Limits of Baumslag-Solitar groups and other families of marked groups with parameters

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Abstract

We give a description and some algebraic properties of groups obtained as limit of Baumslag-Solitar groups marked with a canonical set of generators. We discuss other examples of converging families of marked groups with parameters.

Introduction

The set of marked groups on k generators (see section 1 for definitions) was given a natural topology which turns it to a compact totally disconnected space. This topology received several names: "topology on marked groups", "Cayley topology", "Grigorchuk topology"... The principle is that two marked groups are close if there are large balls of their Cayley graphs which are isomorphic.

This topology has been used for several purposes. Let us cite the following examples :

- Stepin [Step93] used it to prove the existence of amenable but non elementary amenable groups.
- To prove that every finitely generated Kazhdan group is a quotient of some finitely *presented* Kazhdan group, Shalom proved in [Sh00] that Kazhdan's property (T) defines an open subset of the space of marked groups.
- In [CG04], Champetier et Guirardel gave a characterization of limit groups of Sela in terms of the topology on marked groups.

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There also exists several papers about questions whose formulation involves the topology on marked groups language. Let us cite the following ones :

- In [Ch00], Champetier showed that the quotient of the space of marked groups on k generators by the group isomorphism relation is not a standard Borel space. He also studied the closure of non elementary hyperbolic groups.

- In [St05], the second author gave an almost complete characterization of convergent sequences among Baumslag-Solitar groups.

We are interested in the closure of Baumslag-Solitar groups and its elements, which we study for their own right. Theorem 6 of [St05] allows us to define the following elements of the closure (Definition 1.6):

$$\overline{BS}(m,\xi) = \lim_{n\to\infty} BS(m,\xi_n)$$

where $m \in \mathbb{Z}^*$, $\xi \in \mathbb{Z}_m$, ξ_n is any sequence of integers such that $\xi_n \to \xi$ in $\mathbb{Z}_m$ and $|\xi_n| \to \infty$ (for $n \to \infty$).

Outline of the paper and description of results Section 1 contains the material we want to recall and the definitions of the main groups appearing in the article.

In the spirit of [St05], we treat in Section 2 the problem of convergence of Torus knots groups. The solution happens to be simpler, for the one-parameter sequences are all convergent (Proposition 2.1). We also discuss the cases of Baumslag-Solitar groups with changing markings (Theorem 2.4) and other Baumslag's one-relator groups (Proposition 2.5).

Section 3 achieves the characterization, which was not complete in [St05], of convergent sequences of Baumslag-Solitar groups (Theorem 3.7) and gives a necessary and sufficient condition for marked groups $\overline{BS}(m,\xi)$ and $\overline{BS}(m,\eta)$ to be equal (Corollary 3.10).

In Section 4, we show that the map $\mathbb{Z}_m \to \mathcal{G}_2; \xi \mapsto \overline{BS}(m,\xi)$ is continuous and injective on $\mathbb{Z}_m^\times$.

It is well-known that a Baumslag-Solitar group acts on its Bass-Serre tree by automorphisms and on $\mathbb{Q}$ by affine transformations. Section 5 is devoted to construct actions of $\overline{BS}(m,\xi)$. First $\overline{BS}(p,\xi)$ acts affinely on $\mathbb{Q}_p$ when p is a prime (Theorem 5.1). Second, we construct an action of $\overline{BS}(m,\xi)$ by automorphisms on a tree which is in some sense a "limit" of Bass-Serre trees (Theorem 5.3).

The tree action allows us to prove in Section 6 that the $\overline{BS}(m,\xi)$'s are extensions of free groups by the wreath product $\mathbb{Z} \wr \mathbb{Z}$ (Theorem 6.2), so that they have the Haagerup property (Corollary 6.4).

We finally deal in Section 7 with presentations of the $\overline{BS}(m,\xi)$'s. We show first that almost none of them is finitely presented (Proposition 7.1). Second, we exhibit a presentation, which is again related to the tree action (Theorem 7.4).

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1 Definitions and preliminaries

1.1 The ring of m-adic integers

Let $m \in \mathbb{Z}^*$. As seen in [St05], the ring of m-adic integers $\mathbb{Z}_m$ is the projective limit (in the category of topological rings) of the system

$$\ldots \to \mathbb{Z}/m^h\mathbb{Z} \to \mathbb{Z}/m^{h-1}\mathbb{Z} \to \ldots \to \mathbb{Z}/m^2\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$$

where the arrows are the canonical (surjective) homomorphisms. This shows that $\mathbb{Z}_m$ is compact. We collect now some easy facts about m-adic integers which are useful in following sections.

Proposition 1.1 *Let m be a nonzero integer and let $m = \pm p_1^{k_1} \cdots p_\ell^{k_\ell}$ be its decomposition in prime factors:*

(a) One has an isomorphism of topological rings $\mathbb{Z}_m \cong \mathbb{Z}_{p_1} \oplus \ldots \oplus \mathbb{Z}_{p_\ell}$. In particular, for m not prime, the ring $\mathbb{Z}_m$ has zero divisors.

(b) The group of invertible elements of $\mathbb{Z}_m$ is given by

$$\mathbb{Z}_m^\times = \mathbb{Z}_m \setminus (p_1\mathbb{Z}_m \cup \ldots \cup p_k\mathbb{Z}_m) \ ;$$

(c) Any ideal of $\mathbb{Z}_m$ is principal. Moreover any ideal of $\mathbb{Z}_m$ containing a nonzero integer can be written $p_1^{i_1} \cdots p_\ell^{i_\ell}\mathbb{Z}_m$ with $i_1, \ldots i_\ell \in \mathbb{N}$.

(d) Assume $|m| \geq 2$. For any $i_1, \ldots i_\ell \in \mathbb{N}$, one has $\mathbb{Z} \cap p_1^{i_1} \cdots p_\ell^{i_\ell}\mathbb{Z}_m = p_1^{i_1} \cdots p_\ell^{i_\ell}\mathbb{Z}$.

Proof. Notice first that statements (a)-(d) are trivial in case $|m| = 1$ (i.e. $\ell = 0$). We assume then $|m| \geqslant 2$ in what follows.

(a) Consider the following commutative diagrams (for $h \geqslant 2$):

$$\begin{tikzcd}
\mathbb{Z}/m^h\mathbb{Z} \arrow[r, two heads] \arrow[d, "\cong"] & \mathbb{Z}/m^{h-1}\mathbb{Z} \arrow[d, "\cong"] \\
\bigoplus_{s=1}^{\ell} \mathbb{Z}/p_s^{hk_s}\mathbb{Z} \arrow[r, two heads] & \bigoplus_{s=1}^{\ell} \mathbb{Z}/p_s^{(h-1)k_s}\mathbb{Z} .
\end{tikzcd}$$

Passing to projective limits gives $\mathbb{Z}_m \cong \mathbb{Z}_{p_1} \oplus \ldots \oplus \mathbb{Z}_{p_\ell}$.

(b) For any prime p, it is well known that the group of invertible p-adic integers is $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus p\mathbb{Z}_p$. In view of (a), the conclusion follows obviously.

(c) Let I be an ideal of $\mathbb{Z}_m$. It corresponds by (a) to an ideal of $\mathbb{Z}_{p_1} \oplus \ldots \oplus \mathbb{Z}_{p_\ell}$, which has to have the form $I_1 \oplus \ldots \oplus I_\ell$ where I_s is an ideal of $\mathbb{Z}_{p_s}$. Since it is well known, for any prime p, that the nonzero ideals of $\mathbb{Z}_p$ are exactly the $p^h\mathbb{Z}_p$'s for $h \in \mathbb{N}$, one gets, for every s, $I_s = p_s^{i_s}\mathbb{Z}_{p_s}$ or $I_s = 0$. Set $\xi = (\xi_1, \ldots, \xi_\ell)$ with $\xi_s = p_s^{i_s}$ if $I_s = p_s^{i_s}\mathbb{Z}_{p_s}$ and $\xi_s = 0$ if $I_s = 0$. Set $e_1 = (1, 0, \ldots, 0), \ldots, e_\ell = (0, \ldots, 0, 1)$. The ideal I is obviously generated by $e_1\xi, \ldots, e_\ell\xi$, hence by ξ. Consequently, I is a principal ideal.

Assume now that I contains a nonzero integer k. Since k does not vanish in any $\mathbb{Z}_{p_s}$, one has $I = p_1^{i_1}\mathbb{Z}_{p_1} \oplus \ldots \oplus p_\ell^{i_\ell}\mathbb{Z}_{p_\ell}$ with $i_1, \ldots, i_\ell \in \mathbb{N}$ and $\xi = (p_1^{i_1}, \ldots, p_\ell^{i_\ell})$. Setting $\eta_s = \prod_{t \neq s} p_t^{i_t}$, we obtain an invertible element $\eta = (\eta_1, \ldots, \eta_\ell)$ such that $\eta\xi = p_1^{i_1} \cdots p_\ell^{i_\ell} \cdot (1, \ldots, 1)$. This implies $I = p_1^{i_1} \cdots p_\ell^{i_\ell}\mathbb{Z}_m$.

(d) The inclusion $\supseteq$ is obvious. To show the converse, take $n \in \mathbb{Z} \cap p_1^{i_1} \cdots p_\ell^{i_\ell}\mathbb{Z}_m$. For any s, consider the following sequence of (canonical) morphisms

$$\mathbb{Z} \to \mathbb{Z}_m \to \mathbb{Z}/m^{i_s}\mathbb{Z} \to \mathbb{Z}/p_s^{i_s}\mathbb{Z}$$

whose composition is the canonical projection $\mathbb{Z} \to \mathbb{Z}/p_s^{i_s}\mathbb{Z}$. Since n is in $p_1^{i_1} \cdots p_\ell^{i_\ell}\mathbb{Z}_m$, it is in the kernel of all those maps and thus in $p_1^{i_1} \cdots p_\ell^{i_\ell}\mathbb{Z}$. □

Definition 1.2 *Let m be an integer such that $|m| \geqslant 2$ and let $p_1, \ldots, p_\ell$ be its prime factors. If E is a subset of $\mathbb{Z}_m$ containing a nonzero integer, the* greatest common divisor (gcd) *of the elements of E is the (unique) number $p_1^{i_1} \cdots p_\ell^{i_\ell}$ (with $i_1, \ldots, i_\ell \in \mathbb{N}$) such that the ideal generated by E is $p_1^{i_1} \cdots p_\ell^{i_\ell}\mathbb{Z}_m$.*

If $|m| = 1$, we set by convention $gcd(\mathbb{Z}_m) = 1$.

Lemma 1.3 *Let $m \in \mathbb{Z}^*$ and let m' be a divisor of m. Let us write $m' = \pm p_1^{j_1} \cdots p_{\ell'}^{j_{\ell'}}$ and $m = \pm p_1^{k_1} \cdots p_\ell^{k_\ell}$ their decomposition in prime factors ($\ell' \leqslant \ell$ and $j_s \leqslant k_s$ for all $s = 1, \ldots, \ell'$). Let $\pi : \mathbb{Z}_m \to \mathbb{Z}_{m'}$ the morphism induced by projections $\mathbb{Z}/m^h\mathbb{Z} \to \mathbb{Z}/m'^h\mathbb{Z}$ (for $h \geqslant 1$). Then the following holds:*

(a) One has $\pi(n) = n$ for any integer n;

(b) For any $d = \pm p_1^{i_1} \cdots p_{\ell'}^{i_{\ell'}}$ with $i_1, \ldots, i_{\ell'} \in \mathbb{N}$, one has $\pi^{-1}(d\mathbb{Z}_{m'}) = d\mathbb{Z}_m$.

Proof. Assertion (a) is obvious. The morphism π corresponds by Proposition 1.1 (a) to the projection

$$\bigoplus_{s=1}^{\ell} \mathbb{Z}_{p_s} \twoheadrightarrow \bigoplus_{s=1}^{\ell'} \mathbb{Z}_{p_s} \ .$$

For any d, one has then clearly

$$\pi^{-1}(d\mathbb{Z}_{m'}) = \pi^{-1}\left(\bigoplus_{s=1}^{\ell'} p_s^{i_s}\mathbb{Z}_{p_s}\right) = \bigoplus_{s=1}^{\ell'} p_s^{i_s}\mathbb{Z}_{p_s} \oplus \bigoplus_{s=\ell'+1}^{\ell} \mathbb{Z}_{p_s} = d\mathbb{Z}_m,$$

which proves (b). □

1.2 Marked groups and their topology

Introductory expositions of these topics can be found in [Ch00] or [CG04]. We only recall some basics and what we need in following sections.

The free group on k generators will be denoted by $\mathbb{F}_k$, or F_S (with $S = (s_1, \ldots, s_k)$) if we want to precise the names of (canonical) generating elements. A *marked group on k generators* is a pair (Γ, S) where Γ is a group and $S = (s_1, \ldots, s_k) \in \Gamma^k$ is a family which generates Γ. A marked group (Γ, S) comes always with a canonical epimorphism $\phi : \mathbb{F}_S \to \Gamma$, giving an isomorphism of marked groups between $\mathbb{F}_S/\ker\phi$ and Γ. Hence a class of marked groups can always be represented by a quotient of $\mathbb{F}_S$. In particular if a group is given by a presentation, this defines a marking on it. The nontrivial elements of $\mathcal{R} := \ker\phi$ are called *relations* of (Γ, S). Given $w \in \mathbb{F}_k$ we will often write "$w = 1$ in Γ" or "$w \underset{\Gamma}{=} 1$" to say that the image of w in Γ is trivial.

Let $w = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ be a reduced word in $\mathbb{F}_S$ (with $x_i \in S$ and $\varepsilon_i \in \{\pm 1\}$). The integer n is called the *length* of w and denoted $|w|$. If (Γ, S) is a marked group on k generators, and $\gamma \in \Gamma$ the *length* of γ is

$$\begin{aligned} |\gamma|_\Gamma &:= \min\{n : \gamma = s_1 \cdots s_n \text{ with } s_i \in S \sqcup S^{-1}\} \\ &= \min\{|w| : w \in \mathbb{F}_S,\ \phi(w) = \gamma\} \ . \end{aligned}$$

Let $\mathcal{G}_k$ be the set of marked groups on k generators (up to marked isomorphism). Let us recall that the topology on $\mathcal{G}_k$ comes from the following ultrametric: for $(\Gamma_1, S_1) \neq (\Gamma_2, S_2) \in \mathcal{G}_k$ we set $d\big((\Gamma_1, S_1), (\Gamma_2, S_2)\big) := e^{-\lambda}$ where λ is the length of a shortest element of $\mathbb{F}_k$ which vanishes in one group and not in the other one. But what the reader has to keep in mind is the following characterization of convergent sequences.

