Isometric group actions on Hilbert spaces: structure of orbits

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November 8, 2005

Abstract

Our main result is that a finitely generated nilpotent group has no isometric action on an infinite dimensional Hilbert space with dense orbits. In contrast, we construct such an action with a finitely generated metabelian group.

Mathematics Subject Classification: Primary 22D10; Secondary 43A35, 20F69.
Key words and Phrases: Affine actions, Hilbert spaces, minimal actions, nilpotent groups.

1 Introduction

The study of isometric actions of groups on affine Hilbert spaces has, in recent years, found applications ranging from the K-theory of C^*-algebras [HiKa], to rigidity theory [Sh2] and geometric group theory [Sh3, CTV]. This renewed interest motivates the following general problem: *How can a given group act by isometries on an affine Hilbert space?*

This paper is a sequel to [CTV], but can be read independently. In [CTV], we focused, given an an isometric action of a finitely generated group G on a Hilbert space $\alpha : G \to \mathrm{Isom}(\mathcal{H})$, on the growth of the function $g \mapsto \alpha(g)(0)$. Here the emphasis is on the structure of orbits.

In §2, we consider affine isometric actions of $\mathbf{Z}^n$ or $\mathbf{R}^n$. On finite-dimensional Euclidean spaces, the situation is clear-cut: such an action is an orthogonal sum of a bounded action and an action by translations. Even if the general case is more subtle, something remains from the finite-dimensional case. We say that a convex subset of a Hilbert space is *locally bounded* if its intersection with any finite dimensional subspace is bounded.

Theorem. (see Theorem 2.2) *Let either $\mathbf{Z}^n$ or $\mathbf{R}^n$ act isometrically on a Hilbert space $\mathcal{H}$, with linear part π. Let $\mathcal{O}$ be an orbit under this action. Then there exist*

- *a subspace T of $\mathcal{H}$ (the "translation part"), contained in the invariant vectors of π, of finite dimension $\leq n$, and*
- *a closed, locally bounded convex subset U of the orthogonal subspace $T^{\perp}$,*

such that $\mathcal{O}$ is contained in $T \times U$.

In §3, we address a question due to A. Navas: which locally compact groups admit an affine isometric action with *dense* orbits (i.e. a minimal action) on an infinite-dimensional Hilbert space?

The main result of the paper is a negative answer in the case of finitely generated nilpotent groups.

Theorem. (see Theorem 3.15 and its corollaries) *A compactly generated, nilpotent-by-compact group does not admit any affine isometric action with dense orbits on an infinite-dimensional Hilbert space.*

Actually, for compactly generated nilpotent groups, one can describe all affine isometric actions with dense orbits; see Corollary 3.16.

In the course of our proof, we introduce the following new definitions: a unitary or orthogonal representation π of a group is *strongly cohomological* if it satisfies: for every nonzero subrepresentation $\rho \leq \pi$, we have $H^1(G, \rho) \neq 0$. It is easy to observe that the linear part of a affine isometric action with dense orbits is strongly cohomological. The non-trivial step in the proof of the main theorem is the following result.

Proposition. (see Corollary 3.14) *Let π be an orthogonal or unitary representation of a second countable, nilpotent group G. Suppose that π is strongly cohomological. Then π is a trivial representation.*

Another case for which we have a negative answer is the following.

Theorem. (see Theorem 3.18) *Let G be a connected semisimple Lie group. Then G has no isometric action on a nonzero Hilbert space with dense orbits.*

It is not clear how the main theorem can be generalized, in view of the following example.

Proposition. (see Proposition 3.2) *There exists a finitely generated metabelian group admitting an affine isometric action with dense orbits on $\ell^2_{\mathbf{R}}(\mathbf{Z})$.*

Recall that an isometric action $\alpha : G \to \mathrm{Isom}(\mathcal{H})$ *almost has fixed points* if for every $\varepsilon > 0$ and every compact subset $K \subset G$ there exists $v \in \mathcal{H}$ such that $\sup_{g\in K} \|v - \alpha(g)v\| \leq \varepsilon$. There is a link between this notion and strongly cohomological representations.

