A SOBOLEV-LIKE INEQUALITY FOR THE DIRAC OPERATOR

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Abstract. In this article, we prove a Sobolev-like inequality for the Dirac operator on closed compact Riemannian spin manifolds with a Sobolev embedding constant that is nearly optimal. As an application, we give a criterion for the existence of solutions to a nonlinear equation with critical Sobolev exponent involving the Dirac operator. We finally specify a case where this equation can be solved.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 3$. The Sobolev embedding theorem asserts that the Sobolev space H_1^2 of functions $u \in L^2(M)$ such that $\nabla u \in L^2(M)$ embedds continiously in the Lebesgue space $L^N(M)$ (with $N = \frac{2n}{n-2}$). In other words, there exists two constants $A, B > 0$ such that, for all $u \in H_1^2$, we have:

$$\Big(\int_M |u|^N dv(g)\Big)^{\frac{2}{N}} \leq A\int_M |\nabla u|^2 dv(g) + B\int_M u^2 dv(g). \tag{1}$$

It is well known (see [Aub76b]) that the best constant A in this inequality is given by:

$$A = K(n,2)^2 = \frac{4}{n(n-2)}\omega_n^{\frac{2}{n}},$$

where ω_n stands for the volume of the standard n-dimensional sphere. As pointed out by Aubin [Aub76a], the best constant $K(n,2)^2$ plays a fundamental role in the study of the Yamabe problem. This famous problem of Riemannian geometry can be stated as follow: given a compact Riemannian manifold (M^n, g) of dimension $n \geq 3$, can one find a metric conformal to g such that the scalar curvature is constant? This problem has a long and fruitful history and it has been completely solved in several steps by Yamabe [Yam60], Trüdinger [Tru68], Aubin [Aub76a] and finally Schoen [Sch84] using the Positive Mass Theorem coming from General Relativity. The Yamabe problem is in fact equivalent to find a smooth positive solution $u \in C^\infty(M)$ to a nonlinear elliptic equation:

$$L_g u := 4\frac{n-1}{n-2}\Delta_g u + R_g u = \lambda u^{N-1}, \tag{2}$$

where L_g is known as the conformal Laplacian (or the Yamabe operator), Δ_g (resp. R_g) denotes the standard Laplacian acting on functions (resp. the scalar curvature) with respect to the Riemannian metric g and $\lambda \in \mathbb{R}$ is a constant. Indeed, if such a function exists then the metric $\overline{g} = u^{N-2}g$ is conformal to g and satisfies $R_{\overline{g}} = \lambda$. In the sixties,

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Yamabe [Yam60] claims to solve this problem using a variational approach. However, in 1968, Trüdinger [Tru68] points out a mistake in Yamabe's paper and can recover some cases in which Yamabe's theorem is valid. More precisely, Yamabe notices that the positive critical points of the functional:

$$I(f)=\frac{4\frac{n-1}{n-2}\int_M|\nabla f|^2dv(g)+\int_M R_gf^2dv(g)}{\Big(\int_M|f|^Ndv(g)\Big)^{\frac{2}{n}}}$$

defined on the Sobolev space $H_1^N(M)$, are smooth solutions of Equation (2). However a standard variational approach cannot allow to conclude because of the lack of compactness in the Sobolev embedding theorem involved in this approach. So he first shows the existence of smooth positive solutions $u_q\in C^\infty(M)$ (for $1<q<N$) for the nonlinear elliptic equation:

$$4\frac{n-1}{n-2}\Delta_g u_q+R_gu_q=\lambda_q u_q^{q-1}\qquad\text{and}\qquad\int_M u_q^q dv(g)=1,$$

where λ_q is a constant coming from the Lagrange multipliers theorem. Then, he proves the existence of a subsequence of $(\varphi_q)_q$ which converges to a smooth positive solution of (2) when q tends to N. However, this is precisely where the proof failled. Indeed, it is not true in general unless, as shown by Aubin [Aub76a], the following inequality:

$$Y(M,[g])=\inf_{f\neq 0}I(f)<4\frac{n-1}{n-2}K(n,2)^{-2}\tag{3}$$

holds. This condition points out the tight relation between the Yamabe problem and the best constant involved in the Sobolev inequality. Moreover, it is sharp in the sense that for all compact Riemannian manifolds (M^n,g), the following inequality holds (see [Aub76a]):

$$Y(M,[g])\leq 4\frac{n-1}{n-2}K(n,2)^{-2}.\tag{4}$$

In the setting of Spin geometry, a problem similar to the Yamabe problem has been studied in several works of Ammann (see [Amm03c], [Amm03a]), and Ammann, Humbert and others (see [AHGM], [AHM06]). The starting point of all these works is the Hijazi inequality [Hij86], [Hij91] which links the first eigenvalue of two elliptic differential operators: the conformal Laplacian L_g and the Dirac operator D_g. Hijazi's result can be stated as follow:

$$\lambda_1^2(g)\,Vol(M,g)^{\frac{2}{n}}\geq\frac{n}{4(n-1)}Y(M,[g]),\tag{5}$$

where $\lambda_1(g)$ denotes the first eigenvalue of the Dirac operator D_g. Thereafter, B. Ammann defines and studies the spin conformal invariant defined by:

$$\lambda_{\min}(M,[g],\sigma):=\inf_{\overline{g}\in[g]}\lambda_1(\overline{g})Vol(M,\overline{g})^{\frac{1}{n}}\tag{6}$$

and points out that the search of critical metrics for this invariant involves similar analytic problems than those appearing in the Yamabe problem. Indeed, finding a critical metric of this invariant in the *generalized conformal class* is equivalent to prove the existence of a smooth spinor field φ minimizing the functional defined by:

$$\mathcal{F}_g(\psi) = \frac{\Big(\int_M |D_g\psi|^{\frac{2n}{n+1}} dv(g) \Big)^{\frac{n+1}{n}}}{\big| \int_M \langle D_g\psi, \psi \rangle dv(g) \big|}, \tag{7}$$

with Euler-Lagrange equation given by:

$$D_g\varphi = \lambda_{\min}(M, [g], \sigma)|\varphi|^{\frac{2}{n-1}}\varphi. \tag{8}$$

In [Amm03a], the author shows that, like in the Yamabe problem, a standard variational approach does not yield a proof. Indeed, the Sobolev inclusion involved in this approach is precisely the one for which the compacity is lost in the Reillich-Kondrakov theorem. His approach to solve this problem is then similar to the one used in the Yamabe problem. Indeed, he considers a subcritical equation of (8) for which the compacity of the Sobolev inclusion is valid and thus he proves the existence of a sequence of spinor fields solutions of this subcritical equation. Then he shows that there exists a subsequence which converges to a smooth solution of Equation (8). However, this solution can be identically zero and so one might be able to find a criterion which prevents this situation. It is now important to note that one have a similar inequality of (4) in the spinorial setting (see [Amm03b] and [AHGM]), that is:

$$\lambda_{\min}(M, [g], \sigma) \leq \lambda_{\min}(\mathbb{S}^n, [g_{\mathrm{st}}], \sigma_{\mathrm{st}}) = \frac{n}{2}\omega_n^{\frac{1}{n}}, \tag{9}$$

where $(\mathbb{S}^n, g_{\mathrm{st}}, \sigma_{\mathrm{st}})$ stands for the n-dimensional sphere equipped with its standard Riemannian metric g_{st}, its standard spin structure σ_{st} and where $\omega_n = vol(\mathbb{S}^n, g_{\mathrm{st}})$. The criterion obtained by Ammann in [Amm03a] is the same that the one involved in the Yamabe problem that is if (9) is strict then the spinor field solution of (8) cannot be identically zero.

In this paper, we study a more general nonlinear equation involving the Dirac operator (since it also includes Ammann's result in the case of invertible Dirac operator). The proof we give here lies on a Sobolev type inequality for the Dirac operator and thus emphasizes in particular that the same kind of questions of those arising from the Yamabe problem can be asked in the context of spin geometry.

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2. Geometric and Analytic preliminaries

In this section, we give some brief recalls on Spin geometry and on some well-known facts from analysis of partial differential equations on manifolds involving the Dirac operator.