Lemma 1.4 *[St05, Proposition 1] Let $(G_n)_{n\geqslant 0}$ be a sequence of marked groups in $\mathcal{G}_k$. The sequence $(G_n)_{n\geqslant 0}$ is converging if and only if for any $w \in \mathbb{F}_k$, we have either $w = 1$ in G_n for n large enough, or $w \neq 1$ in G_n for n large enough.*

The reader could remark that the latter condition characterizes exactly Cauchy sequences. Another useful statement we will use in the paper is the following:

Lemma 1.5 *[CG04, Lemma 2.3] If a sequence $(G_n)_{n\geqslant 0}$ in $\mathcal{G}_k$ is converging to a marked group $G \in \mathcal{G}_k$ which is given by a finite presentation, then, for n large enough, G_n is a marked quotient of G.*

We address the reader to the given references for proofs.

1.3 Notations and conventions

We give now some notations and conventions which hold in the whole paper. First, recall that we define the Baumslag-Solitar groups by

$$BS(m,n) = \big\langle a, b \;\big|\; ab^m a^{-1} = b^n \big\rangle \;\; (m, n \in \mathbb{Z}^*) \;.$$

Then, we define a new family of groups in the following way:

Definition 1.6 *For $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, one defines a marked group on two generators $\overline{BS}(m,\xi)$ by the formula*

$$\overline{BS}(m,\xi) = \lim_{n\to\infty} BS(m,\xi_n)$$

where ξ_n is any sequence of integers such that $\xi_n \to \xi$ in $\mathbb{Z}_m$ and $|\xi_n| \to \infty$ (for $n \to \infty$).

Notice that $\overline{BS}(m,\xi)$ is well defined for any $\xi \in \mathbb{Z}_m$ by Theorem 6 of [St05]. Note also that for any $n \in \mathbb{Z}^*$, one has $\overline{BS}(m,n) \neq BS(m,n)$. Indeed, the word $ab^m a^{-1} b^{-n}$ represents the neutral element in $BS(m,n)$, but not in $\overline{BS}(m,n)$.

When appearing as marked groups, the free group $\mathbb{F}_2 = \mathbb{F}(a,b)$, the Baumslag-Solitar groups and the groups $\overline{BS}(m,\xi)$ are all (unless stated otherwise) supposed to be marked by the pair (a,b).

Another group which plays an important role in this article is

$$\mathbb{Z} \wr \mathbb{Z} = \mathbb{Z} \ltimes_t \mathbb{Z}[t,t^{-1}] \cong \mathbb{Z} \ltimes_s \bigoplus_{\mathbb{Z}} \mathbb{Z}$$

where the generator of the first copy of $\mathbb{Z}$ acts on $\mathbb{Z}[t,t^{-1}]$ by multiplication by t or, equivalently, on $\bigoplus_{\mathbb{Z}} \mathbb{Z}$ by shifting the indices. This group is assumed (unless specified otherwise) to be marked by the generating pair consisting of elements $(1,0)$ and $(0,t^0)$.

The last groups we introduce here are $\Gamma(m,n) = \mathbb{Z} \ltimes_{\frac{n}{m}} \mathbb{Z}[\frac{\text{gcm}(m,n)}{\text{lcm}(m,n)}]$ $(m,n \in \mathbb{Z}^*)$ where the generator of the first copy of $\mathbb{Z}$ acts on $\mathbb{Z}[\frac{\text{gcm}(m,n)}{\text{lcm}(m,n)}]$ by multiplication by $\frac{n}{m}$. This group is assumed (unless specified otherwise) to be marked by the generating pair consisting of elements $(1,0)$ and $(0,1)$. The latter elements are the images of $(1,0)$ and $(0,t^0)$ by the homomorphism $\mathbb{Z} \wr \mathbb{Z} \to \mathbb{Z} \ltimes \mathbb{Z}[\frac{\text{gcm}(m,n)}{\text{lcm}(m,n)}]$ given by the evaluation $t = \frac{n}{m}$; they are also the images of the elements a and b of $BS(m,n)$ by the homomorphism defined by $a \mapsto (1,0)$ and $b \mapsto (0,1)$. Notice last that the group $\mathbb{Z} \ltimes \mathbb{Z}[\frac{\text{gcm}(m,n)}{\text{lcm}(m,n)}]$ acts affinely on $\mathbb{Q}$ (or $\mathbb{R}$) by $(1,0)\cdot x = \frac{n}{m}x$ and $(0,y)\cdot x = x+y$.

We introduce the homomorphism $\sigma_a : \mathbb{F}_2 \to \mathbb{Z}$ defined by $\sigma_a(a) = 1$ and $\sigma_a(b) = 0$. It factories through all groups $BS(m,n), \overline{BS}(m,\xi), \mathbb{Z} \wr \mathbb{Z}, \mathbb{Z} \ltimes_{\frac{n}{m}} \mathbb{Z}[\frac{\text{gcm}(m,n)}{\text{lcm}(m,n)}]$. The induced morphisms are also denoted σ_a. To end this section we define the homomorphism $^-\colon \mathbb{F}_2 \to \mathbb{F}_2$ given by $\bar{a} = a$ and $\bar{b} = b^{-1}$. Note that it is compatible with quotient maps and also defines homomorphisms $^-\colon BS(m,n) \to BS(m,n)$ for $m,n \in \mathbb{Z}^*$ and $^-\colon \overline{BS}(m,\xi) \to \overline{BS}(m,\xi)$ for $m \in \mathbb{Z}^*, \xi \in \mathbb{Z}_m$.

2 Converging sequences of some marked one-relator groups with parameters

2.1 Limits of torus knots groups

We define the groups

$$T_{m,n} = \left\langle a,b \;\middle|\; a^m = b^n \right\rangle \text{ and } T_m = \left\langle a,b \;\middle|\; [a^m,b]=1 \right\rangle,\ m,n \in \mathbb{Z}^*.$$

When m and n are relatively prime, the group $T_{m,n}$ is the fundamental group of $\mathbb{S}^3 \setminus K_{m,n}$ where $K_{m,n}$ is the knot drawn on the torus $\mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ obtained as the image of the map

$t \longmapsto (mt, nt)$. Given with their natural ordered set of generators (a, b), the groups $T_{m,n}$ and T_m are marked groups of $\mathcal{G}_2$.

Proposition 2.1 *i) The limit of $T_{m,n}$ in $\mathcal{G}_2$, when m and n tend both to infinity, is the free group $\mathbb{F}_2 =< a, b >$ marked with its natural set of generators.*

ii) Assume now that m is fixed. The limit in $\mathcal{G}_2$ of $T_{m,n}$ as n tends to infinity is T_m.

PROOF. Notice first that $T_{m,n}$ is the free product of $\langle a \rangle \simeq \mathbb{Z}$ by $\langle b \rangle \simeq \mathbb{Z}$ with amalgamation over $\langle a^m \rangle \simeq \mathbb{Z}$ and $\langle b^n \rangle \simeq \mathbb{Z}$ (see [LS77], Ch.IV for definition).

Let us prove $i)$. It suffices to show that for any freely reduced word w on $\{a, b\}^{\pm}$, there is some integer $L = L(w)$ such that

(1) $w \underset{\mathbb{F}_2}{=} 1 \Rightarrow w \underset{T_{m,n}}{=} 1$ for all $m, n > L$,

(2) $w \underset{\mathbb{F}_2}{\neq} 1 \Rightarrow w \underset{T_{m,n}}{\neq} 1$ for all $m, n > L$.

As (1) is trivial, we only need to prove (2). Let w be a freely reduced word on $\{a, b\}^{\pm}$. By the normal form theorem ([LS77], Ch.IV, Th.2.6,pp 187) for free products with amalgammation, the image of w in $T_{m,n}$ is trivial for all $m, n > |w|$ if and only if $w \underset{\mathbb{F}_2}{=} 1$, which completes the proof of $i)$.

Let us show the statement $ii)$. By Dycke's theorem, the map $a \longmapsto a, b \longmapsto b$ induces an epimorphism of T_m onto $T_{m,n}$ for all $n, m \in \mathbb{Z}$. It suffices to show that for any reduced word w on $\{a, b\}^{\pm}$, there is some integer $L = L(w)$ such that

(1) $w \underset{T_m}{=} 1 \Rightarrow w \underset{T_{m,n}}{=} 1$ for all $n > L$,

(2) $w \underset{T_m}{\neq} 1 \Rightarrow w \underset{T_{m,n}}{\neq} 1$ for all $n > L$.

As (1) is trivial, we only need to show (2). Let $w = a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_l} b^{\beta_l}$ be a freely reduced word. We claim that we can write

$$w \underset{T_m}{=} a^s v \quad \text{with } s \in \mathbb{Z} \text{ and } v = 1 \text{ or } v = b^{\beta'_1} a^{\alpha'_2} \dots b^{\beta'_{t-1}} a^{\alpha'_t} \tag{2.1}$$

where β'_j is a non-zero sum of some β_i's, $\alpha'_j \in \{1, \dots, m-1\}$ for all $j \in \{2, \dots, t-1\}$ and $\alpha'_t \in \{0, \dots, m-1\}$. We call such a decomposition a *suitable* decomposition of w. We show (2.1) by induction on l. The case $l = 1$ is clear.

We can write for each $i \in \{1,\ldots,l\}$, $\alpha_i = q_i m + r_i$ with $r_i \in \{0,\ldots,m-1\}$. We set $\alpha_i' = r_i$. Hence, $w \underset{T_m}{=} w' = a^{\alpha_1' + \sum_{i=1}^{l} q_i m} b^{\beta_1} a^{\alpha_2'} \cdots b^{\beta_{l-1}} a^{\alpha_l'} b^{\beta_l}$. If $r_i > 0$ for all $i \in \{2,\ldots,l\}$, then the last decomposition is suitable. If not we have then $w \underset{T_m}{=} w''$ where w'' is the free reduction of w' and we apply the induction hypothesis to w''.

Assume that $n > |w|$ and let $w = a^s v$ in T_m with $s \in \mathbb{Z}, v = 1$ or $v = b^{\beta_1'} a^{\alpha_2'} \ldots b^{\beta_t'} a^{\alpha_t'}$ be a suitable decomposition. We have $|\beta_j'| \leq \sum_{i=1}^{t} |\beta_j| \leq |w| < n$, for $j = 1,\ldots,t$. By the normal form theorem for free product with amalgammation, the image of w in $T_{m,n}$ is trivial only if $s = 0$ and $v \underset{\mathbb{F}_2}{=} 1$. In this case, the image of w in T_m is also trivial, which proves statement (2). $\square$

Proposition 2.2 *The groups T_m and $T_{m'}$ are isomorphic if and only if $|m| = |m'|$.*

Proof. We begin by proving that the center $Z(T_m)$ of T_m is the cyclic group $C_m = \langle a^m \rangle$. Obviously, $C_m \leq Z(T_m)$. The presentation $\langle a, b \mid a^m = 1 \rangle$ is a presentation for the quotient T_m/C_m. Thus T_m/C_m is isomorphic to the free product $\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$. As non trivial free products are centerless, we have $Z(T_m/C_m) = \{1\}$. Since the quotient map $T_m \longrightarrow T_m/C_m$ maps central elements on central elements, we have then $Z(T_m) = C_m$. If the groups T_m and $T_{m'}$ are isomorphic, then $T_m/Z(T_m) \simeq \mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$ and $T_{m'}/Z(T_{m'}) \simeq \mathbb{Z} * \mathbb{Z}/m'\mathbb{Z}$. are also isomorphic. This can occur only if $|m| = |m'|$.

Conversely, if $m' = -m$, we check easily that the map $a \longmapsto a^{-1}, b \longmapsto b$ induces an isomorphism between T_m and $T_{m'}$. $\square$

2.2 Limits of Baumslag-Solitar groups with changing markings

Let us note $\Gamma(m,n) = \mathbb{Z} \ltimes_{\frac{n}{m}} \mathbb{Z}[\frac{\text{lcm}(m,n)}{\text{gcm }(m,n)}]$ (these groups are defined in Section 1.3). Notice that $\Gamma(1,n) = BS(1,n)$. It is known from [St05] that the limit of the sequence $(BS(1,n))_{n\geq 1}$ in $\mathcal{G}_2$ is the marked group $\mathbb{Z} \wr \mathbb{Z}$. Let ϕ be the endomorphism of $\mathbb{F}_2$ induced by the map $a \longmapsto a, b \longmapsto b^m$. We notice that it induces endomorphisms $\phi : BS(m,n) \to BS(m,n)$ of Baumslag-Solitar groups.

Remark 2.3 *The morphism $\phi : BS(m,n) \to BS(m,n)$ is an epimorphism if and only if m and n are relativeley prime integers. More precisley, $Im\,\phi \cap \langle b \rangle = \langle b^{gcd(m,n)} \rangle$ (see Lemma A.30 of [Sou00] for a proof). Hence, if $gcd(m,n) = 1$, the image of (a,b) under ϕ^k is the generating set (a, b^{m^k}) of $BS(m,n)$.*

Let us fix a nonzero integer m.

For any $n \in \mathbb{Z}^*$ and any $\ell \in \mathbb{N}$, we denote by $\Gamma_{n,\ell}$ the subgroup of $BS(m,n)$ generated by a and b^{m^ℓ} (We use the latter two elements as marking). Note that if m and n are coprime, then for any $\ell \in \mathbb{N}$ one has $\Gamma_{n,\ell} = BS(m,n)$ (as groups). The result we are aiming at is the following.