Proposition. (see Proposition 3.10) *Let G be a topological group and α an isometric action on a Hilbert space that does not almost have fixed points. Then its linear part π has a nonzero subrepresentation that is strongly cohomological.*

However the converse is not true as shown by the following example.

Proposition. (see Proposition 3.4) *There exists a countable group admitting an affine isometric action with dense orbits, almost having fixed points on $\ell^2_{\mathbf{R}}(\mathbf{N})$ (more precisely, every finitely generated subgroup has a fixed point).*

Acknowledgements. *We thank A. Fathi and A. Navas for a useful indication concerning Theorem 2.2.*

2 Actions of $\mathbf{Z}^n$ and $\mathbf{R}^n$

Let $\mathcal{H}$ be a Hilbert space.

Definition 2.1. A convex subset K of $\mathcal{H}$ is said to be locally bounded if $K \cap F$ is bounded for every finite-dimensional subspace F of $\mathcal{H}$.

Theorem 2.2. *Let $G = \mathbf{Z}^n$ or $\mathbf{R}^n$ act isometrically on a Hilbert space $\mathcal{H}$, with linear part π. Let $\mathcal{O}$ be an orbit under this action. Then there exist*

- *a subspace T of $\mathcal{H}$, contained in $\mathcal{H}^{\pi(G)}$, of finite dimension $\leq n$, and*
- *a closed, locally bounded convex subset U of $T^{\perp}$,*

such that $\mathcal{O}$ is contained in $T \times U$.

Proof. The case of $\mathbf{R}^n$ is reduced to the case of $\mathbf{Z}^n$ by taking a dense, free abelian subgroup of finite rank in $\mathbf{R}^n$.

Let $(\pi, \mathcal{H})$ be a unitary representation of $\mathbf{Z}^n$. Let $b \in Z^1(\mathbf{Z}^n, \pi)$ define an affine action of $\mathbf{Z}^n$ with linear part π, and let $\mathcal{O}$ be an orbit. We can suppose that $0 \in \mathcal{O}$, so that $\mathcal{O}$ is the range of b.

To emphasize the main idea of the proof, let us start with the case when $n = 1$. Write $\mathcal{H}_0 = \mathrm{Ker}(\pi(1) - \mathrm{Id}) = \mathcal{H}^{\pi(G)}$. The representation decomposes as $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Denote by π_0 and π_1 the corresponding subrepresentations of π. The cocycle b decomposes as $b = b_0 + b_1$. Note that b_0 is an additive morphism:

$\mathbf{Z} \to \mathcal{H}_0$; define T as the linear subspace generated by $b_0(1)$. On the other hand, let us show that the sequence $(b_1(k))_{k\in\mathbf{Z}}$ is contained in a locally bounded convex subset of $\mathcal{H}_1$. First, note that

$$\|(\pi(1) - \mathrm{Id})b(k)\| \ \leq\ 2\|b(1)\|.$$

Indeed, since $b(k) = \sum_{j=0}^{k-1} \pi(1)^j b(1)$, we get

$$(\pi(1) - \mathrm{Id})b(k) = (\pi(k) - \mathrm{Id})b(1).$$

Moreover, since $\mu = \pi_1(1) - \mathrm{Id}$ is injective, it follows that the closed convex set $U = \mu^{-1}(B(0, 2\|b(1)\|))$ is locally bounded, and $\mathcal{O}$ is contained in $T \times U$.

Let us turn to the general case. Write $I = \{1, \dots, n\}$. Let $e_1, \dots, e_n$ be the canonical basis of $\mathbf{Z}^n$. Define, for every subset $J \subset I$, a closed subspace $\mathcal{H}_J$ of $\mathcal{H}$, as follows: $\mathcal{H}'_J = \{\xi \in \mathcal{H}, \forall i \in I - J,\ \pi(e_i)\xi = \xi\}$, and $\mathcal{H}_J$ is the orthogonal subspace in $\mathcal{H}'_J$ of $\sum_{K\subsetneq J} \mathcal{H}'_K$. It is immediate that $\mathcal{H}$ is the direct sum of all $\mathcal{H}_J$'s ($J \subset I$), and that $\mathcal{H}_J$ is $\mathbf{Z}^n$-stable, defining a subrepresentation π_J of π.