2.1. **Geometric preliminaries.** For more details on this subject, we refer to [Fri00] or [LM89] for example. Let (M^n, g, σ) be a n-dimensional compact Riemannian manifold equipped with a spin structure denote by σ. It is well-known that on such a manifold one can construct a complex vector bundle of rank $2^{[\frac{n}{2}]}$ denoted by $\Sigma_g(M)$ and called the complex spinor bundle. This bundle is naturally endowed with a spinorial Levi-Civita connection ∇, a pointwise Hermitian scalar product $\langle .,. \rangle$ and a Clifford multiplication ".". There is also a natural elliptic differential operator of order one acting on sections of this bundle, the Dirac operator. This operator is locally given by:

$$D_g\varphi = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}\varphi,$$

for all $\varphi \in \Gamma\big(\Sigma_g(M)\big)$ and where $\{e_1, ..., e_n\}$ is a local g-orthonormal frame of the tangent bundle. It defines a self-adjoint operator whose spectrum is constitued by an unbounded sequence of real numbers. Estimates on the spectrum of the Dirac operator has been and is again the main subject of several works (a non exhaustive list is [Fri80], [Hij86] or [Bär92]). As point out in the introduction, a key result for the following of this paper is the Hijazi inequality. More precisely, O. Hijazi (see [Hij91]) gives an inequality which links the squared of the first eigenvalue of the Dirac operator with the first eigenvalue of the conformal Laplacian. The proof of this inequality relies on the famous Schrödinger-Lichnerowicz formula (see [Hij99] for example) and on the conformal covariance of the Dirac operator. In fact, if we consider a conformal change of the metric we have a canonical identification of the spinor bundle over (M, g) with the one over $(M, \overline{g})$, where $\overline{g}$ is a metric conformal to g (see [Hit74] or [Hij86]). This identification will be denoted by:

$$\begin{array}{ccc} \Sigma_g(M) & \longrightarrow & \Sigma_{\overline{g}}(M) \\ \varphi & \longmapsto & \overline{\varphi}. \end{array} \tag{10}$$

Thus, under this isomorphism, one can rely the Dirac operators D_g and $D_{\overline{g}}$ acting respectively on $\Sigma_g(M)$ and $\Sigma_{\overline{g}}(M)$. More precisely, if $\overline{g} = e^{2u}g \in [g]$ where u is a smooth function, then:

$$D_{\overline{g}}\overline{\varphi} = e^{-\frac{n+1}{2}u}\,\overline{D_g(e^{\frac{n-1}{2}u}\varphi)}, \tag{11}$$

for all $\varphi \in \Gamma\big(\Sigma_g(M)\big)$.

2.2. **Analytic preliminaries.** In this section we give some well-known facts on Sobolev spaces on spinors and on the analysis of differential equations involving the Dirac operator. In the following, we assume that (M^n, g) is a n-dimensional compact Riemannian spin manifolds ($n \geq 2$) such that the Dirac operator is invertible.

The Sobolev space H_1^q is defined as being the completion of the space of smooth spinor fields with respect to the norm:

$$||\varphi||_{1,q} := ||\nabla\varphi||_q + ||\varphi||_q, \tag{12}$$

where $||\ ||_q$ denotes the L^q-norm. However, since our problem involves the Dirac operator, it would be more convenient if on can consider a norm defined from it. According to the following result, it is possible to get such a characterization. Indeed, we have:

Lemma 1. *The map:*

$$\psi \mapsto ||hD_g\psi||_q \tag{13}$$

defines a norm equivalent to the H_1^q-norm for every smooth positive function h on M.

Proof: We easily get from the definition of (13) that this application defines a norm on the space of smooth spinors which is equivalent to the norm defined by:

$$\psi \mapsto ||D_g\psi||_q.$$

Now we show that this norm is equivalent to the H_1^q-norm. A simple computation shows that there exists a positive constant $C_1 > 0$ such that for all smooth spinors ψ we have:

$$||D_g\psi||_q \leq C_1\big(||\nabla\psi||_q + ||\psi||_q\big).$$

On the other hand, with the help of pseudo-differantial operators (see the proof of Lemma 2), it is not difficult to see that there also exists another positive constant $C_2 > 0$ such that:

$$\big(||\nabla\psi||_q + ||\psi||_q\big) \leq C_2||D\psi||_q,$$

which concludes the proof of this lemma. □

Using this result and the fact that the Sobolev space H_1^q is defined as the completion of the space of smooth spinors with respect to the H_1^q-norm, it is clear that one can consider the Sobolev space as defined independently from one of the three preceding norms. It will provide a very usefull tool to solve the nonlinear equation studied in this work. A natural way to prove the existence of solutions for this kind of equation is the variational approch which consists to minimize a certain functional define on an adaptated Sobolev space and then to apply the machinery of Sobolev-Kondrakov embedding theorems, Schauder estimates and a-priori elliptic estimates. Here we will use this approach, and we refer to the works of Ammann ([Amm03c] and [Amm03a]) for proofs of all these results in the context of Spin geometry. However, in order to make this work self-contained, we give the proof of the following result which will be of great help in the next section:

Lemma 2. *If the Dirac operator is invertible then there exists a constant $C > 0$ such that for all $\varphi \in H_1^q$ we have:*

$$||\varphi||_p \leq C||D_g\varphi||_q,$$

where $p^{-1} + q^{-1} = 1$ and $2 \leq p < \infty$.

This result comes from the equivalence of the H_1^q-norm and the norm defined by (13) with $h \equiv 1$, however it is quite interesting to give the proof since it gives, in particular, the proof of this equivalence.

Proof: We show that the operator:

$$D_g^{-1} : L^q \longrightarrow L^p$$

defines a continous map. Since D_g^{-1} is a pseudo-differential operator of order -1 the operator $(Id + \nabla^*\nabla)^{\frac{1}{2}} D_g^{-1}$ is a pseudo-differential operator of order zero. Thus the spinor field:

$$(Id + \nabla^*\nabla)^{\frac{1}{2}} D_g^{-1}\varphi \in L^s,$$

for all $s > 1$, $\varphi \in H_1^q$ and in particular for $s = q$, we get that $(Id + \nabla^*\nabla)^{\frac{1}{2}} D_g^{-1}\varphi \in L^q$. The spinor field $D_g^{-1}\varphi$ is so in the Sobolev space H_1^q which is continously embedded in L^p (using the Sobolev embedding theorem). Then there exists a positive constant $C > 0$ such that:

$$||D_g^{-1}\varphi||_p \leq C||\varphi||_q,$$

and this conclued the proof. $\square$

Remark 1. *(1) For $q = q_D = 2n/(n+1)$ and p_D such that $p_D^{-1} + q_D^{-1} = 1$ the quotient:*

$$\mathcal{C}_g(\varphi) = \frac{||D_g\varphi||_{q_D}}{||\varphi||_{p_D}}$$

is invariant under a conformal change of metric, that is:

$$\mathcal{C}_{\overline{g}}(h^{-\frac{n-1}{2}}\varphi) = \mathcal{C}_g(\varphi) \tag{14}$$

for all $\varphi \in H_1^{q_D}$ and for $\overline{g} = h^2 g \in [g]$. Indeed, an easy computation using the canonical identification (10) between $\Sigma_g M$ and $\Sigma_{\overline{g}} M$ and the formula (11) which links D_g and $D_{\overline{g}}$, leads to (14).

(2) On the n-dimensional sphere $(\mathbb{S}^n, g_{\mathrm{st}})$ endowed with its standard spin structure, the Dirac operator is invertible since the scalar curvature is positive. Then using Lemma 2, there exists a constant $C > 0$ such that for all $\Phi \in H_1^{q_D}$:

$$||\Phi||_{p_D} \leq C||D^{\mathbb{S}^n}\Phi||_{q_D}.$$

Moreover, since the standard sphere $(\mathbb{S}^n \setminus \{q\}, g_{\mathrm{st}})$ (where $q \in \mathbb{S}^n$) is conformally isometric to the Euclidean space $(\mathbb{R}^n, \xi)$, we conclude that for all $\psi \in \Gamma_c\big(\Sigma_\xi(\mathbb{R}^n)\big)$:

$$||\psi||_{p_D} \leq C||D_\xi\psi||_{q_D}$$

where $\Gamma_c\big(\Sigma_\xi(\mathbb{R}^n)\big)$ denotes the space of smooth spinor fields over $(\mathbb{R}^n, \xi)$ with compact support

3. The Sobolev Inequality

In this section, we prove a Sobolev inequality in the spinorial setting. The classical Sobolev inequality (1) shows in particular that the Sobolev space of functions H_1^2 is continiously embedded in $L^{\frac{2n}{n-2}}$. Here one could interpret our result as the inequality involved in the continuous embedding:

$$H_1^{2n/(n+1)} \hookrightarrow H_{1/2}^2$$

where $H_{1/2}^2$ is defined as the completion of the space of smooth spinors with respect to the norm:

$$||\psi||_{\frac{1}{2},2} := \sum_i |\lambda_i|^{\frac{1}{2}} |A_i|^2$$

and where $\psi = \sum_i A_i\psi_i$ is the spectral decomposition of any smooth spinor (see [Amm03c]).

