.12], one has for some constant C and fixed x_0,

$$d(z_1, z_2) \geq C + d(z_1, x_0) + d(z_2, x_0)$$

for all z_1 (resp. z_2) sufficiently close to ξ_1 (resp. ξ_2). Hence $\xi_1 \notin S(\xi_2)$ and $\xi_2 \notin S(\xi_1)$. ■

Corollary 9 *Let X be a strictly pseudoconvex bounded domain with C^2-boundary equipped with Kobayashi's metric. Then $S(\xi) = S^{x_0}(\xi) = \{\xi\}$ for every $\xi \in \partial X$.*

The corollary is consistent with Proposition 4 in view of the δ-hyperbolicity of these domains ([5]). From an estimate of Diederich-Fornaess and some additional arguments it is possible to prove:

Theorem 10 *Let X be a bounded pseudoconvex domain with real-analytic boundary and equipped with Kobayashi's metric. Then $S(\xi) = S^{x_0}(\xi) = \{\xi\}$ for every $\xi \in \partial X$.*

In the case when X is a convex C^2-domain, Abate essentially proved that the Kobayashi stars $S(\xi) \subset Star(\xi)$ (the latter set is defined above). What are the stars for a general bounded pseudoconvex domain with Kobayashi's metric? (Recall that Hilbert's metric is an analogous metric and Teichmüller metric is another example.)

2.3.5 Teichmüller space

Let M be a closed surface of genus $g \geq 2$ and let $\mathcal{S}$ be the set of homotopy classes of simple closed curves on M. Denote by $i(\alpha, \beta)$ the minimal number of intersection of representatives of $\alpha, \beta \in \mathcal{S}$. Let $\mathcal{MF}$ (resp. $\mathcal{PMF}$) be the set of (resp. projective equivalence classes of) measured foliations, which coincides with the closure of the image of the embedding

$$\alpha \mapsto i(\alpha, \cdot)$$

of $\mathcal{S}$ into $\mathbb{R}^{\mathcal{S}}$ (resp. $P\mathbb{R}^{\mathcal{S}}$). The intersection number i extends to a bihomogeneous continuous function on $\mathcal{MF} \times \mathcal{MF}$. A foliation $F \in \mathcal{PMF}$ is called *minimal* if $i(F, \alpha) > 0$ for every $\alpha \in \mathcal{S}$. The Teichmüller space $\mathcal{T}$ of M is embedded into $P\mathbb{R}^{\mathcal{S}}$ by the hyperbolic length function. (Below, V_ϕ stands for the vertical foliation and h_ϕ the horizontal length associated to a quadratic differential ϕ, see [18] for more details.)

Lemma 11 *Let ϕ_n be the quadratic differential corresponding (in the Teichmüller embedding with reference point x_0) to $x_n \in \mathcal{T}$. Assume $\phi_n \to \phi_\infty$, a norm one quadratic differential, and $x_n \to F$ in $\mathcal{PMF}$. Whenever $\beta_n \in \mathcal{S}$ such that $Ext_{x_n}(\beta_n) < D$ and $\beta_n \to H$ in $\mathcal{PMF}$, it holds that*

$$i(V_{\phi_\infty}, H) = 0 = i(F, H).$$

Proof. A proof analysis shows that this is proved in [25]: Denote by ψ_n the terminal differential corresponding to the Teichmüller map from x_0 to x_n. Since

$$h_{\psi_n}(\beta_n) \leq Ext_{x_n}(\beta_n)^{1/2} < D^{1/2},$$

$x_n \to \infty$ in $\mathcal{T}$, and in view of the stretching of the Teichmüller map ($h_{\psi_n} = e^{d(x_0,x_n)} h_{\phi_n}$) we see that

$$\lim_{n\to\infty} i(V_{\phi_n}, \beta_n) = \lim_{n\to\infty} h_{\phi_n}(\beta_n) = 0.$$

Since β_n is a sequence in $\mathcal{S}$ converging to H in $\mathcal{PMF}$, there is a sequence λ_n of bounded positive scalars such that $\lambda_n \beta_n \to H$ in $\mathcal{MF}$. By continuity and homogeneity of i we have $i(V_{\phi_\infty}, H) = 0$.