Theorem 2.4 *Let $m \in \mathbb{Z}^*$. The following statements hold:*

(a) For any n and for $\ell \to \infty$, one has $\Gamma_{n,\ell} \to \Gamma(m,n) = \mathbb{Z} \ltimes_{\frac{n}{m}} \mathbb{Z}[\frac{gcm(m,n)}{lcm(m,n)}]$;

(b) For $|n| \to \infty$ and $\ell \to \infty$, one has $\Gamma_{n,\ell} \to \mathbb{Z} \wr \mathbb{Z}$.

Result (a) was known by Baumslag in [Bau76]. He proved also that $\Gamma(2,3)$ is the metabelianization of $BS(2,3)$ and that it is not finitely presented but has trivial Schur multiplicator. This convergence was used in [AB+03] to prove that $BS(m,n)$, for m and n relatively prime, is not uniformly non-amenable and in [Os02] to show that $BS(m,n)$ is weakly amenable.

Proof. The morphism $\phi : \mathbb{F}_2 \to \mathbb{F}_2$ defines marked epimorphisms $\Gamma_{n,\ell} \to \Gamma_{n,\ell+1}$ for all $n \in \mathbb{Z}^*$ and $\ell \in \mathbb{N}$. Consequently, for all $n \in \mathbb{Z}^*$, $\ell \in \mathbb{N}$ and $w \in \mathbb{F}_2$, one has

$$w \underset{\Gamma_{n,\ell}}{=} 1 \iff \phi^\ell(w) \underset{BS(m,n)}{=} 1 . \tag{2.2}$$

Let us now take $w \in \mathbb{F}_2$. Thanks to (2.2), it is sufficient to prove that:

(i) For any n, one has $w \underset{\Gamma(m,n)}{=} 1 \implies \phi^\ell(w) \underset{BS(m,n)}{=} 1$ for ℓ large enough;

(ii) For any n, one has $w \underset{\Gamma(m,n)}{\neq} 1 \implies \phi^\ell(w) \underset{BS(m,n)}{\neq} 1$ for ℓ large enough;

(iii) One has $w \underset{\mathbb{Z}\wr\mathbb{Z}}{=} 1 \implies \phi^\ell(w) \underset{BS(m,n)}{=} 1$ for $|n|$ and ℓ large enough;

(iv) One has $w \underset{\mathbb{Z}\wr\mathbb{Z}}{\neq} 1 \implies \phi^\ell(w) \underset{BS(m,n)}{\neq} 1$ for $|n|$ and ℓ large enough.

We write $w = a^{i_1} b^{\beta_1} a^{-i_1} \cdots a^{i_k} b^{\beta_k} a^{-i_k} a^\sigma$. Up to conjugate, we may assume that we have $i_1, \ldots, i_k \geqslant 0$. Set now $\ell_w = \max(i_1, \ldots, i_k)$. For all $\ell \geqslant \ell_w$ and $n \in \mathbb{Z}^*$, one has

$$\begin{aligned} \phi^\ell(w) &= a^{i_1} b^{m^\ell \beta_1} a^{-i_1} \cdots a^{i_k} b^{m^\ell \beta_k} a^{-i_k} a^\sigma \\ &\underset{BS(m,n)}{=} b^{m^{\ell-i_1} n^{i_1} \beta_1} \cdots b^{m^{\ell-i_k} n^{i_k} \beta_k} a^\sigma \\ &= b^{m^\ell \sum_{s=1}^k \beta_s \left(\frac{n}{m}\right)^{i_s}} a^\sigma . \end{aligned}$$

Proof of (i). Fix $n \in \mathbb{Z}^*$. The equality $w = 1$ in $\Gamma(m,n)$ implies $\sum_{s=1}^{k} \beta_s \left(\frac{n}{m}\right)^{i_s} = 0$ and $\sigma = 0$. Consequently, one has $\phi^\ell(w) = 1$ in $BS(m,n)$ for all $\ell \geqslant \ell_w$.

Proof of (ii). Fix $n \in \mathbb{Z}^*$. In the case $\sigma \neq 0$, one would have $\phi^\ell(w) \neq 1$ in $BS(m,n)$ for all $\ell \in \mathbb{N}$. Thus we assume $\sigma = 0$ and the inequality $w \neq 1$ in $\Gamma(m,n)$ gives $\sum_{s=1}^{k} \beta_s \left(\frac{n}{m}\right)^{i_s} \neq 0$. Consequently, for any $\ell \geqslant \ell_w$, one has

$$\phi^\ell(w) \underset{BS(m,n)}{=} b^{m^\ell \sum_{s=1}^{k} \beta_s \left(\frac{n}{m}\right)^{i_s}} \underset{BS(m,n)}{\neq} 1 .$$

Proof of (iii). The equality $w = 1$ in $\mathbb{Z} \wr \mathbb{Z}$ implies $\sigma = 0$ and $\sum_{s=1}^{k} \beta_s t^{i_s} = 0$ (as polynomials). Consequently, one has $\phi^\ell(w) = 1$ in $BS(m,n)$ for all $\ell \geqslant \ell_w$ and $n \in \mathbb{Z}^*$.

Proof of (iv). In the case $\sigma \neq 0$, one would have $\phi^\ell(w) \neq 1$ in $BS(m,n)$ for all $\ell \in \mathbb{N}$ and $n \in \mathbb{Z}^*$. Thus we assume $\sigma = 0$ and the inequality $w \neq 1$ in $\mathbb{Z} \wr \mathbb{Z}$ gives $\sum_{s=1}^{k} \beta_s t^{i_s} \neq 0$ (as polynomials). The polynomial $\sum_{s=1}^{k} \beta_s t^{i_s}$ having only finitely many roots, one has $\sum_{s=1}^{k} \beta_s \left(\frac{n}{m}\right)^{i_s} \neq 0$ for $|n|$ large enough. Consequently, for any $\ell \geqslant \ell_w$ and for $|n|$ large enough, one has

$$\phi^\ell(w) \underset{BS(m,n)}{=} b^{m^\ell \sum_{s=1}^{k} \beta_s \left(\frac{n}{m}\right)^{i_s}} \underset{BS(m,n)}{\neq} 1 .$$

This completes the proof. □

2.3 Limits of other Baumslag's one-relator groups

We denote by $BS(m,n,l)$ the group defined by the presentation

$$\left\langle a, b \;\middle|\; (ab^m b^{-1} a^{-n})^l = 1 \right\rangle \text{ with } m, n, l \in \mathbb{Z}^*.$$

This family of one-relator groups with torsion was introduce by Baumslag in [Bau67]. He proved there that such groups are residually finite if $|m| \neq 1 \neq |n|$, n and m are coprime and $l > 1$

Proposition 2.5 *Consider the family of marked groups $BS(m,n,l)$ (endowed with the canonical marking (a,b)) with $m,n \in \mathbb{Z}^*$ and $l \geqslant 2$. One has $BS(m,n,l) \to \mathbb{F}_2$ in $\mathcal{G}_2$ whenever $|m|, |n|$ or $|l|$ tends to infinity.*

Proof. This is a straightforward corollary of the following result of B.B. Newman [LS77, Ch. 5, Pr. 5.28]. If X is a set of letters and w is any word on $X^{\pm}$ which is trivial in the one-relator group with torsion $\left\langle X \;\middle|\; r^k = 1 \right\rangle$, then the length $|w|$ of w with respect to the word metric induced by X is not less than $(k-1)|r|$. □

3 A necessary and sufficient condition for the convergence of sequences of marked Baumslag-Solitar groups

Proposition 3.1 *Let $m, d \in \mathbb{Z}^*$ and $(\xi_n)_{n\geq 0}$ be a sequence of integers such that $|\xi_n| \to \infty$. If $(\xi_n)_{n\geq 0}$ defines a converging sequence in $\mathbb{Z}_m$ then the sequence of marked groups $(BS(md, \xi_n d))_{n\geq 0}$ converges in $\mathcal{G}_2$*

We get this proposition by adaptating quite readily Theorem 6 in [St05]. Nevertheless, we give a proof based on the following lemmas.

Lemma 3.2 *Let $w \in \mathbb{F}_2$. Under the hypothesis of Proposition 3.1, the following alternative holds :*

(a) either $w = b^{\lambda_n}$ in $BS(md, \xi_n d)$ for n large enough;

(b) or w is in $BS(md, \xi_n d) \setminus \langle b \rangle$ for n large enough.

We recall that the map $a \longmapsto (x \mapsto \frac{n}{m}x), b \longmapsto (x \mapsto x+1)$ defines an homomorphism $\psi_n : BS(m,n) \longrightarrow \mathrm{Aff}(\mathbb{R})$.

Lemma 3.3 *[St05, Lemma 7] Let $w = b^{\alpha_0}a^{\varepsilon_1}\ldots a^{\varepsilon_h}b^{\alpha_h}$ with $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathbb{Z}$. We have either $\psi_n(w) = 1$ for $|n|$ large enough or $\psi_n(w) \neq 1$ for $|n|$ large enough. Moreover, if $\psi_l(w) = 1$ for some $|l| > |\alpha_0| + |\alpha_1| \cdots + |\alpha_h|$, then $\psi_n(w) = 1$ for all n.*

The proof of Lemma 3.2 relies on a corollary of Lemma 5 in [St05]. Before to state it, we recall the following statement, whose notations will be used :

Lemma 3.4 *[St05, Lemma 4] Let $m, n, n' \in \mathbb{Z}^*$ and $h \geqslant 1$. If $n \equiv n' \pmod{m^h}$, there exists $s_0, \ldots, s_h$; $s'_0, \ldots, s'_h$; $r_1, \ldots, r_h$, which are unique, such that:*

(i) $0 \leqslant r_i < m\ \forall i$; $s_0 = 1 = s'_0$;

(ii) $s_{i-1}n = s_i m + r_i$ and $s'_{i-1}n' = s'_i m + r_i\ \forall 1 \leqslant i \leqslant h$;

(iii) $s_i \equiv s'_i \pmod{m^{h-i}}\ \forall 0 \leqslant i \leqslant h$.

The reader is addressed to the reference for the proof.

Lemma 3.5 *Let $m, n, n', d \in \mathbb{Z}^*$ and $h \geq t \geq 1$. Assume that $n \equiv n'$ (mod m^h) and let*

$$\alpha = k_0 + k_1 nd + k_2 s_1 nd + \cdots + k_t s_{t-1} nd$$

$$\alpha' = k_0 + k_1 n'd + k_2 s_1' n'd + \cdots + k_t s_{t-1}' n'd$$

where $|k_0| < \min(|n|, |n'|)$ and $s_0, \ldots, s_h; s_0', \ldots, s_h'$ are given by Lemma 3.4. Let us also take $r_1, \ldots, r_h$ as in Lemma 3.4.

(*i*) *We have $\alpha \equiv 0\,(\mathit{mod}\, md)$ if and only if $\alpha' \equiv 0\,(\mathit{mod}\, md)$. If it happens we get $ab^{\alpha}a^{-1} \underset{BS(md,nd)}{=} b^{\beta}$ and $ab^{\alpha'}a^{-1} \underset{BS(md,n'd)}{=} b^{\beta'}$ with*

$$\beta = l_1 nd + l_2 s_1 nd + \cdots + l_{t+1} s_t nd$$

$$\beta' = l_1 n'd + l_2 s_1' n'd + \cdots + l_{t+1} s_t' n'd$$

and

$$l_1 = \frac{1}{m}(k_0 + k_1 r_1 + \cdots + k_t r_t), l_i = k_{i-1} \text{ for } 2 \leq i \leq t+1.$$

(*ii*) *We have $\alpha \equiv 0\,(\mathit{mod}\, nd)$ if and only if $\alpha' \equiv 0\,(\mathit{mod}\, n'd)$. If it happens we get $a^{-1}b^{\alpha}a \underset{BS(md,nd)}{=} b^{\beta}$ and $a^{-1}b^{\alpha'}a \underset{BS(md,n'd)}{=} b^{\beta'}$ with*

$$\beta = l_0 d + l_1 nd + l_2 s_1 nd + \cdots + l_{t-1} s_{t-2} nd$$

$$\beta' = l_0 d + l_1 n'd + l_2 s_1' n'd + \cdots + l_{t-1} s_{t-2}' n'd$$

and

$$l_0 = k_1 m - k_2 r_1 - \cdots - k_t r_{t-1}, l_i = k_{i+1} \text{ for } 1 \leq i \leq t.$$

In particular the l_i's depend only on the common congruence class of n and n' modulo m^h. The case $d = 1$ corresponds to [St05, Lemma 5].

Proof of Lemma 3.5. (*i*) By Lemma 3.4 which ensures that $s_i \equiv s_i'$ (mod m) for all $i = 1, \ldots, t-1$, we have $\alpha \equiv \alpha'\,(\mathit{mod}\, md)$. Assume now that

$$\alpha \equiv 0 \equiv \alpha'\,(\mathit{mod}\, m^h).$$

We set $\overline{\alpha} = \frac{\alpha}{d}$ and $\overline{\alpha}' = \frac{\alpha'}{d}$. We have then

$$\overline{\alpha} = \overline{k_0} + k_1 n + k_2 s_1 n + \cdots + k_t s_{t-1} n$$

$$\overline{\alpha}' = \overline{k_0} + k_1 n' + k_2 s_1 n' + \cdots + k_t s_{t-1} n'$$

where $\overline{k_0} = \frac{k_0}{d}$. By Lemma 5 (i) [St05], we have the following identies

$$ab^{\alpha}a^{-1} = ab^{d\overline{\alpha}}a^{-1} = b^{\overline{\beta}d} = b^{\beta} \text{ in } BS(md, nd)$$

$$ab^{\alpha'}a^{-1} = ab^{d\overline{\alpha}'}a^{-1} = b^{\overline{\beta}'d} = b^{\beta'} \text{ in } BS(md, n'd)$$

with

$$\beta = \overline{\beta}d \text{ and } \overline{\beta} = l_1 n + l_2 s_1 n + \cdots + l_{t+1} s_t n$$

$$\beta' = \overline{\beta}'d \text{ and } \overline{\beta}' = l_1 n' + l_2 s_1' n' + \cdots + l_{t+1} s_t' n'$$

and

$$l_1 = \frac{1}{m}(k_0 + k_1 r_1 + \cdots + k_t r_t), l_i = k_{i-1} \text{ for } 2 \leq i \leq t+1$$

which comes from the proof of [St05, Lemma 5]. Hence (i) is proved.