Let's first examine the case of the sphere which is the starting point of the inequality we want to prove. In fact, it is quite easy to compute that the invariant defined by (6) on the sphere is:

$$\lambda_{\min}(\mathbb{S}^n,[g_{\mathrm{st}}],\sigma_{\mathrm{st}}) = \inf_{\psi} \frac{\left(\int_{\mathbb{S}^n} |D^{\mathbb{S}^n}\psi|^{\frac{2n}{n+1}} dv(g_{\mathrm{st}})\right)^{\frac{n+1}{n}}}{\left|\int_{\mathbb{S}^n}\langle D^{\mathbb{S}^n}\psi,\psi\rangle dv(g_{\mathrm{st}})\right|} = (n/2)\omega_n^{\frac{1}{n}} = 1/K(n). \tag{15}$$

The proof of this fact lies on the Hijazi inequality (5) and on the existence the real Killing spinors on the round sphere (see [Gut86]). Thus using the conformal covariance of (15) and the fact that the sphere (minus a point) is conformally isometric to the Euclidean space, we can conclude that:

$$\Big|\int_{\mathbb{R}^n}\langle D_\xi\psi,\psi\rangle dx\Big| \leq K(n)\Big(\int_{\mathbb{R}^n}|D_\xi\psi|^{\frac{2n}{n+1}}dx\Big)^{\frac{n+1}{n}} \tag{16}$$

for all $\psi \in \Gamma_c\big(\Sigma_\xi(\mathbb{R}^n)\big)$. With this result in mind, we can now state the main result of this section:

Theorem 3. *Let (M^n, g, σ) be an n-dimensional closed compact Riemannian spin manifold and suppose that the Dirac operator is invertible. Then for all $\varepsilon > 0$, there exists a constant B_ε such that:*

$$\Big|\int_M\langle D_g\varphi,\varphi\rangle dv(g)\Big| \leq \Big(K(n)+\varepsilon\Big)\Big(\int_M|D_g\varphi|^{\frac{2n}{n+1}}dv(g)\Big)^{\frac{n+1}{n}} + B_\varepsilon\Big(\int_M|\varphi|^{\frac{2n}{n+1}}dv(g)\Big)^{\frac{n+1}{n}} \tag{17}$$

for all $\varphi \in H_1^{\frac{2n}{n+1}}$ and where $K(n) = 2/(n\omega_n^{\frac{1}{n}})$ with $\omega_n = \mathrm{vol}(\mathbb{S}^n, g_{\mathrm{st}})$.

In order to prove this result, we need some well-known technical results which are summerized in the following lemma:

Lemma 4. *Let $(a_i)_{1\leq i\leq N} \subset \mathbb{R}^+$, $p \in [0,1]$ and $q \geq 1$. The following identities hold:*

(1) $(\sum_{i=1}^N a_i)^p \leq \sum_{i=1}^N a_i^p$

(2) $\sum_{i=1}^N a_i^q \leq (\sum_{i=1}^N a_i)^q$

(3) $\forall\varepsilon > 0,\ \exists C_\varepsilon > 0,\ \forall a, b \geq 0\ :\ (a+b)^p \leq (1+\varepsilon)a^p + C_\varepsilon b^p$

(4) For all functions $f_1,\cdots,f_r : M \to [0,\infty[$, we have:

$$\sum_{i=1}^r\Big(\int_M f_i^p dv(g)\Big)^{\frac{1}{p}} \leq \Big(\int_M\big(\sum_{i=1}^r f_i\big)^p dv(g)\Big)^{\frac{1}{p}}.$$

We can now give the proof of the main result of this part.

Proof of Theorem 3: Let $x \in M$ and $\varepsilon > 0$. Let U (resp. V) be a neighbourhood of $x \in M$ (resp. $0 \in \mathbb{R}^n$) such that the exponentiel map

$$exp_x : V \subset \mathbb{R}^n \longrightarrow U \subset M$$

is a diffeomorphism, then we can identify the spinor bundle over (U,g) with the one over (V,ξ) that is there exists a map:

$$\tau:\Sigma_g(U)\longrightarrow\Sigma_\xi(V) \tag{18}$$

which is a fiberwise isometry (see [BG92]). Moreover the Dirac operators D_g and D_ξ (acting respectively on $\Sigma_g(U)$ and $\Sigma_\xi(V)$) are related by the formula:

$$D_g\varphi(y)=\tau^{-1}\Big(D_\xi\big(\tau(\varphi)\big)\big(exp_x(y)\big)\Big)+\rho(\varphi)(y) \tag{19}$$

for all $y\in U$ and where $\rho(\varphi)\in\Gamma\big(\Sigma_g(U)\big)$ is a smooth spinor such that $|\rho(\varphi)|\leq\varepsilon|\varphi|$. Now since M is compact, we can find a finite sequence $(x_i)_{1\leq i\leq N}\subset M$ and a finite cover $(U_i)_{1\leq i\leq N}$ of M (where U_i is a neighbourhood of $x_i\in M$) such that there exists open sets $(V_i)_{1\leq i\leq N}$ of $0\in\mathbb{R}^n$ and applications τ_i such that (18) and (19) are fulfilled. Moreover without loss of generalities, we can assume that:

$$\frac{1}{1+\varepsilon}\xi\leq g\leq(1+\varepsilon)\xi$$

as bilinear forms, hence the volume forms satisfies:

$$\frac{1}{(1+\varepsilon)^{n/2}}dx\leq dv(g)\leq(1+\varepsilon)^{n/2}dx. \tag{20}$$

Now let $(\eta_i)_{1\leq i\leq N}$ a smooth partition of unity subordinate to the covering $(U_i)_{1\leq i\leq N}$, that is η_i satisfies:

$$\begin{cases}\mathrm{supp}\,(\eta_i)\subset U_i\\ 0\leq\eta_i\leq 1\\ \sum_{i=1}^N\eta_i=1.\end{cases}$$

Now if $\varphi\in\Gamma\big(\Sigma_g(M)\big)$ such that $\int_M\langle D_g\varphi,\varphi\rangle dv(g)>0$, we write:

$$\begin{aligned}(LHS):=\int_M\langle D_g\varphi,\varphi\rangle dv(g)&=\sum_{i=1}^N\int_M\langle\sqrt{\eta_i}D_g(\varphi),\sqrt{\eta_i}\varphi\rangle dv(g)\\&=\sum_{i=1}^N\int_M\langle D_g(\sqrt{\eta_i}\varphi),\sqrt{\eta_i}\varphi\rangle dv(g)\end{aligned}$$

since (LHS) is real and $\mathrm{Re}\langle d(\sqrt{\eta_i})\cdot\varphi,\varphi\rangle=0$. Inequality (20) leads to:

$$(LHS)\leq\big(1+\varepsilon\big)^{\frac{n}{2}}\sum_{i=1}^N\int_{\mathbb{R}^n}\langle\tau_i\big(D_g(\sqrt{\eta_i}\varphi)\big),\tau_i\big(\sqrt{\eta_i}\varphi\big)\rangle dx$$

and then using formula (19), we can write:

$$(LHS)\leq\big(1+\varepsilon\big)^{\frac{n}{2}}\sum_{i=1}^N\Big(\int_{\mathbb{R}^n}\langle D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big),\tau_i(\sqrt{\eta_i}\varphi)\rangle dx+C\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^2dx\Big).$$

On the other hand, since $\tau_i(\sqrt{\eta_i}\varphi) \in \Gamma_c\big(\Sigma_\xi(\mathbb{R}^n)\big)$, Inequality (16) gives:

$$(LHS) \leq \big(1+\varepsilon\big)^{\frac{n}{2}} \sum_{i=1}^{N} \Big(K(n) \big(\int_{\mathbb{R}^n} |D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^{\frac{2n}{n+1}} dx \big)^{\frac{n+1}{n}} + C \int_{\mathbb{R}^n} |\tau_i(\sqrt{\eta_i}\varphi)|^2 dx \Big).$$

Now note that with the help of (4) of Lemma 4 and since $n/(n+1) \leq 1$, the first term in the right hand side of the preceding inequality becomes:

$$\sum_{i=1}^{N} \Big(\int_{\mathbb{R}^n} |D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^{\frac{2n}{n+1}} dx \Big)^{\frac{n+1}{n}} \leq \Big(\int_{\mathbb{R}^n} \big(\sum_{i=1}^{N} |D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^2 \big)^{\frac{n}{n+1}} dx \Big)^{\frac{n+1}{n}}.$$

Using (3) of Lemma 4, we finally get:

$$\begin{aligned}
(LHS)^{\frac{n}{n+1}} \quad &\leq \quad (1+\varepsilon)^{\frac{(n+1)^2+1}{2(n+1)}} K(n)^{\frac{n}{n+1}} \int_{\mathbb{R}^n} \Big(\sum_{i=1}^{N} |D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^2 \Big)^{\frac{n}{n+1}} dx \\
&\quad + (1+\varepsilon)^{\frac{n^2}{2(n+1)}} C_\varepsilon \Big(\sum_{i=1}^{N} \int_{\mathbb{R}^n} |\tau_i(\sqrt{\eta_i}\varphi)|^2 dx \Big)^{\frac{n}{n+1}}.
\end{aligned} \tag{21}$$