The cocycle b decomposes as $b = \sum_J b_J$. Since $\pi_\emptyset$ is a trivial representation, $b_\emptyset$ is given by a morphism: $\mathbf{Z}^n \to \mathcal{H}_\emptyset$. Let T_π denote the (finite-dimensional) subspace generated by $b_\emptyset(\mathbf{Z}^n)$.

Let J be any nonempty subset of I, and fix $i \in J$. Then $\pi_J(e_i) - 1$ is injective. For all $j \notin J$, so that $\pi_J(e_j) = 1$, we have $b_J(e_j) = 0$. Indeed, expanding the relation $b_J(e_i + e_j) = b_J(e_j + e_i)$, we obtain $(\pi(e_i) - 1)b(e_j) = 0$. Thus, the affine action associated to b_J is trivial on all e_j, $j \notin J$. Set $\mu_J = \prod_{j\in J}(\pi_J(e_j) - 1)$. Then μ_J is injective on $\mathcal{H}_J$. Let $\Omega_J \subset \mathcal{H}_J$ be the range of b_J. We easily check that

$$\mu_J\left(b_J\left(\sum_j n_j e_j\right)\right) \leq \sum_{j\in J} 2^n \|b_J(e_j)\|,$$

which is bounded. Thus, Ω_J is contained in $\mu_J^{-1}(B_J)$ for some ball B_J; since μ_J is injective, $\mu_J^{-1}(B_J)$ is a locally bounded convex set. Write $U = \bigoplus_{J\neq\emptyset} \mu_J^{-1}(B_J)$: this is a closed locally bounded convex subset of $\mathcal{H}$, contained in the orthogonal of $\mathcal{H}_\emptyset$. By construction, the orbit Ω of zero for the action associated to b is contained in $T_\pi \times U$. $\square$

3 Actions with dense orbits

We owe the following question to A. Navas.

Question 1 (Navas). Which finitely generated groups acts isometrically on a infinite-dimensional separable Hilbert space with a dense orbit?

More generally, the question makes sense for compactly generated groups. In the case of $\mathbf{Z}^n$ or $\mathbf{R}^n$, the answer is provided by Theorem 2.2.

Corollary 3.1. *Any isometric action with dense orbits of either $\mathbf{Z}^n$ or $\mathbf{R}^n$ on a Hilbert space $\mathcal{H}$, factors through an additive homomorphism with dense image to $\mathcal{H}$ (so that $\mathcal{H}$ is finite-dimensional).* □

3.1 Existence results

Here is a first positive result regarding Navas' question.

Proposition 3.2. *There exists an isometric action of a metabelian 3-generator group on a infinite-dimensional separable Hilbert space, all of whose orbits are dense.*

Proof. Observe that $\mathbf{Z}[\sqrt{2}]$ acts by translations, with dense orbits, on $\mathbf{R}$; so the free abelian group of countable rank $\mathbf{Z}[\sqrt{2}]^{(\mathbf{Z})}$ acts by translations, with dense orbits, on $\ell^2_{\mathbf{R}}(\mathbf{Z})$. Observe now that the latter action extends to the wreath product $\mathbf{Z}[\sqrt{2}] \wr \mathbf{Z} = \mathbf{Z}[\sqrt{2}]^{(\mathbf{Z})} \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts on $\ell^2_{\mathbf{R}}(\mathbf{Z})$ by the shift. That wreath product is metabelian, with 3 generators. □

Corollary 3.3. *There exists an isometric action of a free group of finite rank on a Hilbert space, with dense orbits.* □

In the example given by Proposition 3.2, the given isometric action clearly does not almost have fixed points, i.e. it defines a non-zero element in reduced 1-cohomology. The next result shows that this is not always the case.