(ii) As $|n| > |k_0|$ and $|n'| > |k_0|$, we have $\alpha \equiv 0 \pmod{nd}$ if and only if $k_0 = 0$ if and only if $\alpha' \equiv 0 \pmod{n'd}$. Suppose now that it is the case. By Lemma 5 (ii) [St05], we have then

$$a^{-1}b^{\alpha}a = a^{-1}b^{\overline{\alpha}d}a = b^{\overline{\beta}d} = b^{\beta} \text{ in } BS(md, nd)$$

$$a^{-1}b^{\alpha'}a = ab^{\overline{\alpha}'d}a^{-1} = b^{\overline{\beta}'d} = b^{\beta'} \text{ in } BS(md, n'd)$$

with

$$\beta = \overline{\beta}d \text{ and } \overline{\beta} = l_0 + l_1 n + l_2 s_1 n + \cdots + l_{t-1} s_{t-2} n$$

$$\beta' = \overline{\beta}'d \text{ and } \overline{\beta}' = l_0 + l_1 n' + l_2 s_1' n' + \cdots + l_{t-1} s_{t-2}' n'$$

and

$$l_0 = k_1 m - k_2 r_1 - \cdots - k_t r_{t-1}, l_i = k_{i+1} \text{ for } 1 \leq i \leq t$$

which comes from the proof of [St05, Lemma 5]. Hence (ii) is proved. □

Proof of Lemma 3.2. We define $\Gamma_n = BS(md, \xi_n d)$. Let us write $w = b^{\alpha_0}a^{\varepsilon_1}b^{\alpha_1}\ldots a^{\varepsilon_h}b^{\alpha_h}$ with $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathbb{Z}$, reduced in the sense that $\alpha_i = 0 \Rightarrow \varepsilon_i = \varepsilon_{i+1}$ for all $i \in \{1, \ldots, h-1\}$. We assume (b) not to hold, i.e. $w = b^{\lambda_n}$ in Γ_n of infinitely many n's. Then the sum $\varepsilon_1 + \cdots + \varepsilon_h$ has clearly to be zero (in particular h is even). We have to show that $w = b^{\lambda_n}$ for n large enough.

For n large enough, we may assume that $|\xi_n| > |\alpha_j|$ for all $j \in \{0, \ldots, h\}$ and the ξ_n's are all congruent modulo m^h. We take a value of n such that moreover $w = b^{\lambda_n}$ in Γ_n(there are infinitely many ones) and apply Britton's Lemma. This ensures the existence of an index j such that $\varepsilon_j = 1 = -\varepsilon_{j+1}$ and $\alpha_j \equiv 0 \pmod{md}$ (since $|\xi_n d| > |\alpha_j|$ for all j). For all n large enough, Lemma 3.5 implies

$$w \underset{\Gamma_n}{=} b^{\alpha_0} \dots a^{\varepsilon_{j-1}} b^{\alpha_{j-1}+\beta_j+\alpha_{j+1}} a^{\varepsilon_{j+2}} \dots b^{\alpha_r}$$

with $\beta_j = l_1 \xi_n d = \alpha_j \frac{\xi_n}{m}$ (depending on n). Hence we are allowed to write

$$w \underset{\Gamma_n}{=} b^{\alpha'_{0,n}} a^{\varepsilon'_1} b^{\alpha'_{1,n}} \dots a^{\varepsilon'_{h-2}} b^{\alpha'_{h-2,n}}$$

for n large enough, with $\varepsilon'_i = \pm 1$ and $\alpha'_{i,n} = k'_{0,i} + k'_{1,i} \xi_n d$, where the ε'_i's and $k'_{l,i}$'s do not depend on n.

Now, for n large enough, we may assume that $|\xi_n| > |k'_{0,i}|$ for all $0 \leq j \leq h-2$ (and the ξ_n's are all congruent modulo m^h). Again, we take a value of n such that moreover $w = b^{\lambda_n}$ in Γ_n and apply Britton's Lemma. This ensures the existence of an index j such that either $\varepsilon'_j = 1 = -\varepsilon'_{j+1}$ and $\alpha'_{j,n} \equiv 0 \pmod{md}$, or $\varepsilon'_j = -1 = -\varepsilon'_{j+1}$ and $\alpha'_{j,n} \equiv 0 \pmod{\xi_n d}$. In both cases, while applying Lemma 3.5, we obtain

$$w \underset{\Gamma_n}{=} b^{\alpha''_{0,n}} a^{\varepsilon''_1} b^{\alpha''_{1,n}} \dots a^{\varepsilon''_{h-4}} b^{\alpha''_{h-4,n}}$$

for n large enough, with $\varepsilon''_i = \pm 1$ and $\alpha''_{i,n} = k''_{0,i} + k''_{1,i} \xi_n d + k''_{2,i} s_{1,n} \xi_n d$, where the ε''_i's and $k''_{l,i}$'s do not depend on n.

And so on, and so forth, setting $h' = \frac{h}{2}$, we get finally $w = b^{\alpha^{(h')}_{0,n}}$ in Γ_n for n large enough, with

$$\alpha^{(h')}_{0,n} = k^{(h')}_{0,0} + k^{(h')}_{1,0} \xi_n d + k^{(h')}_{2,0} s_{1,n} \xi_n d + \dots + k^{(h')}_{h',0} s_{h'-1,n} \xi_n d$$

where the $k^{(h')}_{i,0}$'s do not depend on n. It only remains to set $\lambda_n = \alpha^{(h')}_{0,n}$. □

PROOF OF PROPOSITION 3.1. It is easy to show that a word w is equal to 1 in $BS(m,n)$ if and only if it is in the subgroup generated by b and $\psi_n(w) = 1$. Let $w \in \mathbb{F}_2$. Lemmata 3.2 and 3.3 immediately imply that either $w = 1$ in $BS(md, \xi_n d)$ for n large enough or $w \neq 1$ in $BS(md, \xi_n d)$ for n large enough. □

Corollary 3.6 *Let $m, d \in \mathbb{Z}^*$ with $|m| \geq 2$ and let $\xi \in \mathbb{Z}_m$. We fix $\eta \in \mathbb{Z}_{md}$ such that $\pi(\eta) = \xi$, where $\pi : \mathbb{Z}_{md} \longrightarrow \mathbb{Z}_m$ is the map defined in Lemma 1.3.*

Let $h \geq 1$ and $w = b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \dots a^{\varepsilon_{2h}} b^{\alpha_{2h}} \in \mathbb{F}_2$ with $\varepsilon_i = \pm 1$ reduced in the sense that $\alpha_i = 0 \Rightarrow \varepsilon_i = \varepsilon_{i+1}$ for all $i \in \{1, \dots, 2h-2\}$. We set $K_0 = |\alpha_0| + |\alpha_1 + \dots + |\alpha_{2h}|$. We fix an integer n such that $n \equiv \xi \,(\, mod\, m^h \mathbb{Z}_m\,)$ and $|nd| > K_0 (h+1)! |m|^{h-1}$.

Then $w = 1$ in $\overline{BS}(md, \eta d)$ if and only if $w = 1$ in $BS(md, nd)$.

Hence, the word problem is solvable in $\overline{BS}(md, \eta d)$.

PROOF. Reasonning as in the proof of Proposition 3.1 and related lemmata, we see that if $w = 1$ in $\overline{BS}(md, \eta d)$ then there is a sequence of h cancellations for w

$$\begin{array}{ll} (0) & w = b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \dots a^{\varepsilon_h} b^{\alpha_{2h}}, \\ (1) & w' = b^{\alpha'_{0,n}} a^{\varepsilon'_1} b^{\alpha'_{1,n}} \dots a^{\varepsilon'_{h-2}} b^{\alpha'_{h-2,n}}, \\ (2) & w'' = b^{\alpha''_{0,n}} a^{\varepsilon''_1} b^{\alpha''_{1,n}} \dots a^{\varepsilon''_{h-4}} b^{\alpha''_{h-4,n}}, \\ \vdots & \vdots \\ (h) & w^{(h)} = b^{\alpha^{(h)}_{0,n}} = b^0 = 1. \end{array}$$

which occurs in all $BS(md, nd)$ provided that

$$|nd| > |\alpha_j|, |k^{(t)}_{0,i}| \text{ for } 0 \leq j \leq 2h,\ 1 \leq t \leq h, 0 \leq i \leq 2h - 2t \tag{3.1}$$

where the $k^{(t)}_{0,i}$'s are the integers appearing in the proof of Lemma 3.2 and

$$n \equiv \xi \ (\mathrm{mod}\ m^h \mathbb{Z}_m). \tag{3.2}$$

Conversely, if $w = 1$ in $BS(md, nd)$ for some n satisfying both conditions (3.1), (3.2) and

$$|nd| > |\alpha_0| + |\alpha_1| + \cdots + |\alpha_{2h}| = K_0 \tag{3.3}$$

a repeated use of the Britton's lemma in the same way as in the proof of Proposition 3.1 and related lemmata gives rise to a sequence of h cancellations of w leading to the trivial word. By Lemma 3.5 and Lemma 3.3, all these cancellations still occur in any $BS(md, n'd)$ whenever n' satisfies (3.1) and (3.2). Hence, $w = 1$ in $\overline{BS}(md, \eta d)$.

We set $k^{(0)}_{0,i} = |\alpha_i|$ for $i = 0, \dots, 2h$ and $K^{(t)} = \max\limits_{0 \leq s \leq t, 0 \leq i \leq 2h-2s} |k^{(s)}_{0,i}|.$. Because of (3.1) and (3.3), it suffices to bound roughly $K := K^{(h)}$ by $K^{(0)} h! |m|^h$.

For all $t = 0, 1, \dots, h-1$, there is a choosen index $c(t) \in \{1, \dots, 2h - 2t - 1\}$ such that either $\alpha^{(t)}_{c(t),n}$ is congruent to 0 (mod md) (case i) or to 0 (mod nd) (case ii) for all n satisfying (3.1) and (3.2) . We have the identities :

$$\alpha^{(t+1)}_{i,n} = \begin{cases} \alpha^{(t)}_{i,n} & \text{if } 0 \leq i \leq c(t) - 2 \\ \alpha^{(t)}_{c(t)-1,n} + \beta_{c(t)} + \alpha^{(t)}_{c(t)+1,n} & \text{if } i = c(t) - 1 \\ \alpha^{(t)}_{i+2,n} & \text{if } c(t) \leq i \leq 2h \end{cases} \tag{3.4}$$

with $\alpha^{(t)}_{c(t),n} = k^{(t)}_{0,c(t)} + k^{(t)}_{1,c(t)} nd + k^{(t)}_{2,c(t)} s_{1,n} nd + \cdots + k^{(t)}_{t,c(t)} s_{t-1,n} nd$ and $\beta_{c(t)} = l_0 d + l_1 s_{1,n} nd + \cdots + l_{t+1} s_{t,n} nd$ where $l_0, l_1, \dots, l_{t+1}$ are given by Lemma 3.5 and $s_{1,n}, s_{2,n}, \dots, s_{t,n}$ are given by Lemma 3.4. By Lemma 3.5, we know that the integers $l_1, \dots, l_{t+1}$ don't depend on n and we have the following identities :

(case i)

$$l_0 = 0, l_1 = \frac{1}{m}(k^{(t)}_{0,t} + k^{(t)}_{1,c(t)} r_1 + \cdots + k^{(t)}_{t,c(t)} r_t), l_i = k^{(t)}_{i-1,c(t)} \text{ for } 2 \le i \le t+1. \quad (3.5)$$

(case ii)

$$l_0 = k^{(t)}_{1,t} m - k^{(t)}_{2,c(t)} r_1 - \cdots - k^{(t)}_{t,c(t)} r_{t-1}, l_i = k^{(t)}_{i+1,c(t)} \text{ for } 1 \le i \le t-1, l_t = l_{t+1} = 0, \quad (3.6)$$

with $r_1, r_2, \ldots, r_t \in \{0, \ldots, m-1\}$ depend only on the class of n modulo $m^h \mathbb{Z}_m$. Finally, we get from (3.4) :

$$k^{(t+1)}_{j,i} = \begin{cases} k^{(t)}_{j,i} & \text{if } i \le c(t) - 2 \\ k^{(t)}_{j,c(t)-1} + l_j + k^{(t)}_{j,c(t)+1} & \text{if } i = c(t) - 1 \\ k^{(t)}_{j,i+2} & \text{if } c(t) \le i \le h \end{cases} \quad (3.7)$$

We deduce from (3.5),(3.6) and (3.7) that for $t \in \{1, \ldots, h-1\}$ either $K^{(t+1)} \le K^{(t)} + \frac{1}{|m|}|m|(t+1)K^{(t)} + K^{(t)}$ (case i) or $K^{(t+1)} \le K^{(t)} + mtK^{(t)} + K^{(t)}$ (case ii). Hence

$$K^{(t+1)} \le (|m|t + 2)K^{(t)} \text{ for } t \ge 1 \text{ and } K^{(1)} \le 2K^{(0)}$$

because the first cancellation is of type (i). We get then $K \le K^{(0)}(h+1)!|m|^{h-1}$.

□

Theorem 3.7 *Let $m \in \mathbb{Z}^*$ and $(\xi_n)_{n\ge0}$ be sequence of integers such that $|\xi_n| \underset{n\to\infty}{\longrightarrow} \infty$. The sequence $(BS(m, \xi_n))_{n\ge0}$ converges in $\mathcal{G}_2$ if and only if the following conditions both hold :*

(i) *there is some integer d such that $gcd(m, \xi_n) = d$ for all n large enough;*

(ii) *$(\frac{\xi_n}{d})_{n\ge0}$ defines a converging sequence of $\mathbb{Z}_{\frac{m}{d}}$.*

First, we need the following lemma :

Lemma 3.8 *Let $m_i, d_i, k_i \in \mathbb{Z}^*$ for $i = 1, 2$ such that*

$$m_1 d_1 = m_2 d_2, |k_2 d_2| \ne 1, gcd(m_2, k_2) = 1 \text{ and } d_1 \text{ doesn't divide } d_2.$$

Then, the distance between $BS(m_1 d_1, k_1 d_1)$ and $BS(m_2 d_2, k_2 d_2)$ in $\mathcal{G}_2$ is not less than $e^{-\delta}$ with $\delta = 10 + 2d_1 m_1^2$.