We are now going to estimate the first term in the righthand side of Inequality (21). If we let $\gamma_i(\varphi) = d(\sqrt{\eta_i}) \cdot \varphi - \rho_i(\varphi)$, then we can write:

$$\begin{aligned}
\sum_{i=1}^{N} |D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^2 \quad &= \quad \sum_{i=1}^{N} |D_g(\sqrt{\eta_i}\varphi) - \rho_i(\varphi)|^2 \\
&= \quad \sum_{i=1}^{N} |\sqrt{\eta_i} D_g\varphi + \gamma_i(\varphi)|^2
\end{aligned}$$

and the Minkowski's inequality leads to:

$$\begin{aligned}
\sum_{i=1}^{N} |\sqrt{\eta_i} D_g\varphi + \gamma_i(\varphi)|^2 \quad &\leq \quad \Big(\big(\sum_{i=1}^{N} |\sqrt{\eta_i} D_g\varphi|^2 \big)^{\frac{1}{2}} + \big(\sum_{i=1}^{N} |\gamma_i(\varphi)|^2 \big)^{\frac{1}{2}} \Big)^2 \\
&\leq \quad \big(|D_g\varphi| + C|\varphi| \big)^2 \qquad \text{(using (1) of Lemma 4).}
\end{aligned}$$

Thus we have shown that:

$$\int_{\mathbb{R}^n} \Big(\sum_{i=1}^{N} |D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^2 \Big)^{\frac{n}{n+1}} dx \quad \leq \quad (1+\varepsilon)^{\frac{n}{2}} \int_M \big(|D_g\varphi|^2 + C|\varphi|^2 + C|D_g\varphi|\,|\varphi| \big)^{\frac{n}{n+1}} dv(g),$$

and with (2) of Lemma 4, we get:

$$\begin{aligned}
\int_{\mathbb{R}^n} \Big(\sum_{i=1}^{N} |D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^2 \Big)^{\frac{n}{n+1}} dx \quad &\leq \quad (1+\varepsilon)^{\frac{n}{2}} \Big(\int_M \big(|D_g\varphi|^{\frac{2n}{n+1}} dv(g) + C \int_M |\varphi|^{\frac{2n}{n+1}} dv(g) \\
&\quad + C \int_M |D_g\varphi|^{\frac{n}{n+1}} |\varphi|^{\frac{n}{n+1}} dv(g) \Big).
\end{aligned}$$

Now we first apply the Cauchy-Schwarz inequality in the last term of the preceding inequality:

$$\int_M |D_g\varphi|^{\frac{n}{n+1}}|\varphi|^{\frac{n}{n+1}}dv(g) \;\; \leq \;\; \Big(\int_M |D_g\varphi|^{\frac{2n}{n+1}}dv(g)\Big)^{\frac{1}{2}}\Big(\int_M |\varphi|^{\frac{2n}{n+1}}dv(g)\Big)^{\frac{1}{2}}$$

and secondly we use the Young inequality to get:

$$\int_M |D_g\varphi|^{\frac{n}{n+1}}|\varphi|^{\frac{n}{n+1}}dv(g) \;\; \leq \;\; \frac{\varepsilon^2}{2}\int_M |D_g\varphi|^{\frac{2n}{n+1}}dv(g)+\frac{1}{2\varepsilon^2}\int_M |\varphi|^{\frac{2n}{n+1}}dv(g).$$

Finally, we have:

$$\int_{\mathbb{R}^n}\Big(\sum_{i=1}^N |D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^2\Big)^{\frac{n}{n+1}}dx \leq (1+\varepsilon)^{\frac{n}{2}}\Big((1+\frac{\varepsilon^2}{2})\int_M |D_g\varphi|^{\frac{2n}{n+1}}dv(g)+C_\varepsilon\int_M |\varphi|^{\frac{2n}{n+1}}dv(g)\Big).$$

Now we study the second term in the right-hand side of Inequality (21). Hölder's inequality gives:

$$\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^2dx \leq \Big(\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^{\frac{2n}{n-1}}dx\Big)^{\frac{n-1}{2n}}\Big(\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^{\frac{2n}{n+1}}dx\Big)^{\frac{n+1}{2n}}$$

and thus using (1) of Lemma 4 and the preceding inequality lead to:

$$\Big(\sum_{i=1}^N\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^2dx\Big)^{\frac{n}{n+1}} \;\; \leq \;\; \sum_{i=1}^N\Big(\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^{\frac{2n}{n-1}}dx\Big)^{\frac{n-1}{2(n+1)}}\Big(\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^{\frac{2n}{n+1}}dx\Big)^{\frac{1}{2}}.$$

With the help of (2) of Remark 1, there exists a constant $C>0$ such that:

$$\Big(\sum_{i=1}^N\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^2dx\Big)^{\frac{n}{n+1}} \;\; \leq \;\; C\sum_{i=1}^N\Big(\int_{\mathbb{R}^n}|D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^{\frac{2n}{n+1}}dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^{\frac{2n}{n+1}}dx\Big)^{\frac{1}{2}}$$

and then the Young inequality gives:

$$\Big(\sum_{i=1}^N\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^2dx\Big)^{\frac{n}{n+1}} \;\; \leq \;\; C\varepsilon^2\sum_{i=1}^N\int_{\mathbb{R}^n}|D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^{\frac{2n}{n+1}}dx+\frac{C}{\varepsilon^2}\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^{\frac{2n}{n+1}}dx.$$

It is easy to see that:

$$\frac{C}{\varepsilon^2}\sum_{\imath=1}^N\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^{\frac{2n}{n+1}}dx \;\; \leq \;\; C_\varepsilon(1+\varepsilon)^{\frac{n}{2}}\int_M|\varphi|^{\frac{2n}{n+1}}dv(g).$$

Moreover a similar argument that the one used below shows that:

$$\sum_{i=1}^N\int_{\mathbb{R}^n}|D_\xi\big(\tau_i(\sqrt{\eta_i}\varphi)\big)|^{\frac{2n}{n+1}}dx \leq (1+\varepsilon)^{\frac{n}{2}}\int_M\big(|D_g\varphi|^{\frac{2n}{n+1}}+C|\varphi|^{\frac{2n}{n+1}}+|D_g\varphi|^{\frac{n}{n+1}}|\varphi|^{\frac{n}{n+1}}\big)dv(g)$$

and then the Cauchy-Schwarz inequality and the Young inequality lead to:

$$\Big(\sum_{i=1}^N\int_{\mathbb{R}^n}|\tau_i(\sqrt{\eta_i}\varphi)|^2dx\Big)^{\frac{n}{n+1}} \;\; \leq \;\; C\varepsilon^2(1+\varepsilon)^{\frac{n}{2}}\int_M|D_g\varphi|^{\frac{2n}{n+1}}dv(g)+C_\varepsilon\int_M|\varphi|^{\frac{2n}{n+1}}dv(g)$$

Putting all these estimates in (21) gives Inequality (17). $\square$

4. A non linear equation for the Dirac operator

4.1. **A criterion for the existence of solutions.** As a direct application of Theorem 3, we give a sufficient criterion for the existence of solutions for a nonlinear equation involving the Dirac operator. More precisely, the aim of this section is to prove the following result:

Theorem 5. *Let (M^n, g) a n-dimensional compact Riemannian spin manifold and let H be a smooth positive function on M. If the Dirac operator is invertible and if:*

$$\lambda_{\min} < K(n)^{-1}(\max H)^{-\frac{2}{p_D}}, \tag{22}$$

then there exists a spinor field $\varphi \in C^{1,\alpha}\big(\Sigma_g(M)\big) \cap\ C^\infty\big(\Sigma_g(M \setminus \varphi^{-1}(0))\big)$ satisfiying the following nonlinear elliptic equation:

$$D_g\varphi = \lambda_{\min} H|\varphi|^{\frac{2}{n-1}}\varphi \quad \textit{and} \quad \int_M H|\varphi|^{\frac{2n}{n-1}} dv_g = 1. \tag{23}$$

Here we let for $2 \leq q \leq q_D$:

$$\lambda_q = \lambda_q(M, g) := \inf_\psi \Big\{ \frac{\big(\int_M H^{-(q/p)}|D_g\psi|^q dv(g)\big)^{\frac{2}{q}}}{\big|\int_M \langle D_g\psi, \psi\rangle dv(g)\big|} \Big\} = \inf_\psi \frac{||H^{-(1/p)}D_g\psi||_q^2}{\big|\int_M \langle D_g\psi, \psi\rangle dv(g)\big|} \tag{24}$$

where the infimun is taken over all $\psi \in H_1^q$ and where $\lambda_{q_D}(M, g) := \lambda_{\min}$. In the rest of this section, we will let:

$$\mathcal{F}_q(\psi) = \mathcal{F}_{g,q}(\psi) = \frac{||H^{-(1/p)}D_g\psi||_q^2}{\big|\int_M \langle D_g\psi, \psi\rangle dv(g)\big|}.$$