Proposition 3.4. *There exists a countable group Γ with an affine isometric action α on a Hilbert space, such that α has dense orbits, and every finitely generated subgroup of Γ has a fixed point. In particular, the action almost has fixed points.*

Proof. We first construct an uncountable group G and an affine isometric action having dense orbits and almost having fixed points.

In $\mathcal{H} = \ell^2_{\mathbf{R}}(\mathbf{N})$, let A_n be the affine subspace defined by the equations

$$x_0 = 1,\ x_1 = 1, \dots,\ x_n = 1,$$

and let G_n be the pointwise stabilizer of A_n in the isometry group of $\mathcal{H}$. Let G be the union of the G_n's. View G as a discrete group.

It is clear that G almost has fixed points in $\mathcal{H}$, since any finite subset of G has a fixed point. Let us prove that G has dense orbits.

Claim 1. For all $x, y \in \mathcal{H}$, we have $\lim_{n\to\infty} |d(x, A_n) - d(y, A_n)| = 0$.

By density, it is enough to prove Claim 1 when x, y are finitely supported in $\ell^2_{\mathbf{R}}(\mathbf{N})$. Take $x = (x_0, x_1, ..., x_k, 0, 0, ...)$ and choose $n > k$. Then

$$d(x, A_n)^2 = \sum_{j=0}^{k}(x_j - 1)^2 + \sum_{j=k+1}^{n} 1^2 = n + 1 - 2\sum_{j=0}^{k} x_j + \sum_{j=0}^{k} x_j^2,$$

so that $d(x, A_n) = \sqrt{n} + O(\frac{1}{\sqrt{n}})$, which proves Claim 1.

Claim 2. G has dense orbits in $\mathcal{H}$.

Observe that two points $x, y \in \mathcal{H}$ are in the same G_n-orbit if and only if $d(x, A_n) = d(y, A_n)$. Fix $x_0, z \in \mathcal{H}$. We want to show that $\lim_{n\to\infty} d(G_n x_0, z) = 0$. So fix $\varepsilon > 0$; by the first claim, $|d(x_0, A_n) - d(z, A_n)| < \varepsilon$ for n large enough. So we find $y \in \mathcal{H}$ such that $\|y - z\| < \varepsilon$ and $d(x_0, A_n) = d(y, A_n)$. By the previous observation, y is in $G_n x_0$, proving the claim.

Using separability of $\mathcal{H}$, it is now easy to construct a countable subgroup Γ of G also having dense orbits on $\mathcal{H}$. $\square$

Question 2. Does there exist an affine isometric action of a *finitely generated* group on a Hilbert space, having dense orbits and almost having fixed points?

3.2 Non-existence results

Let us show that locally compact, compactly generated nilpotent groups cannot act with dense orbits on an infinite-dimensional separable Hilbert space. We actually prove something slightly stronger.

Definition 3.5. We say that an isometric action of a group G on a metric space (X, d) has *coarsely dense orbits* if there exists $C \geq 0$ such that, for every $x, y \in X$,

$$d(x, G.y) \leq C.$$

Observe that, for an action of a topological group, having coarsely dense orbits is stable under passing to a cocompact subgroup.

Definition 3.6. If G is a topological group and π a unitary representation, we say that π is *strongly cohomological* if every nonzero subrepresentation of π has nonzero first cohomology.

Lemma 3.7. *Let G be a topological group and π a unitary representation, admitting a 1-cocycle b with coarsely dense image. Then π is strongly cohomological.*

Proof. If σ is a nonzero subrepresentation of π, let b_σ be the orthogonal projection of b on $\mathcal{H}_\sigma$, so that $b_\sigma \in Z^1(G,\sigma)$. Then $b_\sigma(G)$ is coarsely dense in $\mathcal{H}_\sigma$, in particular b_σ is unbounded. So b_σ defines a non-zero class in $H^1(G,\sigma)$. □

The following Lemma is Proposition 3.1 in Chapitre III of [Gu2].