Proof of Lemma 3.8. Consider $r = a^2 b^{d_1 m_1^2} a^{-2} b$ and let $w = r\overline{r}$. On one hand, we have $r = b^{d_1 k_1^2+1}$ in $BS(m_1 d_1, k_1 d_1)$, which implies $w = 1$ in $BS(m_1 d_1, k_1 d_1)$. On the other hand, $r = ab^{m_1 d_2 k_2} a^{-1} b$ in $BS(m_2 d_2, k_2 d_2)$. As m_2 and k_2 are coprime integers, we notice that $m_2 d_2$ divides $m_1 d_2 k_2$ if and only if d_1 divides d_2. Under the assumptions of the lemma, the writing $ab^{m_1 d_2 k_2} a^{-1} bab^{-m_1 d_2 k_2} a^{-1} b^{-1}$ is then a reduced form for w in $BS(m_2 d_2, k_2 d_2)$. By Britton's Lemma, $w \neq 1$ in $BS(m_2 d_2, k_2 d_2)$. As $|w| = 10 + 2 d_1 m_1^2$, we get the conclusion. □

Proof of Theorem 3.7. Notice that Theorem 2 of [St05] shows the theorem in the case $m = \pm 1$. We assume then $|m| \geq 2$.

Let us show first that (i) and (ii) are necessary. If (i) doesn't hold, we can find two subsequences $(\xi'_n)_{n\geq 0}$ and $(\xi''_n)_{n\geq 0}$ of $(\xi_n)_{n\geq 0}$ such that $gcd(m, \xi'_n) = d_1, gcd(m, \xi''_n) = d_2, |\xi''_n| > 1$ for all n and d_1 doesn't divide d_2. Then Lemma 3.8 clearly shows that $(BS(m, \xi_n))_{n\geq 0}$ is not a converging sequence in $\mathcal{G}_2$.

To show that (ii) is necessary, we assume now that $(BS(m, \xi_n))_{n\geq 0}$ converges and that there is some $d \in \mathbb{Z}^*$ such that $gcd(m, \xi_n) = d$ for all n large enough. The marked subgroup $\Gamma_{\xi_n, d}$ of $BS(m, \xi_n)$ generated by (a, b^d) is equal to $BS(\frac{m}{d}, \frac{\xi_n}{d})$ endowed with its canonical ordered set of generators (a, b). The sequence of the $BS(\frac{m}{d}, \frac{\xi_n}{d})$'s is then also converging in $\mathcal{G}_2$. By Theorem 3 [St05], the sequence of integers $(\frac{\xi_n}{d})_{n\geq 0}$ defines a converging sequence in $\mathbb{Z}_{\frac{m}{d}}$.

Finally, we see by Proposition 3.1 that (i) and (ii) are also sufficient. □

Remark 3.9 *Theorem 3.7 still holds if we replace* $BS(m, \xi_n)$ *by* $\overline{BS}(m, \xi_n)$ *where* $(\xi_n)_{n\geq 0}$ *is any sequence in* $\mathbb{Z}_m$ *and if we write* $(\pi(\frac{\xi_n}{d}))_{n\geq 0}$ *instead of* $(\frac{\xi_n}{d})_{n\geq 0}$ *in* (ii) *where* $\pi : \mathbb{Z}_m \longrightarrow \mathbb{Z}_{\frac{m}{d}}$ *is the map defined in Lemma 1.3.*

Corollary 3.10 *Let* $m \in \mathbb{Z}^*$ *and let* $\xi, \eta \in \mathbb{Z}_m$*. The equality of marked groups*

$$\overline{BS}(m, \xi) = \overline{BS}(m, \eta)$$

holds if and only if there is some $d \in \mathbb{Z}^*$ *such that* $gcd(\xi, m) = gcd(\eta, m) = d$ *and* $\pi(\frac{\xi}{d}) = \pi(\frac{\eta}{d})$ *in* $\mathbb{Z}_{\frac{m}{d}}$*.*

Proof of Corollary 3.10.

Choose a sequence of integers $(\xi_n)_{n\geq 0}$ such that

$$\xi_{2n} \underset{\mathbb{Z}_m}{\longrightarrow}, \xi, \xi_{2n+1} \underset{\mathbb{Z}_m}{\longrightarrow} \eta \text{ and } |\xi_n| \longrightarrow \infty \text{ as } n \text{ tends to infinity.}$$

One has $\overline{BS}(m,\xi) = \overline{BS}(m,\eta)$ if and only if the sequence $BS(m,\xi_n)$ converges. By Theorem 3.7 it is equivalent to have $\gcd(m,\xi_n) = d$ for n large enough (for some $d \in \mathbb{Z}^*$) and the sequence $\pi(\frac{\xi_n}{d})$ converges in $\mathbb{Z}_{\frac{m}{n}}$. Finally, it is equivalent to have $\gcd(m,\xi) = d = \mathrm{gcm}(m,\eta)$ and $\pi(\frac{\xi}{d}) = \pi(\frac{\eta}{d})$. $\square$

Corollary 3.11 *Let $m \in \mathbb{Z}^*$ and let $\xi, \eta \in \mathbb{Z}_m$ with $\xi \neq \eta$. If no prime factor of m divides both ξ and η, then one has $\overline{BS}(m,\xi) \neq \overline{BS}(m,\eta)$.*

Proof of Corollary 3.11. We may suppose that $\gcd(m,\xi) = \gcd(m,\eta) = d$ by Corollary 3.10). Then $d = 1$ by assumption and $\pi(\xi) = \xi \neq \eta = \pi(\eta)$. By Corollary 3.10, one gets $\overline{BS}(m,\xi) \neq \overline{BS}(m,\eta)$. $\square$

Lemma 3.12 *Let $m, d \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_{md}$. The marked subgroup of $\overline{BS}(md,\xi d)$ generated by (a, b^d) is equal to $\overline{BS}(m,\pi(\xi))$ where $\pi : \mathbb{Z}_{md} \longrightarrow \mathbb{Z}_m$ is the map defined in Lemma 1.3.*

Proof of Lemma 3.12. Let $(\xi_n)_{n\geq 0}$ be a sequence of integers such that $|\xi_n| \longrightarrow \infty, \xi_n \underset{\mathbb{Z}_{md}}{\longrightarrow} \xi$ as n tends to infinity. Since $BS(md,\xi_n d)$ is converging to $\overline{BS}(md,\xi d)$ as n tends to infinity, the marked subgroup G_n of $BS(md,\xi_n d)$ generated by (a,b^d) is converging to the subgroup G of $\overline{BS}(md,\xi d)$ generated by (a,b^d). As $G_n \underset{\mathcal{G}_2}{=} BS(m,\xi_n)$, we have $G \underset{\mathcal{G}_2}{=} \overline{BS}(m,\pi(\xi))$. $\square$

4 Topological inclusion of $\mathbb{Z}_m^\times$ in $\mathcal{G}_2$

In this section, we are to show that the parameterization of the limits of Baumslag-Solitar groups by the m-adic integers is coherent with the topology inherited from $\mathcal{G}_2$. Precise statements are as follows:

Definition 4.1 *For $m \in \mathbb{Z}^*$, let us define the following subsets of $\mathcal{G}_2$:*

$$\begin{aligned} Y_m &= \{BS(m,n) \ : \ n \in \mathbb{Z}^*\} \ ; \\ Z_m &= \left\{\overline{BS}(m,\xi) \ : \ \xi \in \mathbb{Z}_m\right\} \ . \end{aligned}$$

Theorem 4.2 *For all $m \in \mathbb{Z}^*$, the application*

$$\overline{BS}_m \ : \ \mathbb{Z}_m \to Z_m \ ; \ \xi \mapsto \overline{BS}(m,\xi)$$

is continuous, onto and injectiv on $\mathbb{Z}_m^\times$.

For $|m| \geqslant 2$, note that if we endow $\mathbb{Z}$ with the m-adic ultrametric, the analogue map

$$\mathbb{Z}^* \to Y_m \ ; \ n \mapsto BS(m,n)$$

is not continuous anywhere. Indeed, one has $n+p^k \to n$ for $k \to \infty$, while $BS(m, n+p^k) \to \overline{BS}(m,n)$. (We recall that one has $\overline{BS}(m,n) \neq BS(m,n)$, since the word $ab^m a^{-1} b^{-n}$ defines the trivial element in $BS(m,n)$ but not in $\overline{BS}(m,n)$.)

PROOF OF THEOREM 4.2. Let us fix $m \in \mathbb{Z}^*$. Surjectivity of $\overline{BS}_m$ is obvious. As $\mathbb{Z}_m$ is compact, we only have to show that $\overline{BS}_m$ is continuous and injective.

Continuity: Let $(\xi_n)_n$ be a sequence in $\mathbb{Z}_m$ which converges to ξ. A standard diagonal argument shows that for all n, there exists $\xi_n \in \mathbb{Z}$ such that:

(i) $|\xi_n| \geqslant n$;

(ii) $d_m(\xi_n, \xi_n) < \frac{1}{n}$;

(iii) $d\big(\overline{BS}(m,\xi_n), BS(m,\xi_n)\big) < \frac{1}{n}$.

In view of (ii), the sequence $(\xi_n)_n$ is convergent with limit ξ and we get $BS(m,\xi_n) \underset{n\to\infty}{\longrightarrow} \overline{BS}(m,\xi)$ due to (i). We finish by combining this with (iii) to obtain $\overline{BS}(m,\xi_n) \underset{n\to\infty}{\longrightarrow} \overline{BS}(m,\xi)$.

□

We now "particularize to the case of invertible elements" and show that, in this case, the $\overline{BS}$ groups form the boundary of the BS groups. Precise statements are as follows:

Definition 4.3 *For $m \in \mathbb{Z}^*$, we define:*

$$\begin{aligned} X_m &= \big\{BS(m,n) \ : \ n \text{ is relatively prime to } m\big\} \ ; \\ Z_m^\times &= \big\{\overline{BS}(m,\xi) \ : \ \xi \in \mathbb{Z}_m^\times\big\} \ . \end{aligned}$$

By convention, we say that $Z_{\pm 1}^\times$ is empty.

Corollary 4.4 *For all $m \in \mathbb{Z}^*$, the boundary of X_m in $\mathcal{G}_2$ is $Z_m^\times$. It is homeomorphic to the set of invertible m-adic integers.*

PROOF. Theorem 3 of [St05] implies that the elements of $\overline{X_m}$ are the $BS(m,n)$'s with n relatively prime to m and the $\overline{BS}(m,\xi)$ with $\xi \in \mathbb{Z}_m^\times$. One sees easily that the $BS(m,n)$'s are isolated points in $\overline{X_m}$ (consider the word $ab^m a^{-1} b^{-n}$). The equality $\partial X_m = Z_m^\times$ follows immediately. The second statement is a direct consequence of Theorem 4.2. □

5 Actions of limits of marked Baumslag-Solitar groups

It is well known that, being a HNN-extension of $\mathbb{Z}$, a Baumslag-Solitar group $BS(m,n)$ acts naturally on its Bass-Serre tree [Se77]. Also well known is the affine action of $BS(m,n)$ on the real line given by $a \cdot x = \frac{n}{m}x$ and $b \cdot x = x+1$. The purpose of this section is to give similar actions for the groups $\overline{BS}(m,\xi)$. The affine action will not be used in the paper but the tree action will be of interest to prove that the groups $\overline{BS}(m,\xi)$ have the Haagerup property and to exhibit presentations.

Affine action. We restrict to the case where m is a prime number, which we denote by p. Let $\xi \in \mathbb{Z}_p$ and let $(\xi_n)_n$ be a sequence of integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$. For $n \in \mathbb{N}$, we set

$$\psi_n = \psi_{\xi_n} : \begin{cases} BS(p,\xi_n) & \longrightarrow & \mathrm{Aff}(\mathbb{Q}_p) \\ a & \longmapsto & (x \mapsto \frac{\xi_n}{p}x) \\ b & \longmapsto & (x \mapsto x+1) \end{cases}$$

where $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$. These actions are are well defined exactly for the same reasons as the above affine actions on $\mathbb{R}$.

Theorem 5.1 *Let p be a prime number and let $\xi \in \mathbb{Z}_p$, $\xi \neq 0$. There is an affine action of $\overline{BS}(p,\xi)$ on $\mathbb{Q}_p$ defined by*

$$\psi_\xi : \begin{cases} \overline{BS}(p,\xi) & \longrightarrow & \mathrm{Aff}(\mathbb{Q}_p) \\ a & \longmapsto & (x \mapsto \frac{\xi}{p}x) \\ b & \longmapsto & (x \mapsto x+1) \end{cases} .$$

Proof. It is sufficient to show that the affine action of $\mathbb{F}_2$ on $\mathbb{Q}_p$ given by

$$\psi : \begin{cases} \mathbb{F}_2 & \longrightarrow & \mathrm{Aff}(\mathbb{Q}_p) \\ a & \longmapsto & (x \mapsto \frac{\xi}{p}x) \\ b & \longmapsto & (x \mapsto x+1) \end{cases}$$

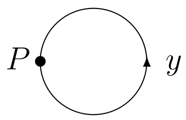
satisfies $\psi(w) = \mathrm{id}$ for all w such that $w = 1$ in $\overline{BS}(p,\xi)$.