For the second equality note that it is known that $x_n \to F$ in $\mathcal{PMF}$ implies that there is a sequence $r_n \to 0$ such that $i(r_n x_n, \cdot) \to i(F, \cdot)$ in $\mathcal{MF}$. From definitions we also have

$$i(x_n, \beta_n) \leq A_g^{1/2} Ext_{x_n}(\beta_n)^{1/2} < A_g^{1/2} D^{1/2},$$

which by the same argument as before now also shows the second equality. ∎

The following result can be viewed as a generalization of Lemma 1.4.2 in [18] (there seems to be a misprint in their statement however) and is obtained by the same method of proof.

Theorem 12 *Let X be the Teichmüller space of a compact surface and equipped with the Teichmüller metric d. Let $\overline{X}$ be the Thurston compactification $X \cup \mathcal{PMF}$. For $F \in \mathcal{PMF}$, a minimal foliation, we have*

$$S(F) \subset \{G : i(F, G) = 0\}.$$

Proof. Given $y_n \to G \in S^{x_0}(F)$, select $x_n \to F$ such that $d(y_n, x_n) \leq d(y_n, x_0) + C$ for all n and some $C > 0$. From continuity and Mumford compactness, it is a fact that sequences β_n as in Lemma 11 corresponding to x_n always exist. Assume now that F is minimal. It is then known (due to Rees) that, $i(F, G) = 0$ if and only if G is minimal and equivalent to F. Hence V_{ϕ_0}, F and H as in Lemma 11 are all equivalent minimal foliations. Fix these. Note that $\lambda_n \to 0$ here because of the minimality. Let θ_n (resp. ψ_n) denote the initial (resp. terminal) quadratic differential of the Teichmüller map from x_0 to y_n. We have

$$\begin{aligned} i(V_{\theta_n}, \lambda_n \beta_n) &= \lambda_n h_{\theta_n}(\beta_n) \\ &= \lambda_n e^{-d(y_n, x_0)} h_{\psi_n}(\beta_n) \\ &\leq \lambda_n e^{-d(y_n, x_0)} D e^{d(y_n, x_n)} \to 0, \end{aligned}$$

where the last inequality follows from Kerckhoff's formula for Teichmüller distances. Thus $i(V_{\theta_0}, H) = 0$, which implies what we want, since $i(F, G) =$

0 is an equivalence relation for minimal foliations and because of Lemma 11. Finally since the set on the right in the proposition is closed, we have $i(F,G) = 0$ for all $G \in S(F)$. ■

Corollary 13 *If $F \in \mathcal{UE}$ (which is a subset of full Lebesgue measure), then $S(F) = S^{x_0}(F) = \{F\}$.*

Conjecturally, for any $F \in \mathcal{PMF}$, it holds that $S(F) = \{G : i(F,G) = 0\}$.

3 Dynamics of isometries

3.1 Definitions

Let X be a metric space. By an *isometry* we here mean a distance preserving bijection. If the action of the isometries of X extends to an action by homeomorphisms of $\overline{X}$ we call the bordification an *Isom*(X)*-bordification*. Note that every proper metric space has a (typically nontrivial) metrizable Isom(X)-compactification by horofunctions ([4], [2]).

A subset D of isometries is called *bounded* (resp. *unbounded*) if Dx_0 is a bounded (resp. unbounded) set. A single isometry g is called *bounded* (resp. *unbounded*) if $\{g^n\}_{n>0}$ is bounded (resp. unbounded). Note that these definitions are independent of x_0.

Under the assumption that $\overline{X}$ is an Isom(X)-bordification, the isometries of X act on the stars $S(\xi)$ as can be seen from:

$$\begin{aligned} gH(W,C) &= \{z : d(g^{-1}z, W) \leq d(g^{-1}z, x_0) + C\} \\ &= \{z : d(z, gW) \leq d(z, gx_0) + C\}, \end{aligned}$$

which is included in $H(gW, C+d(x_0, gx_0))$ and contains $H(gW, C-d(x_0, gx_0))$. Hence we have $gS(\xi) = S(g\xi)$ and it is plain that g preserves star distances. Note that we also have an action on the faces.