Remark 2. *Using Lemma 2, we have $\lambda_q > 0$.*

A standard variational approach for the study of (23) cannot allow to conclude because of the lack of compactness of the inclusion $H_1^{q_D}$ in L^{p_D}. The method we use here consists in proving the existence of solutions for subcritical equations where the compactness of the Sobolev embedding theorem is valid and then to make converge this sequence to a solution of (23). We begin with the existence of solutions for subcritical equations, that is:

Proposition 6. *For all $q \in (q_D, 2)$, there exists a spinor field $\varphi_q \in C^{1,\alpha}(\Sigma_g M) \cap C^\infty(\Sigma_g(M \setminus \varphi_q^{-1}(0)))$ such that:*

$$D_g\varphi_q = \lambda_q H|\varphi_q|^{p-2}\varphi_q \qquad (E_q)$$

where $p \in \mathbb{R}$ is such that $p^{-1} + q^{-1} = 1$. Moreover, we have:

$$\int_M H|\varphi_q|^p dv_g = 1.$$

Proof: The proof of this result is divided in two parts. In a first step, se show that there exists a spinor field $\varphi_q \in H_1^q$ satisfying (E_q), and then we will show that this solution has the desired regularity. For the rest of this proof, we fix $q \in (q_D, 2)$.

First step: We prove the existence of a spinor field $\varphi_q \in H_1^q$ satisfying (E_q). First we study the functional defined by:

$$\mathcal{F}_q : \mathcal{H}_1^q := \big\{\psi \in H_1^q \ / \int_M \langle D_g\psi, \psi\rangle dv(g) = 1\big\} \longrightarrow \mathbb{R}.$$

It is clear that $\mathcal{H}_1^q$ is non empty, take for exemple a smooth eigenspinor ψ_1 associated to the first positive eigenvalue $\lambda_1 > 0$ of the Dirac operator and thus $(\lambda_1)^{-(1/2)}||\psi_1||_2^{-1}\psi_1 \in \mathcal{H}_1^q$. On the other hand, since $\mathcal{F}_q(\psi) \geq 0$ for all $\psi \in \mathcal{H}_1^q$, we can consider a minimizing sequence (ψ_i) for $\mathcal{F}_q$, that is a sequence such that $\mathcal{F}_q(\psi_i) \to \lambda_q$ with $(\psi_i) \subset \mathcal{H}_1^q$. It is clear that this sequence is bounded in H_1^q and thus there exists a spinor field $\psi_q \in H_1^q$ such that:

- $\psi_i \to \psi_q$ strongly in L^p with $p^{-1} + q^{-1} = 1$ (by the Reillich-Kondrakov theorem)
- $\psi_i \to \psi_q$ weakly in H_1^q (by reflexivity of the Sobolev space H_1^q).

Moreover, we can write:

$$\int_M \langle D_g\psi_q, \psi_q\rangle dv(g) = \int_M \langle D_g\psi_q, \psi_q - \psi_i\rangle dv(g) + \int_M \langle D_g\psi_q, \psi_i\rangle dv(g)$$

and first note that:

$$\Big|\int_M \langle D_g\psi_q, \psi_q - \psi_i\rangle dv(g)\Big| \leq ||D_g\psi_q||_q||\psi_q - \psi_i||_p \longrightarrow 0$$

where we have used the Hölder inequality and the strong convergence in L^p. One can also easily check that the map:

$$\Phi \longmapsto \int_M \langle D_g\psi_q, \Phi\rangle dv(g)$$

defines a continuous linear form on H_1^q and then the weak convergence in H_1^q gives:

$$\int_M \langle D_g\psi_q, \psi_q\rangle dv(g) = 1,$$

that is $\psi_q \in \mathcal{H}_1^q$. By weak convergence in H_1^q and because of Lemma 1, we also have:

$$||H^{-(1/p)}D_g\psi_q||_q^2 \leq \liminf_{i\to\infty} ||H^{-(1/p)}D_g\psi_i||_q^2 = \lambda_q$$

and thus $\lambda_q = \mathcal{F}_q(\psi_q)$. We have finally shown that there exists $\psi_q \in \mathcal{H}_1^q$ which reach λ_q. Now for all smooth spinors Φ, we compute that:

$$\frac{d}{dt}_{|t=0}||D_g(\psi_q + t\Phi)||_q^2 \ = \ 2\lambda_q^{\frac{2-q}{2}} \int_M \mathrm{Re}\langle H^{-(q/p)}|D_g\psi_q|^{q-2}D_g\psi_q, D_g\Phi\rangle dv(g)$$

and:

$$\frac{d}{dt}_{|t=0}\int_M \mathrm{Re}\langle D_g(\psi_q + t\Phi), (\psi_q + t\Phi)\rangle dv(g) = 2\int_M \mathrm{Re}\langle \psi_q, D_g\Phi\rangle dv(g)$$

which, by the Lagrange multipliers theorem, gives the existence of a real number α such that:

$$\lambda_q^{\frac{2-q}{2}} \int_M \mathrm{Re}\langle H^{-(q/p)}|D_g\psi_q|^{q-2}D_g\psi_q, D_g\Phi\rangle dv(g) \ = \ \alpha\int_M \mathrm{Re}\langle \psi_q, D_g\Phi\rangle dv(g).$$

Moreover, since ψ_q is a critical point for $\mathcal{F}_q$, we get that $\alpha = \lambda_q$, that is:

$$\int_M \langle \lambda_q^{q/2}\psi_q - H^{-(q/p)}|D_g\psi_q|^{q-2}D_g\psi_q, D_g\Phi\rangle dv(g) = 0.$$

To sum up, we proved the existence of a spinor field $\psi_q \in \mathcal{H}_1^q$ satifying weakly the equation:

$$|D_g\psi_q|^{q-2}D_g\psi_q = \lambda_q^{q/2}H^{q/p}\psi_q.$$

If we let $\varphi_q = \lambda_q^{1/2}\psi_q$, we can easily check that $\varphi_q \in H_1^q$ satisfies (E_q) (here we used the relations $|\psi_q| = \lambda_q^{-(q/2)}H^{-(q/p)}|D_g\psi_q|^{q/p}$ and $|D_g\psi_q|^{2-q} = (\lambda_q^{q/2}H^{q/p}|\psi_q|)^{p-2}$). On the other hand, since:

$$\int_M \langle D_g\psi_q, \psi_q\rangle dv(g) = 1,$$

and since the spinor field φ_q is a solution of (E_q), we deduce that:

$$\int_M H|\varphi_q|^p dv_g = 1.$$

Second step: We show that $\varphi_q \in C^{1,\alpha}\big(\Sigma_g(M)\big) \cap\ C^\infty\big(\Sigma_g(M \setminus \varphi_q^{-1}(0))\big)$. The proof of this result uses the classical "bootstrap argument". Indeed, the spinor field φ_q is in the Sobolev space H_1^q which is continuously embedded in L^{p_1} with $p_1 = nq/(n-q)$, by the Sobolev embedding theorem. The Hölder inequality implies that $H|\varphi_q|^{p-2}\varphi \in L^{p_1/(p-1)}$ and then elliptic a-priori estimates (see [Amm03a]) gives $\varphi \in H_1^{p_1/(p-1)}$. Once again, the Sobolev embedding theorem implies that $\varphi_q \in L^{p_2}$ with

$$p_2 = np_1/(n(p-1) - p_1),$$

if $n(p-1) > p_1$ or $\varphi_q \in L^s$ for all $s > 1$ if $n(p-1) \le p_1$. Note that since $q > q_D$, we can easily check that $p_2 > p_1$ and thus we have a better regularity for the spinor field φ_q. In fact, if we continue this argument, we can show that $\varphi_q \in L^{p_i}$ for all i, where p_i is the sequence of real numbers defined by:

$$p_i := \begin{cases} \frac{np_{i-1}}{n(p-1)-p_{i-1}} & \text{if} \quad n(p-1) > p_{i-1} \\ \\ +\infty & \text{if} \quad n(p-1) \le p_{i-1}. \end{cases}$$

A classical study of this sequence leads to the existence of a rank $i_0 \in \mathbb{N}$ such that $p_{i_0} = +\infty$ and thus we can conclude that $\varphi_q \in L^s$ for all $s > 1$. The elliptic a-priori estimate gives that $\varphi_q \in H_1^s$ for all $s > 1$ and if we apply the Sobolev embedding theorem, one get that $\varphi_q \in C^{0,\alpha}$ for all $\alpha \in (0,1)$. Hence $f|\varphi_q|^{p-2}\varphi_q \in C^{0,\alpha}$ as well, and the Schauder estimate (see [Amm03a]) gives $\varphi_q \in C^{1,\alpha}$. It is clear that one can carry on this argument on $M \setminus \varphi_q^{-1}(0)$ to obtain $\varphi_q \in C^\infty(M \setminus \varphi_q^{-1}(0))$. □