Take $(\xi_n)_n$ a sequence of integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$. For $n \in \mathbb{N}$, we denote again ψ_n the action corresponding to the above action of $BS(p,\xi_n)$. We have that $\psi_n(a)(x) = \frac{\xi_n}{p}x \to \psi(a)(x)$ and $\psi_n(b)(x) = x+1 \to \psi(b)(x)$ for $n \to \infty$ $(x \in \mathbb{Q}_p)$. It follows easily that $\psi_n(w)(x) \to \psi(w)(x)$ for all $w \in \mathbb{F}_2$. Now, if $w = 1$ in $\overline{BS}(p,\xi)$, it implies $w = 1$ in $BS(p,\xi_n)$ for n sufficiently large. For those values of n, the equality $\psi_n(w)(x) = x$ holds for all $x \in \mathbb{Q}_p$, so that we get $\psi(w) = \mathrm{id}$. $\square$

Remark 5.2 *The $\mathbb{Z}$-action by translations on $\mathbb{R}$ is proper, but the $\mathbb{Z}$-action by translations on $\mathbb{Q}_p$ is not. In particuler, the action of $\overline{BS}(p,\xi)$ on $\mathbb{Q}_p$ is not proper, even if restricted to the subgroup generated by b.*

Tree action. We are to produce a tree on which the group $\overline{BS}(m,\xi)$ will act in a reasonable way. This tree will be constructed from the Bass-Serre trees of the groups $BS(m,\xi_n)$. It will be shown that the tree we construct does not depend on the auxiliary sequence $(\xi_n)_n$.

We recall that $BS(m,n)$ is the fundamental group of the following graph of groups (G,Y) [Se77, Section 5.1].



$$G_P = \langle b \rangle \cong \mathbb{Z};\ G_y = \langle c \rangle \cong \mathbb{Z}$$
$$c^y = b^m;\ c^{\bar{y}} = b^n$$

Notice that, a is the element of $\pi_1(G,Y,P)$ associated to the edge y. To be precise, we set the *Bass-Serre tree* of $BS(m,n)$ to be the universal covering associated to (G,Y), the maximal subtree P and the orientation given by the edge $\bar{y}$ [Se77, Section 5.3]. We choose the edge $\bar{y}$ instead of y to minimize the dependence on n of the set of tree edges. Denoting by T the Bass-Serre tree of $BS(m,n)$, one has:

$$V(T) = BS(m,n)/\langle b \rangle \ ;\ E(T) = BS(m,n)/\pi_{\tilde{y}} \sqcup BS(m,n)/\pi_{\bar{\tilde{y}}} \ ;\ \overline{w\pi_{\tilde{y}}} = w\pi_{\bar{\tilde{y}}}$$

where $\pi_{\tilde{y}} = \langle b^m \rangle = \pi_{\bar{\tilde{y}}}$. The origin and terminal vertex are given by:

$$\begin{array}{rclcrcl} o(w\pi_{\tilde{y}}) & = & wa^{-1}\langle b \rangle & ; & t(w\pi_{\tilde{y}}) & = & w\langle b \rangle \ ; \\ o(w\pi_{\bar{\tilde{y}}}) & = & w\langle b \rangle & ; & t(w\pi_{\bar{\tilde{y}}}) & = & wa^{-1}\langle b \rangle \ . \end{array}$$

We choose an orientation on T which is preserved by the $BS(m,n)$-action by setting

$$E_+(T) = BS(m,n)/\pi_{\bar{\tilde{y}}} = BS(m,n)/\langle b^m \rangle \ .$$

Given $m \in \mathbb{Z}^*$, $\xi \in \mathbb{Z}_m$ and $(\xi_n)_n$ a sequence of integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$, we denote by H_n (resp H_n^m) the subgroup of $BS(m,\xi_n)$ generated by b (resp b^m) and by T_n the Bass-Serre tree of $BS(m,\xi_n)$. We set

$$\begin{array}{rclcl} Y & = & \left(\prod_{n \in \mathbb{N}} V(T_n) \right) / \sim & = & \left(\prod_{n \in \mathbb{N}} BS(m,\xi_n)/H_n \right) / \sim \\ Y^m & = & \left(\prod_{n \in \mathbb{N}} E_+(T_n) \right) / \sim & = & \left(\prod_{n \in \mathbb{N}} BS(m,\xi_n)/H_n^m \right) / \sim \end{array}$$

where $\sim$ is defined by $(x_n)_n \sim (y_n)_n \iff \exists n_0\ \forall n \geqslant n_0 :\ x_n = y_n$ in both cases. We now define an oriented graph $X = X_{m,\xi}$ by

$$\begin{aligned}
V(X) &= \{x \in Y : \exists w \in \mathbb{F}_2 \text{ such that } (x_n)_n \sim (wH_n)_n\} \\
E_+(X) &= \{y \in Y^m : \exists w \in \mathbb{F}_2 \text{ such that } (y_n)_n \sim (wH_n^m)_n\} \\
o\big((wH_n^m)_n\big) &= (wH_n)_n = \big(o(wH_n^m)\big)_n \\
t\big((wH_n^m)_n\big) &= (wa^{-1}H_n)_n = \big(t(wH_n^m)\big)_n
\end{aligned}$$

The map o is well defined since $(vH_n^m)_n \sim (wH_n^m)_n$ implies $(vH_n)_n \sim (wH_n)_n$. In the other hand, the map t is well defined since $(vH_n^m)_n \sim (wH_n^m)_n$ implies $v^{-1}w \in H_n^m$ for n large enough, whence $(va^{-1})^{-1}(wa^{-1}) = av^{-1}wa^{-1} \in H_n$ for those values of n. It follows that $(va^{-1}H_n)_n \sim (wa^{-1}H_n)_n$. The graph X is thus well defined and the free group $\mathbb{F}_2$ acts obviously on it by left multiplications. The statement we want to prove is the following:

Theorem 5.3 *Let $m \in \mathbb{Z}^*$, $\xi \in \mathbb{Z}_m$ and $(\xi_n)_n$ a sequence of integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$. The graph $X = X_{m,\xi}$ (seen here as unoriented) has the following properties:*

(a) It is a tree;

(b) It does not depend (up to equivariant isomorphism) on the choice of the sequence $(\xi_n)_n$;

(c) The obvious action of $\mathbb{F}_2$ on X factors through the canonical projection $\mathbb{F}_2 \to \overline{BS}(m,\xi)$.

Before the proof, we state an immediate consequence of [St05, Lemma 6]:

Lemma 5.4 *Let $(vH_n)_n$ and $(wH_n)_n$ be two vertices of the graph X. If $vH_n = wH_n$ for infinitely many values of n, then $(vH_n)_n = (wH_n)_n$ in X.*

Proof of theorem 5.3. **(a)** Let us show first that the graph X is connected. We show by induction on $|w|$ that any vertex $(wH_n)_n$ ($w \in \mathbb{F}_2$) is connected to $(H_n)_n$. The case $|w| = 0$ is trivial. If $|w| = \ell > 0$, there exists $x \in \{a^{\pm 1}, b^{\pm 1}\}$ such that wx has length $\ell - 1$. By induction hypothesis, it is sufficient to show that $(wH_n)_n$ is connected to $(wxH_n)_n$. If $x = a$, then the edge $(wxH_n^m)_n$ connects $(wxH_n)_n$ to $(wH_n)_n$, if $x = a^{-1}$, then then the edge $(wH_n^m)_n$ connects $(wH_n)_n$ to $(wxH_n)_n$ and if $x = b^{\pm 1}$, then one has even $(wH_n)_n = (wxH_n)_n$.

Second, we show that X has no circuit. We assume by contradiction that X has a circuit whose vertices and edges are

$$(v_0H_n)_n, (w_1H_n^m)_n, (v_1H_n)_n, \ldots, (w_\ell H_n^m)_n, (v_\ell H_n)_n = (v_0H_n)_n \ .$$

(The formulas $(w_1H_n^m)_n \neq \overline{(w_\ell H_n^m)_n}$ and $(w_{i+1}H_n^m)_n \neq \overline{(w_iH_n^m)_n}$ are assumed to hold.) For any n, the sequence

$$v_0H_n, w_1H_n^m, v_1H_n, \ldots, w_\ell H_n^m, v_\ell H_n$$

forms a path in the tree T_n. For infinitely many values of n, we have moreover $w_1H_n^m \neq \overline{w_\ell H_n^m}$, $w_{i+1}H_n^m \neq \overline{w_iH_n^m}$ and $v_\ell H_n = v_0H_n$ by construction of X. We have obtained a circuit in some T_n, in contradiction with the fact it is a tree.

(b) We show now that X does not depend (up to equivariant isomorphism) on the choice of the sequence $(\xi_n)_n$. Take another sequence $(k'_n)_n$ satisfying both $|k'_n| \to \infty$ and $k'_n \to \xi$ in $\mathbb{Z}_m$ and consider the associated tree X'. We construct the sequence $(k''_n)_n$ given by

$$k''_n = \begin{cases} k_{\frac{n}{2}} & \text{if } n \text{ is even} \\ k'_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

which satisfies again both $|k''_n| \to \infty$ and $k''_n \to \xi$ in $\mathbb{Z}_m$ and the associated tree X''. There are obvious equivariant surjective graph morphisms $X'' \to X$ and $X'' \to X'$. We have to show the injectivity of these morphisms, which we can check on vertices only, for we are dealing with trees. But Lemma 5.4 precisely implies the injectivity on vertices.

(c) Take $w \in \mathbb{F}_2$ such that $w = 1$ in $\overline{BS}(m, \xi)$. We have to prove that w acts trivially on X. As X is a simple graph, we only have to prove that w acts trivially on $V(X)$. Let $(vH_n)_n$ be a vertex of X. For n large enough, we have $w = 1$ in $BS(m, \xi_n)$, so that $wvH_n = vH_n$. Hence we have $w \cdot (vH_n)_n \sim (vH_n)_n$, as desired. □

Remark 5.5 *The action of $\overline{BS}(m, \xi)$ on X is transitive and the stabilizer of the vertex $(H_n)_n$ is the subgroup of elements which are powers of b in all but finitely many $BS(m, \xi_n)$. It does not coincide with the subgroup of $\overline{BS}(m, \xi)$ generated by b, since the element ab^ma^{-1} is not in the latter subgroup, but stabilizes the vertex.*

We end this section by statements about the structure of the tree $X_{m,\xi}$. The first one is the analogue of Lemma 5.4 for edges.

Lemma 5.6 *Let $(vH_n^m)_n$ and $(wH_n^m)_n$ be two edges of the graph X. If one has $vH_n^m = wH_n^m$ for infinitely many values of n, then $(vH_n^m)_n = (wH_n^m)_n$ in X.*

PROOF. By assumption, one has $vH_n = wH_n$ and $va^{-1}H_n = wa^{-1}H_n$ for infinitely many values of n. By Lemma 5.4, we get $(vH_n)_n = (wH_n)_n$ and $(va^{-1}H_n)_n = (wa^{-1}H_n)_n$. The edges $(vH_n^m)_n$ and $(wH_n^m)_n$ having the same origin and terminal vertex, they are equal, since X is a simple graph. $\square$

Proposition 5.7 *Let $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$. Each vertex of the tree $X_{m,\xi}$ has exactly $|m|$ outgoing edges. More precisely, (given a sequence $(\xi_n)_n$ of nonzero integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$) the edges outgoing from the vertex $(wH_n)_n$ are exactly $(wH_n^m)_n, (wbH_n^m)_n, \ldots, (wb^{|m|-1}H_n^m)_n$.*

PROOF. It suffices to treat the case $w = 1$. The edges $(H_n^m)_n$, $(bH_n^m)_n, \ldots$, $(b^{|m|-1}H_n^m)_n$ are clearly outgoing from $(H_n)_n$ and distinct. Let now $(vH_n^m)_n$ be an edge outgoing from $(H_n)_n$. In particular, we have $(vH_n)_n = (H_n)_n$, so that $v = b^{\lambda_n}$ in $BS(m, \xi_n)$ for n large enough. There exists necessarily $\lambda \in \{0, \ldots, |m| - 1\}$ such that we have $\lambda_n \equiv \lambda \pmod m$ for infinitely many values of n, so that $vH_n^m = b^\lambda H_n^m$ for infinitely many values of n. By Lemma 5.6, we get $(vH_n^m)_n = (b^\lambda H_n^m)_n$ and we are done. $\square$

6 A structure theorem

We recall that $\mathbb{Z} \ltimes_{\frac{n}{m}} \mathbb{Z}[\frac{\mathrm{gcm}(m,n)}{\mathrm{lcm}(m,n)}]$ act affinely on $\mathbb{R}$ and that it is a marked quotient of both $BS(m, n)$ and $\mathbb{Z} \wr \mathbb{Z}$ (see Section 1.3).

Proposition 6.1 *For any $m \in \mathbb{Z}^ast$ amd $\xi \in \mathbb{Z}_m$, the morphism of marked groups $q\ : \mathbb{F}_2 \twoheadrightarrow \mathbb{Z} \wr \mathbb{Z}$ factors through a morphism $q_{m,\xi}\ : \overline{BS}(m, \xi) \twoheadrightarrow \mathbb{Z} \wr \mathbb{Z}$.*

PROOF. Take $(\xi_n)_{n\geq 0}$ a sequence of integers such that one has $|\xi_n| \underset{n\to\infty}{\longrightarrow} \infty$ and $\xi_n \underset{n\to\infty}{\longrightarrow} \xi$ in $\mathbb{Z}_m$. One has the following diagram

$$\begin{array}{ccc} BS(m, k_n) & \xrightarrow{n\to\infty} & \overline{BS}(m, \xi) \\ \downarrow & & \\ \Gamma(m, k_n) & \xrightarrow{n\to\infty} & \mathbb{Z} \wr \mathbb{Z} \end{array}$$

by Definition 1.6 and Theorem 2.4. Given $w \in \mathbb{F}_2$ with $w = 1$ in $\overline{BS}(m, \xi)$, it is now clear that $w = 1$ in $\mathbb{Z} \wr \mathbb{Z}$, so that the proposition holds.

$\square$

We now are able to state the main results of this section, which are the following:

Theorem 6.2 *Consider the exact sequence (where $N_{m,\xi}$ is the image of N in $\overline{BS}(m,\xi)$)*

$$1 \longrightarrow N_{m,\xi} \longrightarrow \overline{BS}(m,\xi) \xrightarrow{q_{m,\xi}} \mathbb{Z} \wr \mathbb{Z} \longrightarrow 1 \ .$$

For any $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, the group $N_{m,\xi} = \ker q_{m,\xi}$ is free.