3.2 A contraction lemma

The following observation lies behind the construction of Dirichlet fundamental domains (see e.g. [28]): For any isometry g it holds that

$$g(H_x^{g^{-1}y}) = H_{gx}^y.$$

This leads to a contraction lemma, which in spite of its simplicity and fundamental nature, we have not been able to locate in the literature:

Lemma 14 *Let g_n be a sequence of isometries such that $g_n x_0 \to \xi^+$ and $g_n^{-1} x_0 \to \xi^-$ in a bordification $\overline{X}$ of X. Then for any neighborhoods V^+ and V^- of ξ^+ and ξ^- respectively, there exists $N > 0$ such that*

$$g_n(X \setminus H(V^-)) \subset H(V^+)$$

for all $n \geq N$.

Proof. Given neighborhoods V^+ and V^- as in the statement, by assumption there is an N such that $g_n x_0 \in V^+$ and $g_n^{-1} x_0 \in V^-$ for every $n \geq N$. For any $z \in X$ outside $H(V^-)$, so $d(z, v) > d(z, x_0)$ for every $v \in V^-$, we have

$$d(g_n z, V^+) \leq d(g_n z, g_n x_0) = d(z, x_0) < d(z, g_n^{-1} x_0) = d(g_n z, x_0)$$

for every $n \geq N$. ■

Here is a version of the contraction phenomenon when the isometries act on the boundary:

Proposition 15 *Assume that $\overline{X}$ is an Isom(X)-compactification. Let g_n be a sequence of isometries such that $g_n x_0 \to \xi^+$ and $g_n^{-1} x_0 \to \xi^-$ in $\overline{X}$. Then for any $z \in \overline{X} \setminus S^{x_0}(\xi^-)$,*

$$g_n z \to S^{x_0}(\xi^+).$$

Moreover, the convergence is uniform outside neighborhoods of $S^{x_0}(\xi^-)$.

Proof. Since z does not belong to $S^{x_0}(\xi^-)$ there is some neighborhood V^- of ξ^- such that $z \notin \overline{H(V^-)}$. As the latter is a closed set, there is an open neighborhood U of z disjoint from $\overline{H(V^-)}$. Given a neighborhood V^+ of ξ^+ we therefore have for all sufficiently large n that $g_n(U \cap X) \subset H(V^+)$ for all $n > N$. Since g_n are homeomorphisms we have that $g_n z \subset \overline{H(V^+)}$ as required. The proposition now follows in view of Lemma 1. ■

3.3 Individual isometries

Let g be an isometry of X and let

$$a_n = d(g^n x_0, x_0).$$

A subsequence $n_i \to \infty$ is called *special* for g if there is a constant $C \geq 0$ such that $a_{n_i} > a_m - C$ for all i and $m < n_i$. Note that being special clearly passes to subsequences and by the triangle inequality it is independent of x_0 (see (1) below) and invariant under the shift $\{n_i\} \mapsto \{n_i + N\}$ for fixed integer N.

Let $A^{x_0}(g)$ denote the limit points of $g^n x_0$ along the special subsequences. The *characteristic set* $F(g)$ of g is the face of $A^{x_0}(g)$.

Proposition 16 *Assume that $\overline{X}$ is a sequentially compact bordification of X and g an isometry of X. Then*

$$\overline{\{g^n x_0\}_{n>0}} \cap \partial X \subset \{\eta : F(g) \in S(\eta)\}.$$

If in addition $\overline{X}$ is star-reflexive, then

$$\overline{\{g^n x_0\}_{n>0}} \cap \partial X \subset \bigcap_{\xi \in A^{x_0}(g)} S(\xi).$$

Proof. Suppose g is unbounded and let n_i be a special sequence for g (it is a simple fact, see [19], that special subsequences exist if and only if g is unbounded) and such that $g^{n_i} x_0$ converges to some point $\xi \in \partial X$. Observe that for any positive $k < n_i$ it holds that

$$d(g^{n_i} x_0, g^k x_0) = d(g^{n_i - k} x_0, x_0) = a_{n_i - k} < a_{n_i} + C = d(g^{n_i} x_0, x_0) + C.$$

Now suppose we have a convergent sequence $g^{k_j} x_0 \to \eta \in \partial X$, which means that given a neighborhood V of η, we can find j large so that $g^{k_j} x_0 \in V$. Now from the above inequality we get that for all large enough i

$$g^{n_i} x_0 \in H(\{g^{k_j} x_0\}, C) \subset H(V, C).$$

Therefore $\xi \in \overline{H(V,C)}$ and since V was an arbitrary neigborhood we have $\xi \in S(\eta)$. (Note that in particular this means that $A^{x_0}(g)$ and $F(g)$ are nonempty.) Finally, assuming star-rexlexivity we have showed that $\eta \in S(\xi)$ for every special limit point ξ. ■