Lemma 3.8. *Let π be a unitary representation of G that does not contain the trivial representation. Let z be a central element of G. Suppose that $1-\pi(z)$ has a bounded inverse (equivalently, 1 does not belong to the spectrum of $\pi(z)$). Then $H^1(G,\pi)=0$.* □

Proof. If $g \in G$, expanding the equality $b(gz) = b(zg)$, we obtain that $(1-\pi(z))b(g)$ is bounded by $2\|b(z)\|$, so that b is bounded by $2\|(1-\pi(z))^{-1}\|\|b(z)\|$. □

Lemma 3.9. *Let G be a locally compact, second countable group, and π a strongly cohomological representation. Then π is trivial on the centre $Z(G)$.*

Proof. Fix $z \in Z(G)$. As G is second countable, we may write $\pi = \int_{\hat{G}}^{\oplus} \rho \, d\mu(\rho)$, a disintegration of π as a direct integral of irreducible representations. Let $\chi : \hat{G} \to S^1 : \rho \mapsto \rho(z)$ be the continuous map given by the value of the central character of ρ on z. For $\varepsilon > 0$, set $X_\varepsilon = \{\rho \in \hat{G} : |\chi(\rho) - 1| > \varepsilon\}$ and $\pi_\varepsilon = \int_{X_\varepsilon}^{\oplus} \rho \, d\mu(\rho)$, so that π_ε is a subrepresentation of π. Since $|\rho(z)-1|^{-1} < \varepsilon^{-1}$ for $\rho \in X_\varepsilon$, the operator

$$(\pi_\varepsilon(z) - 1)^{-1} = \int_{X_\varepsilon}^{\oplus} (\rho(z)-1)^{-1} \, d\mu(\rho)$$

is bounded. We are now in position to apply Lemma 3.8, to conclude that $H^1(G,\pi_\varepsilon) = 0$. By definition, this means that π_ε is the zero subrepresentation, meaning that the measure μ is supported in $\hat{G} - X_\varepsilon$. As this holds for every $\varepsilon > 0$, we see that μ is supported in $\{\rho \in \hat{G} : \rho(z) = 1\}$, to the effect that $\pi(z) = 1$. □

Proposition 3.10. *Let G be a topological group, and π a unitary representation of G. Suppose that $\overline{H^1}(G,\pi) \neq 0$. Then π has a nonzero subrepresentation that is strongly cohomological.*

Proof. Suppose the contrary. Then, by an standard application of Zorn's Lemma, π decomposes as a direct sum $\pi = \bigoplus_{i \in I} \pi_i$, where $H^1(G,\pi_i) = 0$ for every $i \in I$, so that $\overline{H^1}(G,\pi) = 0$ by Proposition 2.6 in Chapitre III of [Gu2]. □

Remark 3.11. The converse is false, even for finitely generated groups: indeed, it is known (see [Gu1]) that every nonzero representation of the free group F_2 has non-vanishing H^1, so that every unitary representation of F_2 is strongly cohomological. But it turns out that F_2 has an irreducible representation π such that $\overline{H^1}(F_2,\pi) = 0$ (see Proposition 2.4 in [MaVa]).

Corollary 3.12. *Let G be a locally compact, second countable group, and let π be a unitary representation of G without invariant vectors. Write $\pi = \pi_0 \oplus \pi_1$, where π_1 consists of the $Z(G)$-invariant vectors. Then*

(1) π_0 does not contain any strongly cohomological subrepresentation (in particular, $\overline{H^1}(G, \pi_0) = 0$);

(2) every 1-cocycle of π_1 vanishes on $Z(G)$, so that $H^1(G, \pi_1) \simeq H^1(G/Z(G), \pi_1)$.