Remark 3. *If we assume that $p \ge 2$, the regularity of the spinor field φ_q can be improved to $C^{2,\alpha}$ on the whole of M.*

In the following, we want to prove the existence of a solution for the equation (E_{q_D}). However, we cannot argue like in the proof of Proposition 6 because of the lack of compacity of the embedding $H_1^{q_D} \hookrightarrow L^{p_D}$ which is precisely the one involving in our problem. The idea is to adapt the proof of the Yamabe problem (see for example [LP87]). Indeed we will prove that one can extract a subsequence from the sequence of solutions (φ_q) which converges to a weak solution of Problem (23) (see Lemma 7). Then in Lemma 8, we will get the desired regularity for this solution and finally in Lemma 9, using Inequality (17) of Theorem 3, we will be able to exclude the trivial solution. So we first have:

Lemma 7. *There exists a sequence (q_i) which tends to q_D and such that the corresponding sequence (φ_{q_i}), solution of (E_{q_i}), converges to a weak solution $\varphi \in H_1^{q_D}$ of (23).*

Proof: It is clear that without loss in generalities, one can suppose that the volume of the manifold (M, g) is equal to 1. Otherwise, because of the conformal covariance of equation (23), we just have to change the metric with a homothetic one (and so a conformal one). In a similar way, we can also assume (because of a rescaling argument) that the maximum of the function H is equal to 1. Now we prove that the sequence (φ_q) is uniformaly bounded in $H_1^{q_D}$. Indeed, since $q \geq q_D$, the Hölder inequality gives:

$$||H^{-(1/p_D)} D_g \varphi_q||^2_{q_D} \leq ||H^{-(1/p_D)} D_g \varphi_q||^2_q.$$

On the other hand, $p \leq p_D$ implies that:

$$||H^{-(1/p_D)} D_g \varphi_q||^2_{q_D} \leq \lambda_q^2.$$

Now the variational characterization of λ_q and the Hölder inequality yield to:

$$||H^{-(1/p_D)} D_g \varphi_q||^2_{q_D} \leq \lambda_q^2 \leq \lambda_1^2 (\min H)^{-1}$$

and thus we can conclude that (φ_q) is uniformally bounded in $H_1^{q_D}$. Then there exists a sequence (q_i) which tends to q_D and a spinor field $\varphi \in H_1^{q_D}$ such that:

- $\varphi_{q_i} \to \varphi$ weakly in $H_1^{q_D}$ (by reflexivity of the Sobolev space $H_1^{q_D}$)
- $\varphi_{q_i} \to \varphi$ a.e. on M.

Moreover, since (φ_{q_i}) is bounded in $H_1^{q_D}$, the Sobolev embedding theorem implies that it is bounded in L^{p_D}, and then $H|\varphi_{q_i}|^{p_i-2}\varphi_{q_i}$ is bounded in $L^{p_D/(p_i-1)}$. However, since $p_D/(p_D-1) < p_D/(p_i-1)$, the sequence $H|\varphi_{q_i}|^{p_i-2}\varphi_{q_i}$ is also bounded in $L^{p_D/(p_D-1)}$. Using this fact and since

$$H|\varphi_{q_i}|^{p_i-2}\varphi_{q_i} \to H|\varphi|^{p_D-2}\varphi \quad \text{a.e. on } M,$$

we finally get that:

$$H|\varphi_{q_i}|^{p_i-2}\varphi_{q_i} \to H|\varphi|^{p_D-2}\varphi \quad \text{weakly in } L^{p_D/(p_D-1)}$$

and so weakly in L^1. Now note that for all smooth spinor fields Φ, the map:

$$\psi \longmapsto \int_M \langle D_g \psi, D_g \Phi \rangle dv(g)$$

defines a continuous linear form on $H_1^{q_D}$ and thus by weak convergence in $H_1^{q_D}$, we obtain:

$$\int_M \langle D_g \varphi_{q_i}, D_g \Phi \rangle dv(g) \underset{i \to +\infty}{\longrightarrow} \int_M \langle D_g \varphi, D_g \Phi \rangle dv(g).$$

By weak convergence in L^1, we also have:

$$\int_M \langle H|\varphi_{q_i}|^{p_i-2}\varphi_{q_i}, D_g\Phi\rangle dv(g) \underset{i\to+\infty}{\longrightarrow} \int_M \langle H|\varphi|^{p_D-2}\varphi, D_g\Phi\rangle dv(g).$$

Now using the variational characterization (24) of λ_q and the fact that the function:

$$q \longmapsto ||D_g\Phi||_q$$

is continuous, we easily conclude that $q \longmapsto \lambda_q$ is also continuous. Combining all the preceding statements with the fact that φ_{q_i} is a solution of (E_{q_i}) leads to:

$$\int_M \langle D_g\varphi, D_g\Phi\rangle dv(g) = \lambda_{\min} \int_M \langle H|\varphi|^{2/(n-1)}\varphi, D_g\Phi\rangle dv(g),$$

for all smooth spinor fields Φ, that is $\varphi \in H_1^{q_D}$ is a weak solution of (23). □

We then state a regularity Lemma which is proved in [Amm03a] and thus we omit the proof here.

Lemma 8. *The spinor field φ given in Lemma 7 satisfies $\varphi \in C^{1,\alpha}\big(\Sigma_g(M)\big) \cap C^\infty\big(\Sigma_g(M \setminus \varphi^{-1}(0))\big)$.*

As point out by Trüdinger in the context of the Yamabe problem, one cannot exclude from this step the case where the spinor field φ obtained in Lemma 7 and 8 is identically zero. In [Amm03a], B. Ammann prove that if (22) (with H constant) is fulfilled then φ is non trivial. We give a similar result for Equation (23) which generalizes the one of Ammann in the case where the Dirac operator is invertible. The proof we present here is based on the Sobolev-type inequality obtained in Theorem 3. More precisely, we get:

Lemma 9. *If (22) is fullfilled, the spinor φ obtained in Lemma 7 and 8 is non identically zero and satisfies:*

$$\int_M H|\varphi|^{\frac{2n}{n-1}} dv_g = 1.$$

Proof: Let $\varphi_q \in C^{1,\alpha}(\Sigma M) \cap\ C^\infty\big(\Sigma\big(M \setminus \varphi_q^{-1}(0)\big)\big)$ be a solution of Equation (E_q), that is which satisfies:

$$D_g\varphi_q = \lambda_q H|\varphi_q|^{p-2}\varphi_q$$

and $\int_M H|\varphi|^{\frac{2n}{n-1}} dv_g = 1$ for all $q \in (q_D, 2)$ (where p is such that $p^{-1} + q^{-1} = 1$). Since $q > q_D$, the Hölder inequality yields to:

$$\Big(\int_M |D_g\varphi_q|^{q_D} dv(g)\Big)^{2/q_D} \leq \big(\max H\big)^{2/p}\Big(\int_M |H^{-(1/p)} D_g\varphi_q|^q dv(g)\Big)^{2/q} Vol(M,g)^{2(q-q_D)/(qq_D)}$$

and then using (E_q), we get:

$$\int_M |H^{-(1/p)} D_g\varphi_q|^q dv(g) = \lambda_q^q,$$

which finally gives:

$$\Big(\int_M |D_g\varphi_q|^{q_D} dv(g)\Big)^{2/q_D} \leq \big(\max H\big)^{2/p} \lambda_q^2\, Vol(M,g)^{2(q-q_D)/(qq_D)} \tag{25}$$

On the other hand, applying Theorem 3 for the spinor fields φ_q yields to:

$$\int_M \langle D_g\varphi_q,\varphi_q\rangle dv(g)=\lambda_q\leq\Big(K(n)+\varepsilon\Big)\Big(\int_M |D_g\varphi_q|^{q_D}dv(g)\Big)^{2/q_D}+B_\varepsilon\Big(\int_M|\varphi_q|^{q_D}dv(g)\Big)^{2/q_D}$$

where $B_\varepsilon>0$ is a positive constante. Using (25) in the preceding inequality leads to:

$$1\leq\Big(K(n)+\varepsilon\Big)\big(\max H\big)^{2/p}\lambda_q\, Vol(M,g)^{2(q-q_D)/(qq_D)}+B_\varepsilon\Big(\int_M|\varphi_q|^{q_D}dv(g)\Big)^{2/q_D},$$