Proposition 17 *Assume that $\overline{X}$ is a sequentially compact bordification of X. The subset $F(g) \subset \partial X$ is canonically associated to an isometry g. It is empty if and only if g is bounded.*

Proof. From the triangle inequality we get

$$|d(g^k x, x) - d(g^k x_0, x_0)| \leq 2d(x, x_0), \tag{1}$$

which implies in view of Lemma 2 that $F(g)$ is independent of x_0. The last part follows from Proposition 16. ■

Proposition 18 *Let g be an (unbounded) isometry and $\overline{X}$ an Isom(X)-bordification. Then for every limit point $\xi \in \partial X$ of the orbit it holds that g fixes the corresponding star, that is, $S(g\xi) = S(\xi)$ and the corresponding face, that is, $F(g\xi) = F(\xi)$. Moreover, $F(g)$, when it exists, is also fixed by g.*

Proof. Since by continuity

$$g\xi = g(\lim_{k\to\infty} g^{n_k} x_0) = \lim_{k\to\infty} g^{n_k}(gx_0)$$

we have that $S(g\xi) = S(\xi)$ in view of Lemma 2. If $\xi \in S(\eta)$, then $g\xi \in S(g\eta)$ and again we have $\xi \in S(g\eta)$. Since g is a bijection, the final part of the proposition follows. ■

4 Groups of isometries

4.1 Generalizations of Hopf's theorem on ends

The following extends Hopf's theorem that the number of ends of a finitely generated group is either 0, 1, 2, or ∞:

Proposition 19 *Assume that $\overline{X}$ is a sequentially compact Isom(X)-bordification. Let G be a group of isometries fixing a finite set $F \subset \partial X$, that is, $GF = F$. If F is not contained in two stars, then G is bounded.*

Proof. By passing to a finite index subgroup (which does not effect the boundedness) we can assume that G fixes F pointwise. Now suppose there is a sequence g_n in G such that $g_n^{\pm 1} x_0 \to \xi^{\pm} \in \partial X$. Then F must be contained in $S(\xi^+) \cup S(\xi^-)$ since otherwise there is is a point in F which on the one hand should be contracted towards $S(\xi^+)$ under g_n, but on the other hand it is fixed by G. ■

To see how this implies Hopf's theorem: If two boundary points belong to different ends, then their stars are disjoint. So if one has a finitely generated group with finite number of ends, then applying the proposition with F being the set of ends, one obtains that the number of ends must be at most two.

By the same method of proof:

Proposition 20 *Assume that $\overline{X}$ is a sequentially compact Isom(X)-bordification. Let G be a group of isometries which fixes some collection of stars S_i in the sense that $GS_i = S_i$ for every i. Suppose that for any two arbitrary stars, there is always an i such that S_i is disjoint from these two stars. Then G is bounded.*

These two statements can perhaps be useful to rule out the existence of compact quotients of certain Riemannian manifolds or complex domains.

4.2 Commuting isometries

The proof of Proposition 18 in fact shows the following:

Proposition 21 *Let g be an isometry and $\overline{X}$ an Isom(X)-bordification. Suppose that $g^{n_i}x_0 \to \xi \in \partial X$ and let $Z(g)$ denote the centralizer of g in Isom(X). Then $Z(g)S(\xi) = S(\xi)$, $Z(g)F(\xi) = F(\xi)$, and $Z(g)F(g) = F(g)$ (when it exists).*

4.3 Free subgroups

Proposition 22 *Assume that $\overline{X}$ is compact. Let g and h be two isometries such that $g^{\pm n_k} \to \xi^{\pm} \in \partial X$, $h^{\pm m_l} \to \eta^{\pm} \in \partial X$ for some subsequences n_k and m_l. Assume that $S(\xi^+) \cup S(\xi^-)$ and $S(\eta^+) \cup S(\eta^-)$ are disjoint. Then the group generated by g and h contains a noncommutative free subgroup.*