Proof. (1) follows by combining lemma 3.9 and Proposition 3.10. For (2), we use the idea of proof of Theorem 3.1 in [Sh2]: if $b \in Z^1(G, \pi_1)$, then for every $g \in G$, $z \in Z(G)$,

$$\pi_1(g)b(z) + b(g) = b(gz) = b(zg) = b(g) + b(z)$$

as $\pi_1(z) = 1$. So $\pi_1(g)b(z) = b(z)$; this forces $b(z) = 0$ as π has no G-invariant vector. So b factors through $G/Z(G)$. □

Observe that Corollary 3.12 provides a new proof of Shalom's Corollary 3.7 in [Sh2]: under the same assumptions, every cocycle in $Z^1(G, \pi)$ is almost cohomologous to a cocycle factoring through $G/Z(G)$ and taking values in a subrepresentation factoring through $G/Z(G)$.

From Corollary 3.12 we immediately deduce

Corollary 3.13. *Let G be a locally compact, second countable, nilpotent group, and let π be a representation of G without invariant vectors. Let (Z_i) be the ascending central series of G ($Z_0 = \{1\}$, and Z_i is the centre modulo Z_{i-1}). Let σ_i denote the subrepresentation of G on the space of Z_i-invariant vectors, and finally let π_i be the orthogonal of σ_{i+1} in σ_i, so that $\pi = \bigoplus \pi_i$.*

Then $H^1(G, \pi_i) \simeq H^1(G/Z_i, \pi_i)$ for all i, and π is not a strongly cohomological subrepresentation. In particular, $\overline{H^1}(G, \pi) = 0$. □

Note that the latter statement is a result of Guichardet [Gu1, Théorème 7], which can be stated as: G has Property H_T (i.e. every unitary representation with non-vanishing reduced cohomology contains the trivial representation). If we define Property H_{CT} to be: *every strongly cohomological representation is trivial*, then, as a corollary of Proposition 3.10, Property H_{CT} implies Property H_T; we have actually proved that locally compact, second countable nilpotent groups have Property H_{CT}.

Corollary 3.14. *If G is a locally compact, second countable nilpotent group, and π is a strongly cohomological representation, then π is a trivial representation.* □

Theorem 3.15. *Let G be a locally compact, second countable nilpotent group. Then G has a isometric action on a (real) Hilbert space $\mathcal{H}$ with coarsely dense orbits if and only there exists a continuous morphism: $u : G \to (\mathcal{H}, +)$ with coarsely dense image.*

Proof. Suppose that such an action exists, and let π be its linear part. By lemma 3.7, π is strongly cohomological, hence trivial by Corollary 3.14. So the action is given by a morphism $u : G \to (\mathcal{H}, +)$ with coarsely dense image. The converse is obvious. $\square$

The following generalizes Corollary 3.1.

Corollary 3.16. *Let G be a locally compact, compactly generated nilpotent group, and let $\mathcal{H}$ be a (real) Hilbert space. Then*

- *G has a isometric action on $\mathcal{H}$ with coarsely dense orbits if and only $\mathcal{H}$ has finite dimension k, and G has a quotient isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$, with $n + m \geq k$.*
- *G has a isometric action on $\mathcal{H}$ with dense orbits if and only $\mathcal{H}$ has finite dimension k, and G has a quotient isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$, with* $\max(n + m - 1, n) \geq k$.

Proof. Since G is σ-compact, by [Com, Theorem 3.7] there exists a compact normal subgroup N such that G/N is second countable.

Let α be an affine isometric action of G with coarsely dense orbits. Then G/N has an isometric action with coarsely dense orbits on the set of $\alpha(N)$-fixed points (which is nonempty as N is compact). So we can assume that G is second countable.

Let u be the morphism $G \to \mathcal{H}$ as in Theorem 3.15. Let W be its kernel, so that $A = G/W$ is a locally compact, abelian group, which embeds continuously, coarsely densely in a Hilbert space. By standard structural results, A has an open subgroup, containing a compact subgroup K, such that A/K is a Lie group. Since K embeds in a Hilbert space, it is necessarily trivial, so that A is an abelian Lie group without compact subgroup. So A is isomorphic to $\mathbf{R}^n \times \mathbf{Z}^m$ for some integers n, m. Since A embeds coarsely densely in $\mathcal{H}$, the latter must have finite dimension $k \leq n + m$.