Remark 6.3 *The second derived subgroup of $\overline{BS}(m,\xi)$ is then a free group. Thus $\overline{BS}(m,\xi)$ enjoys the same property as the generalized Baumslag-Solitar groups [Kr90, Corollary 2]*

Corollary 6.4 *For any $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, the group $\overline{BS}(m,\xi)$ has the Haagerup property and is residually solvable.*

Proof of Corollary 6.4. Looking at the exact sequence

$$1 \longrightarrow N_{m,\xi} \longrightarrow \overline{BS}(m,\xi) \xrightarrow{q} \mathbb{Z} \wr \mathbb{Z} \longrightarrow 1 \ ,$$

we see that the quotient group is amenable (it is even metabelian) and the kernel group has the Haagerup property by Theorem 6.2. We conclude by [CC$^+$01, Example 6.1.6]. As a free group is residually solvable, $\overline{BS}(m,\xi)$ is then the extension of residually solvable group by a solvable one and hence is residually solvable. $\square$

Proof of Theorem 6.2. Take $m \in \mathbb{Z}^*$, $\xi \in \mathbb{Z}_m$ and $(\xi_n)_n$ a sequence of integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$. Set X to be the tree constructed in section 5. By [Se77, Section 3.3, Theorem 4], it is sufficient to prove that $N_{m,\xi}$ acts freely on X, i.e. that any $w' \in N$ which stabilizes a vertex satisfies $w = 1$ in $\overline{BS}(m,\xi)$.

Let us take $w' \in N$ and $(vH_n)_n$ a vertex of X which is stabilized by w'. Thus $w = v^{-1}w'v$ stabilizes the vertex $(H_n)_n$, i.e. w is a power of b in all but finitely many $BS(m,\xi_n)$'s. On the other hand, as $w \in N$, Proposition 6.1 implies that the polynomial P_w associated to w is zero. Hence, (5) implies $\psi_{\xi_n}(w) = \mathrm{id}$ for all but finitely many ξ_n's. Together, these two facts imply $w = 1$ (and $w' = 1$) in $\overline{BS}(m,\xi)$. $\square$

7 Presentations for the groups $\overline{BS}(m,\xi)$

7.1 Infinite presentability of the groups $\overline{BS}(m,\xi)$

Our first goal in this Section is to prove:

Proposition 7.1 *For any $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m \setminus m\mathbb{Z}_m$, the group $\overline{BS}(m, \xi)$ is not finitely presented.*

Notice that the Proposition excludes also the existence of a finite presentation of a group $\overline{BS}(m, \xi)$ ($\xi \in \mathbb{Z}_m \setminus m\mathbb{Z}_m$) with another generating set. See for instance [dlH00, Proposition V.2]. By Corollary 3.10, there is only one remaining case, the case $\xi = 0$, where it is still unknown wether $\overline{BS}(m, 0)$ is finitely presented or not. We nevertheless make the following remark.

Remark 7.2 *For $|m| = 1$, the limits $\overline{BS}(\pm 1, \xi)$ are not finitely presented.*

Proof. The result [St05, Theorem 2] implies $\overline{BS}(\pm 1, \xi) = \mathbb{Z} \wr \mathbb{Z}$ for the unique element $\xi \in \mathbb{Z}_{\pm 1}$ and the result [Bau61] of Baumslag on the presentations of wreath products ensures that $\mathbb{Z} \wr \mathbb{Z}$ is not finitely presented. $\square$

Lemma 7.3 *Let $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m \setminus m\mathbb{Z}_m$. Let ℓ be the maximal exponent in the decomposition of m in prime factors and set $d = gcd(m, \xi)$, $m_1 = m/d$.*

(a) There exists a sequence $(\xi_n)_n$ in $\mathbb{Z}^*$ such that for all $n \geqslant 1$ one has $|\xi_n| > |\xi_{n-1}|$ and

$$\begin{aligned} \xi_n &\equiv \xi \pmod{m^n \mathbb{Z}_m} ; \\ \xi_n &\not\equiv \xi \pmod{m_1^{\ell n+1} d\mathbb{Z}_m} . \end{aligned}$$

(b) This sequence satisfies $|\xi_n| \to \infty$, $\xi_n \to \xi$ and

$$\begin{aligned} \xi_n &\equiv \xi_r \quad \pmod{m^n} \quad && \forall r \geqslant n ; \\ \xi_n &\not\equiv \xi_{\ell n+1} \pmod{m_1^{\ell n+1} d} && \forall n . \end{aligned}$$

Proof. **(a)** Let p be a prime factor of m_1 (there exists one, for $\xi \notin m\mathbb{Z}_m$). The sequence (ξ_n) is constructed inductively. We choose for ξ_0 any nonzero integer such that $\xi_0 - \xi \notin m\mathbb{Z}_m$. At the n-th step, we begin by noticing that the exponent of p in the decomposition of m^n (respectively $m_1^{\ell n+1} d$) is at most ℓn (respectively at least $\ell n + 1$). Hence, $m_1^{\ell n+1} d$ is not a multiple of m^n, so that there exists $\alpha \in \mathbb{Z}$ with $\xi \equiv \alpha \pmod{m^n}$ but $\xi \not\equiv \alpha \pmod{m_1^{\ell n+1} d}$.

Notice now that we may replace α by any element of the class $\alpha + m^n m_1^{\ell n+1} d\mathbb{Z}$, so that it suffices to choose ξ_n among the elements β in the latter class which satisfy $|\beta| > |\xi_{n-1}|$.

(b) The properties $\xi_n \equiv \xi \pmod{m^n \mathbb{Z}_m}$ and $|\xi_n| > |\xi_{n-1}|$ imply clearly $\xi_n \to \xi$, $|\xi_n| \to \infty$ and $\xi_n \equiv \xi_r \pmod{m^n}$ for $r \geqslant n$ (for the latter one, Proposition 1.1 (d) is used).

Finally, combining the properties $\xi_{\ell n+1} \equiv \xi \pmod{m^{\ell n+1}\mathbb{Z}_m}$ and $\xi_n \not\equiv \xi \pmod{m_1^{\ell n+1} d\mathbb{Z}_m}$ gives $\xi_n \not\equiv \xi_{\ell n+1} \pmod{m_1^{\ell n+1} d}$. $\square$

Proof of Proposition 7.1. The hypothesis $\xi \in \mathbb{Z}_m \setminus m\mathbb{Z}_m$ implies $m \geqslant 2$. Take ℓ, d, m_1 and a sequence (ξ_n) as in Lemma 7.3. One has then $BS(m, \xi_n) \to \overline{BS}(m, \xi)$. It is thus sufficient by Lemma 1.5 to prove that the $BS(m, \xi_n)$'s are not marked quotients of $\overline{BS}(m, \xi)$ (for n large enough).

Notice now that (for n large enough) one has $\gcd(m, \xi_n) = d$ since $\xi_n \equiv \xi \pmod{m\mathbb{Z}_m}$ holds. Set the words

$$w_n = a^{n+1} b^m a^{-1} b^{-\xi_n} a^{-n} b a^{n+1} b^{-m} a^{-1} b^{\xi_n} a^{-n} b^{-1}.$$

Part (b) of Lemma 7.3 combined with Lemma 3 of [St05] give $w_n = 1$ in $BS(m, \xi_r)$ for all $r \geqslant n$, hence $w_n = 1$ in $\overline{BS}(m, \xi)$ (for n large enough). On the other hand we get $w_{\ell n+1} \neq 1$ in $BS(m, \xi_n)$ the same way, so that $BS(m, \xi_n)$ is not a marked quotient of $\overline{BS}(m, \xi)$ (for n large enough). $\square$

7.2 Defining a presentation for $\overline{BS}(m, \xi)$

For $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, we define the set $\mathcal{R} = \mathcal{R}_{m,\xi}$ by

$$\begin{aligned} \mathcal{R}_{m,\xi} \;=\; \big\{ w\bar{w} : w = a b^{\alpha_1} \cdots a b^{\alpha_k} a^{-1} b^{\alpha_{k+1}} \cdots a^{-1} b^{\alpha_{2k}} \\ \text{with } k \in \mathbb{N}^*, \alpha_i \in \mathbb{Z}\ (i = 1, \ldots, 2k) \text{ and } w \cdot v_0 = v_0 \big\} \end{aligned}$$

where v_0 is the favoured vertex $(H_n)_n$ of the tree $X_{m,\xi}$ defined in Section 5.

Recall that the stabilizer of the vertex v_0 consists of elements which are powers of b in all but finitely many $BS(m, \xi_n)$, where $(\xi_n)_n$ is any sequence of integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$ (see Remark 5.5). It follows that we have $w \cdot v_0 = v_0 \Leftrightarrow \bar{w} \cdot v_0 = v_0$.

The aim of the present Section is to prove the following statement.

Theorem 7.4 *For all $m \in \mathbb{Z}^*$ and $\xi \in \mathbb{Z}_m$, the marked group $\overline{BS}(m, \xi)$ admits the presentation $\langle a, b \mid \mathcal{R}_{m,\xi} \rangle$.*

Set $\Gamma = \langle a, b \mid \mathcal{R}_{m,\xi} \rangle$ for this Section. It is obvious that the elements of $\mathcal{R}_{m,\xi}$ are trivial in $\overline{BS}(m, \xi)$, so that one has a marked (hence surjective) homomorphism $\Gamma \to \overline{BS}(m, \xi)$. Theorem 7.4 is thus reduced to the following proposition, which gives the injectivity.

Proposition 7.5 *Let w be a word on the alphabet $\{a, a^{-1}, b, b^{-1}\}$. If one has $w = 1$ in $\overline{BS}(m, \xi)$, then the equality $w = 1$ also holds in Γ.*

Before to prove Proposition 7.5, we need to introduce some notions what can give rise to geometric interpretations.

From words to paths. Let us call *path* (in a graph) any finite sequence of vertices such that each of them is adjacent to the preceding one. Let w be any word on the alphabet $\{a, a^{-1}, b, b^{-1}\}$. It defines canonically a path in the Cayley graph of Γ (or $\overline{BS}(m,\xi)$) which starts at the trivial vertex. Let us denote those paths by $p_\Gamma(w)$ and $p_{\overline{BS}}(w)$. The map $\Gamma \to \overline{BS}(m,\xi)$ defines a graph morphism which sends the path $p_\Gamma(w)$ onto the path $p_{\overline{BS}}(w)$.

The word w defines the same way a finite sequence of vertices in $X_{m,\xi}$ starting at v_0 and such that each of them is equal or adjacent to the preceding one. Indeed, let f be the map $V\big(Cay(\overline{BS}(m,\xi),(a,b))\big) \to V(X_{m,\xi})$ defined by $f(g) = g \cdot v_0$ for any $g \in \overline{BS}(m,\xi)$. If g, g' are adjacent vertices in $Cay(\overline{BS}(m,\xi),(a,b))$, then $f(g)$ and $f(g')$ are either adjacent (case $g' = ga^{\pm 1}$), or equal (case $g' = gb^{\pm 1}$). The sequence associated to w is the image by f of the path $p_{\overline{BS}}(w)$. Now, deleting consecutive repetitions in this sequence, we obtain a path that we denote by $p_X(w)$.

It follows that if the word w satisfies $w = 1$ in $\overline{BS}(m,\xi)$ (or, stronger, $w = 1$ in Γ), then the path $p_X(w)$ is closed (i.e. its last vertex is v_0).

Height and Valleys. Recall that one has a homomorphism σ_a from $\overline{BS}(m,\xi)$ onto $\mathbb{Z}$ given by $\sigma_a(a) = 1$ and $\sigma_a(b) = 0$. Given a vertex v in $X_{m,\xi}$, we call *height* of v the number $h(v) = \sigma_a(g)$ where g is any element of $\overline{BS}(m,\xi)$ such that $g \cdot v_0 = v$. It is easy to check that any element g' of $\overline{BS}(m,\xi)$ which defines an elliptic automorphism of $X_{m,\xi}$ satisfies $\sigma_a(g') = 0$, so that the height function is well-defined. It is clear from construction that the height difference between two adjacent vertices is 1.

Given $L \geqslant 1$ and $k \geqslant 1$, we call (L,k)-*valley* any path p in $X_{m,\xi}$ such that one has:

- $p = (v_0, v_1, \ldots, v_L = \nu_0, \nu_1, \ldots, \nu_{2k})$, where v_0 is the favoured vertex;
- $h(v_0) = 0 = h(\nu_k)$ and $h(\nu_0) = -k = h(\nu_{2k})$;
- $h(v) < 0$ for any other vertex v of p;
- $\nu_0 = \nu_{2k}$.

Given a (L,k)-valley $p = (v_0, v_1, \ldots, v_L = \nu_0, \nu_1, \ldots, \nu_{2k})$, the subpaths $(\nu_0, \ldots, \nu_k)$ and $(\nu_k, \ldots, \nu_{2k} = \nu_0)$ have to be geodesic, for the heigth difference between $\nu_0 = \nu_{2k}$ and ν_k is k. Thus, one has $\nu_1 = \nu_{2k-1}, \ldots, \nu_{k-1} = \nu_{k+1}$.

Lemma 7.6 *Let w be a word on the alphabet $\{a, a^{-1}, b, b^{-1}\}$ such that the path $p_X(w)$ is a (L,k)-valley, say $p_X(w) = (v_0, v_1, \dots, v_L = \nu_0, \nu_1, \dots, \nu_{2k})$. There exists a word w' such that the equality $w' = w$ holds in Γ and the path $p_X(w')$ is $(v_0, v_1, \dots, v_L)$.*

Proof. We argue by induction on L.

Case $L = 1$: In that case, one has $k = 1$, $p_X(w) = (v_0, v_1 = \nu_0, \nu_1, \nu_2)$. Up to replacing w by a word which defines the same element in $\mathbb{F}_2$ (hence in Γ) and the same path in X, we may assume to have $w = b^{\alpha_0} a^{-1} b^{\alpha_1} a b^{\beta_1} a^{-1} b^{\beta_2}$. Set $r = ab^{-\beta_1} a^{-1} b^{-\alpha_1} a b^{\beta_1} a^{-1} b^{\alpha_1}$. Since ν_0 and ν_2 are equal, the subword $ab^{\beta_1}a^{-1}$ (of w) defines a closed subpath in X, so that we obtain $ab^{-\beta_1} a^{-1} b^{-\alpha_1} \cdot v_0 = v_0$. Consequently, we get $r \in \mathcal{R}$, whence $r = 1$ in Γ. Inserting r in next to last position, we obtain

$$w \underset{\Gamma}{=} b^{\alpha_0 + \beta_1} a^{-1} b^{\alpha_1 + \beta_2} =: w' \ .$$

This equality also implies that the paths $p_X(w)$ and $p_X(w')$ have the same endpoint. Hence one has $p_X(w') = (v_0, \nu_2) = (v_0, v_1)$ and we are done.