Proof. By a compactness argument (similar to that in the proof of Lemma 1) we can find large enough K such that

$$H(\{g^{n_k}x_0\}_{k>K}) \cup H(\{g^{-n_k}x_0\}_{k>K})$$

and

$$H(\{h^{m_l}x_0\}_{l>K}) \cup H(\{h^{-m_l}x_0\}_{l>K})$$

are disjoint. From the contraction observations in subsection 3.2 and the usual freeness criterion ([15]), the proposition is proved. ∎

By a similar proof one has:

Proposition 23 *Assume that $\overline{X}$ is compact. Let g and h be two isometries such that $g^{\pm n_k} \to \xi^{\pm} \in \partial X$, $h^{\pm m_l} \to \eta^{\pm} \in \partial X$ for some subsequences n_k and m_l. Assume that $S(\xi^+)$, $S(\eta^+)$ and $S(\xi^-) \cup S(\eta^-)$ are disjoint. Then the group generated by g and h contains a noncommutative free semigroup.*

4.4 A metric Furstenberg lemma

The following can be viewed as an analog of Furstenberg's lemma (see [32, 3.2.1]):

Lemma 24 *Assume that $\overline{X}$ is a metrizable Isom(X)-compactification such that $S(\xi) = S^{x_0}(\xi)$ for every $\xi \in \partial X$. Let $g_n \in Isom(X)$ and μ, ν be two probability measures on ∂X. Suppose that $g_n\mu \to \nu$ (in the standard weak topology). Then either g_n is bounded or the support of ν is contained in two stars.*

Proof. We assume that g_n is unbounded and by compactness we select a subsequence so that $g_n x_0 \to \xi^+$, $g_n^{-1}x_0 \to \xi^-$, and $g_n\xi^- \to \xi$. We then have that $g_n S(\xi^-) \to S(\xi)$ in view of Lemma 3. Write $\mu = \mu_1 + \mu_2$ where $\mu_1(\partial X \setminus S(\xi^-)) = 0$ and $\mu_2(S(\xi^-)) = 0$. By compactness we can further assume that $g_n\mu_i \to \nu_i$ and $\nu = \nu_1 + \nu_2$. Since μ_1 is supported on $S(\xi^-)$, it follows that ν_1 is supported on $S(\xi)$. Suppose that f is a continuous function vanishing on $S(\xi^+)$. Then

$$\int f(\eta)d\nu_2 = \lim_{n\to\infty}\int f(\eta)d(g_n\mu_2) = \lim_{n\to\infty}\int f(g_n\eta)d\mu_2 = 0$$

by the dominated convergence theorem in view of Proposition 15. Hence we have showed that supp$\nu \subset S(\xi) \cup S(\xi^+)$ as required. ∎

This lemma might be useful for analyzing amenable groups of isometries (let $\mu = \nu$ be an invariant measure). For example, it could provide an alternative approach to a theorem of Burger-Schroeder [8] extended by Adams-Ballmann [1] dealing with CAT(0)-spaces.

4.5 Random walks

Let (Ω, ν) be a measure space with $\nu(\Omega) = 1$ and L a measure preserving transformation. Given a measurable map $w : \Omega \to \mathrm{Isom(X)}$ we let

$$u(n, \omega) = w(\omega)w(L\omega)...w(L^{n-1}\omega).$$

Let $a(n, \omega) = d(x_0, u(n, \omega)x_0)$ and assume that

$$\int_\Omega a(1, \omega)d\nu(\omega) < \infty.$$

For a fixed ω we call a subsequence $n_i \to \infty$ *special* for ω if there is a constant C such that $a(n_i, \omega) > a(m, L^{n_i - m}\omega) - C$ for all i and $m < n_i$.

Let $F(\omega)$ denote the face of all limit points of $u(n,\omega)x_0$ in ∂X along special subsequences.

Theorem 25 *Assume that $\overline{X}$ is a metrizable Isom(X)-compactification. Suppose that*

$$\liminf_{n\to\infty} \frac{1}{n} \int_\Omega a(n,\omega) d\nu(\omega) > 0.$$

Then for a.e. ω,

$$u(n,\omega)x_0 \to \{\eta : F(\omega) \subset S(\eta)\},$$

and the a.e. defined assignment $\Psi : \omega \mapsto F(\omega)$ has the property that $w(\omega)F(L\omega) = F(\omega)$.