If the action has dense orbits, then either $m = 0$ and $n \geq k$, or $m \geq 1$ and $m \geq k - n + 1$; this means that $k \leq \max(n + m - 1, n)$. Conversely, if $k \leq n + m - 1$, then, since $\mathbf{Z}$ has a dense embedding in the torus $\mathbf{R}^k/\mathbf{Z}^k$, $\mathbf{Z}^{k+1}$ has a dense embedding in $\mathbf{R}^k$, and this embedding can be extended to $\mathbf{R}^n \times \mathbf{Z}^m$. $\square$

From Corollary 3.16, we immediately deduce

Corollary 3.17. *A compactly generated, nilpotent-by-compact group does not admit any isometric action with coarsely dense orbits on an infinite-dimensional Hilbert space.* □

Proposition 3.2 on the one hand, and Corollary 3.17 on the other, isolate the first test-case for Navas'question:

Question 3. Can a polycyclic group admit an affine isometric action with dense orbits on an infinite-dimensional Hilbert space?

Let us prove a related result for semisimple groups.

Theorem 3.18. *Let G be a connected, semisimple Lie group. Then G cannot act on a Hilbert space $\mathcal{H} \neq 0$ with coarsely dense orbits.*

Proof. Suppose by contradiction the existence of such an action α, and let π denote its linear part. Then π is strongly cohomological. By Lemma 3.9, π is trivial on the centre of G. Thus the centre acts by translations, generating a finite-dimensional subspace V of $\mathcal{H}$. The action induces a map $p : G \to \mathrm{O}(V) \ltimes V$. Since G is semisimple, the kernel of p contains the sum G_{nc} of all noncompact factors of G, and thus factors though the compact group G/G_{nc}. Thus $H^1(G, V) = 0$, and since π is strongly cohomological, this implies that $V = 0$.

It follows that α is trivial on the centre of G, so that we can suppose that G has trivial centre. Then G is a direct product of simple Lie groups with trivial centre. We can write $G = H \times K$ where K denotes the sum of all simple factors S of G such that $\alpha(S)(0)$ is bounded (in other words, $H^1(S, \pi|_S) = 0$). Then the restriction of α to H also has coarsely dense orbits. Moreover, every simple factor of H acts in an unbounded way, so that, by a result of Shalom [Sh1, Theorem 3.4][1], the action of H is proper. That is, the map $i : H \to \mathcal{H}$ given by $i(h) = \alpha(h)(0)$ is metrically proper and its image is coarsely dense. By metric properness, the subset $X = i(H) \subset \mathcal{H}$ satisfies: X is coarsely dense, and every ball in X (for the metric induced by $\mathcal{H}$) is compact.

Suppose that $\mathcal{H}$ is infinite dimensional and let us deduce a contradiction. For some $d > 0$, we have $d(x, X) \leq d$ for every $x \in \mathcal{H}$. If $\mathcal{H}$ is infinite dimensional, there exists, in a fixed ball of radius $7d$, infinitely many pairwise disjoint balls $B(x_n, 3d)$ of radius $3d$. Taking a point in $X \cap B(x_n, 2d)$ for every n, we obtain a closed, infinite and bounded discrete subset of X, a contradiction.

[1]Shalom only states the result for a simple group, but the proof generalizes immediately. See for instance [CLTV] for another proof, based on the Howe-Moore Property.

Thus $\mathcal{H}$ is finite dimensional; since every simple factor of H is non-compact, it has no non-trivial finite dimensional orthogonal representation, so that the action is by translations, and hence is trivial, so that finally $\mathcal{H} = \{0\}$. $\square$

Remark 3.19. The same argument shows that a semisimple, linear algebraic group over any local field, cannot act with coarsely dense orbits on a Hilbert space.

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