Now if q tends to q_D, we obtain:

$$1\leq\Big(K(n)+\varepsilon\Big)\big(\max H\big)^{2/p_D}\lambda_{\min}+B_\varepsilon\Big(\int_M|\varphi|^{q_D}dv(g)\Big)^{2/q_D},$$

and since, by hypothesis:

$$\lambda_{\min}<K(n)^{-1}\big(\max H\big)^{-(2/p_D)}\quad\Longrightarrow\quad\big(\max H\big)^{2/p_D}\lambda_{\min}\,K(n)<1,$$

we can conclude that, for $\varepsilon>0$ small enough, the norm $||\varphi||_{q_D}>0$ and thus φ is not identically zero. $\square$

Remark 4. *Note that we recover the result of Ammann proved in* [Amm03a] *for $H=cste$ (under the assumption that the Dirac operator has a trivial kernel).*

4.2. **An upper bound for $\lambda_{\min}$.** In this section, we prove a general upper bound for $\lambda_{\min}$. Namely, we get:

Theorem 10. *Let (M^n,g) be a n-dimensional compact Riemannian spin manifold with $n\geq 3$. If $H\in C^\infty(M)$ is a smooth positive function on M, then the following inequality holds:*

$$\lambda_{\min}\leq K(n)^{-1}(\max_M H)^{-\frac{2}{p_D}}.$$

The proof of Theorem 10 lies on the construction of an adaptated test spinor which will be estimated in the variational characterization of $\lambda_{\min}$. We first note that $\lambda_{\min}$ is a conformal invariant, therefore we can work with any metric within the conformal class of the Riemannian metric g. Indeed, we have:

Proposition 11. *The number $\lambda_{\min}$ is a conformal invariant of (M,g).*

Proof: We can easily compute that for $\overline{g}=u^2g\in[g]$ we have:

$$\mathcal{F}_{\overline{g},q_D}(\overline{\psi})=\mathcal{F}_{g,q_D}(u^{\frac{n-1}{2}}\psi)$$

and then because of the variational characterization (24) of $\lambda_{\min}(M,\overline{g})$, its conformal covariance follows direcly. $\square$

For a sake of completness, we briefly recall the work of Ammann, Grosjean, Humbert and Morel [AHGM] which describe in particular the construction of the test-spinor. We first need to construct a trivialization of the spinor bundle which is adaptated for our problem.

Let $(x_1, ..., x_n)$ be the Riemmanian normal coordinates given by the exponential map at $p \in M$:

$$\begin{array}{rccc} \exp_p : & U \subset T_pM \simeq \mathbb{R}^n & \longrightarrow & V \subset M \\ & (x_1, ..., x_n) & \longmapsto & m. \end{array}$$

Now if we consider the smooth map $m \mapsto G_m := \big(g_{ij}(m)\big)$ which associates to any point $m \in V$ the matrix of the coefficients of the metric g at this point in the basis $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$, then one can find an unique symmetric matrix B_m (which depends smoothly of m) such that $B_m^2 = G_m^{-1}$. Thus, at each point $m \in V$ we can built an isometry between $\mathbb{R}^n$ and the tangent space T_mM defined by:

$$\begin{array}{rccc} \mathrm{B}_m : & \big(\mathrm{T}_{\exp_p^{-1}(m)}U \simeq \mathbb{R}^n, \xi\big) & \longrightarrow & \big(T_mV, g_m\big) \\ & (a^1, ..., a^n) & \longmapsto & \sum_{i,j} b_i^j(m) a^i \frac{\partial}{\partial x_j}(m) \end{array}$$

which induce an identification between the two SO_n-principal bundles of orthonormal frames over (U, ξ) and (V, g). Thereafter, this identification can be lifted to the $Spin_n$-principal bundles of spinorial frames over (U, ξ) and (V, g) and then gives an isometry:

$$\begin{array}{ccc} \Sigma_\xi(U) & \longrightarrow & \Sigma_g(V) \\ \varphi & \longmapsto & \overline{\varphi}. \end{array}$$

This identification has already been used in Section 3 and was denoted by τ. However, for a sake of clarity, we will denote this map by " $\overline{\cdot}$ ". Now we let:

$$e_i := b_i^j \frac{\partial}{\partial x_j},$$

such that $\{e_1, ..., e_n\}$ defines an orthonormal frame of (TV, g) and thus, via the preceding identification, one can relate the Dirac operator acting on $\Sigma_\xi(U)$ with the one acting on $\Sigma_g(V)$. More precisely, if D_ξ and D_g denotes those Dirac operators, we have:

$$D_g\overline{\psi} = \overline{D_\xi\psi} + \sum_{i,j=1}^{n} \big(b_i^j - \delta_i^j\big)\overline{\partial_i \cdot \nabla_{\partial_j}\psi} + \mathbf{W} \cdot \overline{\psi} + \mathbf{V} \cdot \overline{\psi}, \tag{26}$$

where $\mathbf{W} \in \Gamma\big(\mathbb{C}l_g(TV)\big)$ and $\mathbf{V} \in \Gamma(TV)$. With a little work, on can compute the expansion of $\mathbf{W}$ and $\mathbf{V}$ in a neighbourhood of $p \in V$. Indeed, for all $m \in V$ and if r denotes the distance from m to p, we have:

$$b_i^j = \delta_i^j - \frac{1}{6} R_{i\alpha\beta j}(p) x^\alpha x^\beta + O(r^3) \tag{27}$$

$$\mathbf{V} = \Big(- \frac{1}{4} (Ric)_{\alpha k}(p) x^\alpha + O(r^2)\Big) e_k \tag{28}$$

$$|\mathbf{W}| = O(r^3), \tag{29}$$

where R_{ijkl} (resp. $(Ric)_{ik}$) are the components of the Riemann (resp. Ricci) curvature tensor. Now consider the smooth spinor field defined on (U, ξ) by:

$$\psi(x) = f^{\frac{n}{2}}(x)(1 - x) \cdot \psi_0$$

where $f(x) = \frac{2}{1+r^2}$ (with $r^2 = x_1^2 + \cdots + x_n^2$) and $\psi_0 \in \Sigma_\xi(U)$ is a constant spinor which can be chosen such that $|\psi_0| = 1$. A straightforward computation show that:

$$D_\xi \psi = \frac{n}{2} f\psi, \quad |\psi|^2 = f^{n-1} \quad \text{and} \quad |D_\xi\psi|^2 = \frac{n^2}{4} f^{n+1}. \tag{30}$$

With these constructions, we can prove the main statement of this section.

Proof of Theorem 10: We construct a test spinor and estimate this spinor field in the variational characterization of $\lambda_{\min}$. Let $\varepsilon > 0$ and ψ the spinor field described above, then we define:

$$\psi_\varepsilon(x) := \eta\psi\big(\frac{x}{\varepsilon}\big) \in \Gamma\big(\Sigma_\xi(\mathbb{R}^n)\big) \tag{31}$$

where $\eta = 0$ on $\mathbb{R}^n \setminus B_p(2\delta)$, $\eta = 1$ on $B_p(\delta)$ and $0 < \delta < 1$ is chosen such that $B_p(2\delta) \subset U$. Since the support of the spinor field ψ_ε lies in the open set U of $\mathbb{R}^n$, one can use the trivialization described previously to obtain a spinor field $\overline{\psi_\varepsilon}$ over (M, g). On the other hand, because of the conformal covariance of $\lambda_{\min}$, we can assume that the metric g satisfies $Ric(p)_{ij} = 0$. Now we can start the estimate:

$$\begin{aligned} D_g\overline{\psi}_\varepsilon(x) \;&=\; \overline{\nabla}\eta \cdot \overline{\psi}_\varepsilon\big(\frac{x}{\varepsilon}\big) + \frac{\eta}{\varepsilon}\frac{n}{2} f\big(\frac{x}{\varepsilon}\big)\overline{\psi}_\varepsilon\big(\frac{x}{\varepsilon}\big) + \eta\sum_{i,j}(b_i^j - \delta_i^j)\overline{\partial_i \cdot \nabla_{\partial_j}\big(\psi(\frac{x}{\varepsilon})\big)} \\ &\quad + \eta\mathbf{W} \cdot \overline{\psi}_\varepsilon\big(\frac{x}{\varepsilon}\big) + \eta\mathbf{V} \cdot \overline{\psi}_\varepsilon\big(\frac{x}{\varepsilon}\big) \end{aligned}$$

where $|\mathbf{W}| = O(r^3)$ and $|\mathbf{V}| = O(r^2)$ (since $Ric(p)_{ij} = 0$). Using [AHGM], we have:

$$|D_g\overline{\psi}_\varepsilon|^2(x) \leq \frac{n^2}{4\varepsilon^2} f^{n+1}\big(\frac{x}{\varepsilon}\big) + Cr^4 f^{n-1}\big(\frac{x}{\varepsilon}\big) + \frac{C}{\varepsilon} r^2 f^{n-\frac{1}{2}}\big(\frac{x}{\varepsilon}\big) = \frac{n^2}{4\varepsilon^2} f^{n+1}\big(\frac{x}{\varepsilon}\big)\big(1 + \Lambda(x)\big)$$

where $\Lambda(x) = C\varepsilon^2 r^4 f^{-2}\big(\frac{x}{\varepsilon}\big) + C\varepsilon r^2 f^{-\frac{3}{2}}\big(\frac{x}{\varepsilon}\big)$. Now note that for all $u \geq -1$:

$$(1+u)^{\frac{n}{n+1}} \leq 1 + \frac{n}{n+1}u,$$

then we get:

$$|D_g\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}(x) \leq \Big(\frac{n}{2\varepsilon}\Big)^{\frac{2n}{n+1}} f^n\big(\frac{x}{\varepsilon}\big) + \frac{n}{n+1}\Big(\frac{n}{2\varepsilon}\Big)^{\frac{2n}{n+1}} f^n\big(\frac{x}{\varepsilon}\big)\Lambda(x). \tag{32}$$