Proof. Proposition 4.2 in [20] guarantees that special subsequences exist for a.e. ω. From this point on, the first part of the theorem is proved in the same way as Proposition 16. The second part is proved by noting that if $\{n_i\}$ is special for ω, then $\{n_i - 1\}$ is special for $L\omega$ (cf. [20, p. 117]). ■

Let $\mathcal{S}$ be the space of closed nonempty subsets of ∂X with Hausdorff's topology and denote by $\mathcal{F}$ the closure in $\mathcal{S}$ of the set of nonempty faces of ∂X. Specializing to the case when $u(n,\cdot)$ is a random walk we have:

Corollary 26 *Let μ be a probability measure on a discrete group of isometries Γ. In the case $(\Omega,\nu) = \prod_{-\infty}^{\infty}(\Gamma,\mu)$ with L being the shift, and under the assumptions of the theorem, the measure space $(\mathcal{F},\Psi_*(\nu))$ is a μ-boundary of Γ.*

Proof. Consider the path space $\Gamma^{\mathbb{Z}_+}$ with the induced probability measure $\mathbb{P}$ from the random walk defined by μ starting at e. Note that Γ naturally acts on $\mathcal{F}$. The map Ψ gives rise to a map Π defined on the path space rather than Ω. This measurable map is time shift invariant (special subsequences are independent of the base point x_0) and Γ-equivariant from the theorem. Now see [17, 1.5]. ■

In several situations, e.g. when X is δ-hyperbolic or CAT(0) (under some reasonable conditions) the μ-boundary obtained in the corollary is in fact isomorphic to the Poisson boundary, see [17] and [20].

4.6 Discrete groups

An isometric action of a group Γ is (*metrically*) *proper* if for every $x \in X$ and every closed ball B centered at x, the set $\{g \in \Gamma : gx \in B\}$ is finite. A pair of stars S_1 and S_2 are *maximal* if the only union of two stars containing them is $S_1 \cup S_2$.

Lemma 27 *Assume that $\overline{X}$ is a Hausdorff Isom(X)-compactification and that ∂X is not the union of two stars. Suppose that g and h are two unbounded isometries generating a proper action and that $h^{\pm n_j} x_0 \to \xi^{\pm}$ with $S(\xi^+)$ and $S(\xi^-)$ disjoint and maximal. If g fixes $S(\xi^-)$, then $h^k = gh^l g^{-1}$ for two nonzero integers k and l, and g fixes a star contained in $S(\xi^+)$.*

Proof. (Cf. [22].) Since $\overline{X}$ is a compact Hausdorff space we can find two disjoint neighborhoods U^+ and U^- of $S(\xi^+)$ and $S(\xi^-)$ respectively, so that $E := \overline{X} \setminus (U^+ \cup U^-)$ is nonempty and not contained in X. Since g is a homeomorphism fixing $S(\xi^{\pm})$ we can moreover suppose that

$$hU^- \cap U^+ = \emptyset. \tag{2}$$

Because h^{-n_j} contracts toward $S(\xi^-)$ (Proposition 15) and g is a homeomorphism fixing $S(\xi^-)$ we have that

$$gh^{-n_j}(E) \subset U^-$$

for all large j. In view of (2) we can find a $k = k(j)$ such that $h^{k(j)} g h^{-n_j} E \cap E$ is nonempty. Let $g_j = h^{k(j)} g h^{-n_j}$. Note that

$$g_j S(\xi^-) = S(\xi^-) \tag{3}$$

and since $g_j S(\xi^+) = h^{k(j)} g S(\xi^+)$, $gS(\xi^+) \cap gS(\xi^-) = \emptyset$, and $k(j) \to \infty$,

$$g_j S(\xi^+) \to S(\xi^+). \tag{4}$$

In view of (3), (4), and the assumptions on $S(\xi^{\pm})$ we have that if $g_{j_k}^{\pm} x_0 \to \eta^{\pm} \in \partial X$, then either $S(\eta^{\pm}) = S(\xi^{\pm})$ or $S(\eta^{\pm}) = S(\xi^{\mp})$. In either case this contradicts that $g_j E \cap E$ is nonempty for all large j. Therefore g_j is bounded and by properness we have $g_j = g_i$ for many i, j different. This means that $h^k = gh^l g^{-1}$ for two nonzero integers k and l. Hence

$$h^k S(g\xi^+) = gh^l g^{-1} g S(\xi^+) = gS(\xi^+) = S(g\xi^+)$$

and we conclude that $S(g\xi^+) \subset S(\xi^+)$, since $gS(\xi^-)$ equals all of $S(\xi^-)$. ∎

It is instructive to compare Lemma 27 with the case of a Baumslag-Solitar group $< g, h : h^k = gh^l g^{-1} >$ acting on its Cayley graph.