Induction step: We assume $L > 1$ to hold. Up to replace w by a word which defines the same element in $\mathbb{F}_2$ (hence in Γ) and the same path in X, we may write

$$w = b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \cdots a^{\varepsilon_L} b^{\alpha_L} \cdot ab^{\beta_1} \cdots ab^{\beta_k} a^{-1} b^{\beta_{k+1}} \cdots a^{-1} b^{\beta_{2k}}$$

with $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathbb{Z}$ for all i. We distinguish two cases:

(1) the vertex v_{L-1} is higher than v_L (i.e. $\varepsilon_L = -1$);

(2) the vertex v_{L-1} is lower than v_L (i.e. $\varepsilon_L = 1$).

Case (1): Set

$$\begin{aligned} r \; = \; & ab^{-\beta_{2k-1}} \cdots ab^{-\beta_k} a^{-1} b^{-\beta_{k-1}} \cdots a^{-1} b^{-\beta_1} a^{-1} b^{-\alpha_L} \cdot \\ & ab^{+\beta_{2k-1}} \cdots ab^{+\beta_k} a^{-1} b^{+\beta_{k-1}} \cdots a^{-1} b^{+\beta_1} a^{-1} b^{+\alpha_L} \ . \end{aligned}$$

Since ν_0 and ν_{2k} are equal, the subword $ab^{\beta_1} \cdots ab^{\beta_k} a^{-1} b^{\beta_{k+1}} \cdots a^{-1} b^{\beta_{2k-1}} a^{-1}$ (of w) defines a closed subpath in X, so that (when considered as a word on its own right) it stabilizes the vertex v_0. Inverting it, we get

$$ab^{-\beta_{2k-1}} \cdots ab^{-\beta_k} a^{-1} b^{-\beta_{k-1}} \cdots a^{-1} b^{-\beta_1} a^{-1} \cdot v_0 = v_0 \ .$$

It implies $r \in \mathcal{R}$, whence $r = 1$ in Γ. Inserting r in next to last position, we obtain

$$\begin{aligned} w \underset{\Gamma}{=} w^* \quad := \quad & b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \cdots a^{\varepsilon_{L-1}} b^{\alpha_{L-1}+\beta_{2k-1}} \cdot \\ & a b^{\beta_{2k-2}} \cdots a b^{\beta_k} a^{-1} b^{\beta_{k-1}} \cdots a^{-1} b^{\beta_1} \cdot \\ & a^{-1} b^{\alpha_L + \beta_{2k}} \ . \end{aligned}$$

We write $w^* = w'' a^{-1} b^{\alpha_L+\beta_{2k}}$. Since the words w and w^* begin the same way and since one has $w = w^*$ in Γ, the path $p_X(w^*)$ has the form

$$(v_0, v_1, \dots, v_{L-1} = \omega_0, \omega_1, \dots, \omega_{2k-2}, v_L) \ .$$

Since v_{L-1} is higher than v_L, we have $h(v_{L-1}) = -(k-1)$. Contemplating w^*, one sees that we have $h(\omega_{k-1}) = 0$, $h(\omega_{2k-2}) = -(k-1)$, so that the subpaths $(\omega_0, \dots, \omega_{k-1})$ and $(\omega_{k-1}, \dots, \omega_{2k-2}, v_L))$ are geodesic. On the other hand, the geodesic between ω_{k-1} and v_L passes through $v_{L-1} = \omega_0$, so that we have $\omega_{2k-2} = v_{L-1}$. It follows that $p_X(w'')$ has the form $(v_0, v_1, \dots, v_{L-1} = \omega_0, \omega_1, \dots, \omega_{2k-2} = v_{L-1})$, so that it is a $(L-1, k-1)$-valley. We apply the induction hypothesis to w'' and get a word w''' such that $w'' = w'''$ holds in Γ and $p_X(w''') = (v_0, \dots, v_{L-1})$. It suffices to set $w' = w''' a^{-1} b^{\alpha_L+\beta_{2k}}$ to conclude.

Case (2): In this case, we have $h(v_L) = -k$ and $h(v_{L-1}) = -(k+1)$, so that the edge linking these vertices goes from v_L to v_{L-1}. By Proposition 5.7, there exists $\lambda \in \{0, \dots\ |m|-1\}$ such that $w b^\lambda a^{-1} \cdot v_0 = v_{L-1}$. Set $w^* = w b^\lambda a^{-1}$ and $w' = w^* a b^{-\lambda}$, so that $w = w'$ holds in Γ. The path $p_X(w^*)$ is a $(L-1, k+1)$-valley, so that we may apply the induction hypothesis to w^*. This gives a word w'' such that $w'' = w^*$ in Γ and $p_X(w'') = (v_0, \dots, v_{L-1})$. We conclude by setting $w' = w'' a b^{-\lambda}$. □

Proof of Proposition 7.5. Let w be a word which defines the trivial element in $\overline{BS}(m, \xi)$. Up to replacing w by a word which defines the same element in $\mathbb{F}_2$ (hence in Γ), we may write

$$w = b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \cdots a^{\varepsilon_\ell} b^{\alpha_\ell}$$

with $\varepsilon_i = \pm 1$ and $\alpha_i \in \mathbb{Z}$ for all i. The path $p_X(w)$ is closed and has length ℓ. We argue by induction on ℓ.

Case $\ell = 0$: In this case, one has $w = b^{\alpha_0}$, so that $b^{\alpha_0} = 1$ in $\overline{BS}(m, \xi)$. It follows $\alpha_0 = 0$, hence $w = 1$ in Γ.

Induction step ($\ell > 0$): Let us write $p_X(w) = (v_0, v_1, \dots, v_{\ell-1}, v_\ell = v_0)$. Up to replace the word w by a cyclic conjugate, we may assume (without changing ℓ) that $h(v_i) \leqslant 0$

for all i. Denote by $k_0 = 0 < k_1 < \ldots < k_s = \ell$ the indices such that $h(v_k) = 0$. We now distinguish two possibilities:

(1) The path $p_X(w)$ turns back at some v_{k_i} (i.e. $\exists i$ with $1 \leqslant i \leqslant s-1$ such that $v_{k_i+1} = v_{k_i-1}$.)

(2) The path $p_X(w)$ does not turn back at any v_{k_i} (i.e. $\forall i$ with $1 \leqslant i \leqslant s-1$ one has $v_{k_i+1} \neq v_{k_i-1}$.)

Case(1): Let i be an index (with $1 \leqslant i \leqslant s-1$) such that $v_{k_i+1} = v_{k_i-1}$. The subword $w' = a^{\varepsilon_{k_{i-1}+1}} b^{\alpha_{k_{i-1}+1}} \cdots a^{\varepsilon_{k_i}} b^{\alpha_{k_i}} a^{\varepsilon_{k_i+1}}$ defines by construction a $(k_i - k_{i-1} - 1, 1)$-valley in the tree X. Lemma 7.6 furnishes then a word w'' such that $w'' = w'$ holds in Γ and the path $p_X(w'')$ is strictly shorter than $p_X(w')$. We construct a word w^* by replacing w' by w'' in w. The path $p_X(w^*)$ is strictly shorter than $p_X(w)$ and one has $w^* = w$ in Γ. Applying the induction hypothesis to w^*, we get $w^* = 1$ in Γ, hence $w = 1$ in Γ.

Case (2): Suppose first that $s = 1$. All vertices of $p_X(w)$ but v_0 and v_L have strictly negative height. It follows that $\alpha_0 = -\alpha_\ell$, for we have $w = 1$ in $\mathbb{Z} \wr \mathbb{Z}$ by hypothesis. Since we also have $\varepsilon_1 = -1$ and $\varepsilon_\ell = 1$ by construction, we get

$$ab^{-\alpha_0} w b^{\alpha_0} a^{-1} \underset{\Gamma}{=} b^{\alpha_1} a^{\varepsilon_2} b^{\alpha_2} \cdots a^{\varepsilon_{\ell-1}} b^{\alpha_{\ell-1}} =: w' \ .$$

Since the path $p_X(w')$ is strictly shorter than $p_X(w)$, we may apply the induction hypothesis to w' and we obtain $w' = 1$, hence $w = 1$, in Γ.

The case $s = 2$ is impossible, for it would imply that the path $p_X(w)$ turns back at v_{k_1}, which is incompatible with case (2).

From now on, we suppose $s \geqslant 3$. There exists some i in $\{1, \ldots, s-2\}$ such that one has $v_{k_i} = v_{k_{i+1}}$ (think at the v_{k_i} which is at most far from v_0). We set

$$w^* := a^{\varepsilon_{k_i+1}} b^{\alpha_{k_i+1}} \cdots a^{\varepsilon_{k_{i+1}-1}} b^{\alpha_{k_{i+1}-1}} a^{\varepsilon_{k_{i+1}}}$$

(it is a subword of w). Let us now fix a sequence $(\xi_n)_n$ of nonzero integers such that $|\xi_n| \to \infty$ and $\xi_n \to \xi$ in $\mathbb{Z}_m$. Then, we consider the morphisms $\psi_n : BS(m, \xi_n) \to \mathrm{Aff}(\mathbb{R})$ given by $\psi_n(a)(x) = \frac{\xi_n}{m} x$ and $\psi_n(b)(x) = x+1$. The word w^* stabilizing v_0 by construction, we get $w^* = b^{\lambda_n}$ in $BS(m, \xi_n)$, which implies $\psi_n(w^*) = x + \lambda_n$ (with $\lambda_n \in \mathbb{Z}$), for n large enough. In the other hand, seeing the word w^* itself, we get

$$\lambda_n = \sum_{j=k_i+1}^{k_{i+1}-1} \alpha_j \left(\frac{\xi_n}{m}\right)^{h(v_j)}$$

for those values of n. Then, taking absolute values, this gives

$$|\lambda_n| \leqslant \sum_{j=k_i+1}^{k_{i+1}-1} |\alpha_j \left(\frac{\xi_n}{m}\right)^{h(v_j)}| \leqslant \frac{|m|}{|\xi_n|} \sum_{j=k_i+1}^{k_{i+1}-1} |\alpha_j| \ .$$

It follows that $|\lambda_n| < 1$ holds for n large enough, since $|\xi_n|$ tends to ∞. For those values of n, we get $w^* = b^0 = 1$ in $BS(m, \xi_n)$. Consequently, we get $w^* = 1$ in $\overline{BS}(m, \xi)$.

By construction (we use the assumption $i \in \{1, \ldots, s-2\}$), the path $p_X(w^*)$ is strictly shorter than $p_X(w)$, so that we apply the induction hypothesis to w^* and get $w^* = 1$ in Γ. Erasing, w^* in w, one gets

$$w \underset{\Gamma}{=} w' = b^{\alpha_0} a^{\varepsilon_1} b^{\alpha_1} \cdots a^{\varepsilon_{k_i}} b^{\alpha_{k_i} + \alpha_{k_{i+1}}} a^{\varepsilon_{k_{i+1}+1}} b^{\alpha_{k_{i+1}+1}} \cdots a^{\varepsilon_\ell} b^{\alpha_\ell} \ .$$

Applying the induction hypothesis to w', we get $w = 1$ in Γ. □

References

[AB+03] G. N. Arzhantseva, J. Burillo, L.Reeves, M. Lustig, H. Short and E. Ventura. Uniform non-Amenability. Preprint 2003.

[Bau61] G. Baumslag, Wreath products and finitely presented groups, Math. Zeitschr. 75, 22-28 (1961).

[Bau67] G. Baumslag, Residually finite one-relator groups, Bull. Amer. Math. Soc. 74 618-620 (1967).

[Bau76] G. Baumslag, R. Strebel, Some finitely generated, infinitely related metabelian groups with trivial multiplicator. J. Algebra 40 (1976), no. 1, 46–62. (Reviewer: Ian M. Chiswell) 20F05.

[Ch00] C. Champetier, L'espace des groupes de type fini, Topology **39** (2000), 657–680.

[CG04] C. Champetier and V. Guirardel. Limit groups as limits of free groups: compactifying the set of free groups. Preprint, available on arXiv:math.GR/0401042 v2.

[CC+01] P. A. Cherix, M. Cowling, P. Jolissaint, P. Julg and A. Valette. *Groups with the Haagerup property. Gromov's a-T-menability.*. Birkhäuser, 2001.

[dlH00] P. de la Harpe, Topics in geometric group theory, Chicago Lectures in Mathematics, The University of Chicago Press 2000.

[GJ03] Ś. R. Gal and T. Januszkiewicz. New a-t-menable HNN-extensions. *Journal of Lie Theory*, 13:383–385, 2003.

[Kr90] P. H. Kropholler, Baumslag-Solitar groups and some other groups of cohomological dimension two, Comment. Math. Helvetici **65** (1990) 547-558.

[LS77] R.C. Lyndon, P.E. Schupp, Combinatorial Group Theory, Springer 1977.

[Os02] D. V. Osin, Weakly amenable groups, Contemporarary Mathematics, **298**, (2002), 105–112.

[Se77] J. P. Serre. *Arbres, amalgammes,* SL_2. Astérisque 46, Société mathématique de France, 1977.

[Sou00] E. Souche, Thèse de Doctorat, Annexe 3, pp 69–90.

[Sh00] Y Shalom. Rigidity of commensurators and irreducible lattices. *Inventiones mathematicae*, 141:1–54, 2000.

[St05] Y. Stalder. Convergence of Baumslag-Solitar groups. Preprint, available on arXiv:math.GR/0403241 v3, to appear in Bulletin of the Belgian Mathematical Society – Simon Stevin.

[Step93] A. M. Stepin, Approximation of groups and group actions, the Cayley topology. Ergodic theory of Z^d actions (Warwick, 1993–1994), 475–484.

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