An *axis* of an isometry is an invariant geodesic line on which the isometry acts by translation. We say that an isometry h fixes an endpoint of a geodesic line c if there is a $C > 0$ such that $d(h(c(t)), c(t)) < C$ for all $t > 0$ or all $t < 0$.

Proposition 28 *Let g, h be two isometries generating a group which acts properly on X. Assume that g has an axis c and that h fixes an endpoint of c. Then $[h, g^N] = 1$ for some $N > 0$.*

Proof. Letting $x_0 = c(0)$ we have that:

$$d(x_0, g^{-n}hg^n x_0) = d(g^n x_0, hg^n x_0) = d(c(nd_g), hc(nd_g)) < C$$

for all $n > 0$ (or $n < 0$). As the action of the group is proper, we must then have that for some $m \neq n$

$$g^{-m}hg^m = g^{-n}hg^n$$

or in other words there is a number $N > 0$ such that $h = g^{-N}hg^N$. ■

5 Some consequences

A few comparisons with previous results were pointed out along the way. Here are some further remarks:

5.1 Hyperbolic bordifications

In this situation, the previous two sections merely provide an alternative exposition of well-known facts, see e.g. [7] for the theory of ends, [28] for classical hyperbolic geometry, [12] for hyperbolic groups, and [22] for non-locally compact spaces.

5.2 Hilbert's metric

For these metric spaces, it seems that several results obtained in this paper cannot be found in the literature. An interesting example is the planar domain bounded by a triangle which has a transitive automorphism group. Proposition 19 gives a criterion in terms of the simplicial diameter for when the automorphism group of a proper convex cone is small. For example, it implies that the automorphism group of a planar domain bounded by an n-gon is bounded if $n \geq 7$. However, in dimension 2 a rather complete result can be found in [14]. The literature on symmetric or homogeneous cones is vast. For recent works on cones where the automorphism group admits a cocompact lattices, see [6], and see [27] for an instance where Hilbert's metric can be a tool in the study of Coxeter groups.

5.3 CAT(0)-spaces

All results in section 4 specialized to the CAT(0)-setting seems to be new except Theorem 25, Propositions 21 and 28. Moreover, in view of Propositions 7 and 15 (or their proofs in the non-proper case) we have:

Theorem 29 *Let X be a complete CAT(0)-space. Let g_n be a sequence of isometries such that $g_n x_0 \to \xi^+ \in \partial X$ and $g_n^{-1} x_0 \to \xi^- \in \partial X$. Then for any $\eta \in \overline{X}$ with $\angle(\eta, \xi^-) > \pi/2$ we have that*

$$g_n \eta \to \{\zeta : \angle(\xi^+, \zeta) \leq \pi/2\}$$

(in the sense that $\limsup \angle(\xi^+, g_n\eta) \leq \pi/2$ *when X is not proper). Assuming that X is proper, the convergence is uniform outside neighborhoods of $S(\xi^-)$.*

Applied to iterates of a single isometry $g_n := h^{k_n}$, the theorem partially extends (since it also deals with parabolic isometries) a lemma of Schroeder [4] generalized by Ruane [29] to include also singular CAT(0)-spaces.

Combining Propositions 7 and 22 yields the following result which generalizes the main theorem in [29] (because no group is here assumed to act cocompactly and properly):

Theorem 30 *Let X be a proper CAT(0)-space. If g and h are two unbounded isometries with limit points ξ^-, ξ^+ and η^-, η^+ respectively (not necessarily all distinct), with $Td(\{\xi^\pm\}, \{\eta^\pm\}) > \pi$, then the group generated by g and h contains a noncommutative free subgroup.*

Compare the following consequence of Proposition 28 to [3, Lemma 4.5]:

Proposition 31 *Let X be a complete CAT(0)-space and g a hyperbolic isometry with an axis c. Assume that h is an isometry which fixes one endpoint of c and that g and h generate a group acting properly. Then h fixes both endpoints of c.*

Proof. From Proposition 28 we have $h = g^{-N} h g^{N}$. Therefore

$$h(c(\pm\infty)) = \lim_{n\to\infty} h g^{\pm nN} x_0 = \lim_{n\to\infty} g^{\pm nN} h x_0 = c(\pm\infty).$$

■

Note that [20] implies a partial converse of the main theorem in [1]: if a group Γ acts on a CAT(0)-space with finite critical exponent and the limit set is countable, then Γ is amenable. In addition, our metric Furstenberg lemma together with [20] could give a generalization of Theorem 1 in [8], and provide a construction of boundary maps as a first step towards superrigidity. We hope to return to these matters elsewhere.

5.4 Holomorphic maps

We obtain the following new Wolff-Denjoy type theorem (cf. [19], [24]):

Theorem 32 *Let X be a C^2 bounded domain in $\mathbb{C}^n$, $f : X \to X$ a holomorphic map, and d the Kobayashi distance. Then $F(f)$ contains at most one point of strong pseudoconvexity. If in addition X is strictly pseudoconvex (or real analytic, pseudoconvex), then for any z either $f^m(z)$ stays away from ∂X ($F(f) = \emptyset$), or $\lim_{m\to\infty} f^m(z) = \xi$ for some $\xi \in \partial X$ ($F(f) = \{\xi\}$).*

Proof. In the proofs of Propositions 16 and 17 we in fact only need that $d(gx, gy) \leq d(x, y)$ for all $x, y \in X$, which holds for holomorphic maps. It is known ([24, Cor. 4.1.12]) that (X, d) is proper in the strictly pseudoconvex case, therefore we can use [9] to guarantee that, unless the orbit $\{f^n(z)\}_{n>0}$ is bounded, it accumulates only in ∂X. The rest is now clear in view of Theorem 8. ■

5.5 Mapping class groups

Although the arguments in this paper provide (especially if all stars of the Teichmüller spaces can be identified) an alternative explanation of some theorems on the mapping class groups of surfaces obtained notably in [16] and [26], it might however be preferable to study the action directly on the Thurston boundary (or the curve complex) as is done in those works. It is conceivable, see subsection 2.3.5, that the set of simple closed curves $\mathcal{S} \subset \mathcal{PMF}$ with the star-distance restricted to it is in fact exactly the curve complex.

Note that pseudo-Anosov elements of the mapping class groups, hyperbolic isometries of a δ-hyperbolic space, and rank 1 isometries (see [2], [3]) of a CAT(0)-space, are all examples of a *strictly hyperbolic isometry*, that is, an isometry g for which $g^n x_0 \to \xi^+$ and $g^{-n} x_0 \to \xi^-$ as $n \to \infty$, such that $S(\xi^+) = \{\xi^+\}$ not equal to $S(\xi^-) = \{\xi^-\}$.

5.6 Groups with a word metric

The observation in subsection 3.2 together with the ping-pong lemma ([15]) yields the following freedom criterion: Let g and h be two elements of order at least 3 in a group Γ with word metric $||\cdot||$ and let Λ be the subgroup generated by g and h. If for any $a \in \Lambda$ at least one of $||ag^{\pm 1}||$ and $||ah^{\pm 1}||$ is strictly greater than $||a||$, then $\Lambda \cong F_2$. (There is a similar criterion for free semigroups.)

Let Γ be a finitely generated group, μ the uniformly distributed probability measure on a finite generating set A, X the Cayley graph associated to A and $\overline{X}$ the horofunction compactification. If Γ is nonamenable, then Corollary 26 provides a (probably often nontrivial) μ-boundary for (Γ, μ).

Let Γ denote a finitely generated group with a boundary $\partial\Gamma$. Consider the incidence geometry (cf. [31]) defined by the points and the stars in $\partial\Gamma$, and acted upon by Γ. In general or for some specific group, what can this geometry be? In view of section 3, for example torsioness, subexponential growth, or amenability of Γ implies strong restrictions. It may be interesting to extend the existing theory of convergence groups to the more general setting where one has nontrivial incidence geometry mixed in. A first instance of what we essentially have in mind are the 'biconvergence groups' to be found in [11].

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