On the other hand, since $p \in M$ is a point where H is maximum, we have:

$$H(x) = H(p) + O(r^2)$$

which yields to:

$$H(x)^{-\frac{n-1}{n+1}} = H(p)^{-\frac{n-1}{n+1}}\big(1 + O(r^2)\big). \tag{33}$$

An integration combining (32) and (33) gives:

$$\int_{B_p(2\delta)} H^{-\frac{n-1}{n+1}} |D_g\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}} dv(g) \leq \Big(\frac{n}{2\varepsilon}\Big)^{\frac{2n}{n+1}} H(p)^{-\frac{n-1}{n+1}}\big(\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D}\big)$$

where:

$$\begin{aligned}
\mathbf{A} &= \int_{B_p(2\delta)} f^n\big(\frac{x}{\varepsilon}\big)dv(g) \\
\mathbf{B} &= C\int_{B_p(2\delta)} f^n\big(\frac{x}{\varepsilon}\big)\Lambda(x)\,dv(g) \\
\mathbf{C} &= C\int_{B_p(2\delta)} r^2 f^n\big(\frac{x}{\varepsilon}\big)\,dv(g) \\
\mathbf{D} &= C\int_{B_p(2\delta)} r^2 f^n\big(\frac{x}{\varepsilon}\big)\Lambda(x)\,dv(g).
\end{aligned}$$

Since the function f is radially symmetric, we have:

$$\mathbf{A} = \int_0^{2\delta} f^n\big(\frac{x}{\varepsilon}\big)\omega_{n-1}G(r)r^{n-1}dr$$

where

$$G(r) = \int_{\mathbb{S}^{n-1}} \sqrt{|g|_{rx}}d\sigma(x) \quad \text{with} \quad |g|_y := \det g_{ij}(y).$$

Now using the fact that $Ric_{ij}(p) = 0$, one can easily compute that (see [Heb97], for example):

$$G(r) \leq 1 + O(r^4).$$

Thus, a direct computation shows that if $n \geq 3$:

$$\mathbf{A} = \omega_{n-1}I\varepsilon^n + o(\varepsilon^n),$$

where we let $I = \int_0^{+\infty} r^{n-1}f^n(r)dr$. In the same way, we can prove that for $n \geq 3$:

$$\mathbf{B} = \mathbf{C} = \mathbf{D} = o(\varepsilon^n).$$

In brief, we get that:

$$\int_{B_p(2\delta)} H^{-\frac{n-1}{n+1}}|D_g\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}dv(g) = \big(\frac{n}{2}\big)^{\frac{2n}{n+1}}\big(\omega_{n-1}I\big)H(p)^{-\frac{n-1}{n+1}}\varepsilon^{\frac{n(n-1)}{n+1}}\big(1+o(1)\big),$$

hence:

$$\Big(\int_M H^{-\frac{n-1}{n+1}}|D_g\overline{\psi}_\varepsilon|^{\frac{2n}{n+1}}dv(g)\Big)^{\frac{n+1}{n}} = \big(\frac{n}{2}\big)^2\big(\omega_{n-1}I\big)^{\frac{n+1}{n}}H(p)^{-\frac{n-1}{n}}\varepsilon^{n-1}\big(1+o(1)\big).$$

The denominator of the functional $\lambda_{\min}$ can also be estimated and similar calculations give (see [AHGM]):

$$\int_M \langle D_g\overline{\psi}_\varepsilon, \overline{\psi}_\varepsilon\rangle dv(g) = \frac{n}{2}\omega_{n-1}I\varepsilon^{n-1} + o(\varepsilon^{n+1})$$

for $n \geq 3$. Combining these estimates leads to:

$$\lambda_{\min} \leq K(n)^{-1}\big(\max_M H\big)^{-\frac{2}{p_D}}\big(1+o(1)\big)$$

which prove the announced result. $\square$

Remark 5. *We can derive a similar result for the case of* 2*-dimensional manifolds. More precisely, using the proof of* [AHM06] *one can show that if* (M^2, g) *is a smooth surface we have:*

$$\lambda_{\min} \leq 2\sqrt{\pi}\,(\max_M H)^{-2}.$$

Remark 6. *This result is in the spirit of the one obtained by Aubin in* [Aub76a] *for the conformal Laplacian. Indeed, in this article, the author proves that on a* n*-dimensional compact Riemannian manifold with* $n > 4$*, and if* f, h *are smooth positive functions on* M *such that:*

$$h(p) - R_g(p) + \frac{n-4}{2}\frac{\Delta_g f(p)}{f(p)} < 0,$$

where $f(p) = \max_{x\in M} f(x)$*, then the nonlinear equation:*

$$4\frac{n-1}{n-2}\Delta_g u + hu = f u^{\frac{n+2}{n-2}}$$

admits a smooth positive solution. One could hope to obtain a similar criterion for the equation we study. However, if one carries out the computations in the proof of Theorem 10, we obtain for $n \geq 5$*:*

$$\lambda_{\min} = K(n)^{-1}\big(\max_M H\big)^{-\frac{2}{p_D}}\big(1 + \frac{n-1}{2n(n-2)}\frac{\Delta H(p)}{H(p)}\,\varepsilon^2 + o(\varepsilon^2)\big).$$

Thus we cannot conclude anything since at a point $p \in M$ *where* H *is maximum we have* $\Delta H(p) \geq 0$*.*

4.3. **An existence result.** To conclude this section, we give conditions on the manifold (M^n, g) and on the function $H \in C^\infty(M)$ which ensure that (22) is fulfilled and thus that there exists a solution to the nonlinear Dirac equation (23) (applying Theorem 5). The condition on H is a technical one given by:

There is a maximum point $p \in M$ *at which all partial derivaties of* H *of order less than or equal to* $(n-1)$ *vanish.* (34)

The result we obtain is the following:

Theorem 12. *Suppose* (M^n, g) *is a locally conformally flat manifold and* $H \in C^\infty(M)$ *a smooth positive function on* M *for which (34) holds. Assume that the Dirac operator is invertible and the mass endomorphism has a positive eigenvalue. Then there exists a spinor field solution of the nonlinear Dirac equation (23).*

This result is quite close to the work of Escobar and Schoen [ES86] and lies on the construction of Ammann, Humbert and Morel [AHM06] of the mass endomorphism.

Now we briefly recall the construction of the mass endormorphism. For more details, we refer to [AHM06]. Consider a point $p \in M$ and suppose that there is a neighborhoud U of p which is flat. Thus, because we assumed that the Dirac operator has a trivial kernel, one can easily show that the Green function G_D of the Dirac operator has the following expansion in U:

$$\omega_{n-1} G_D(x,p)\psi_0 = -\frac{x-p}{|x-p|^n}\cdot\psi_p + v(x,p)\psi_p$$

for all $x \in U$ and where $v(\,.\,,p)\psi_p$ is a smooth harmonic spinor near p with $\psi_p \in \Sigma_p(M)$. The mass operator is defined as the self-adjoint endomorphism of the fiber $\Sigma_p(M)$ given by:

$$\alpha_p(\psi_p) = v(p,p)\psi_p.$$

This operator shares many properties with the mass of the Green function of the conformal Laplacian. One of them is that the sign of its eigenvalues is invariant under conformal changes of metrics which preserves the flatness near p. With this construction, we can prove the main result of this section.

Proof of Theorem 12: We have to construct a test-spinor which will be estimated in the variational characterization of $\lambda_{\min}$. Then the assumption on the mass endomorphism will enable to conclued that (22) is fulfilled and the result will follow from Theorem 5. The test-spinor is exactly the one used in [AHM06]. In order to make this paper self-contained, we have chosen to briefly recall this construction. First, since $\lambda_{\min}$ is a conformal invariant of (M^n, g) which is locally conformally flat, one can suppose that the metric is flat near a point $p \in M$ where (34) is fulfilled. Now for $\varepsilon > 0$ we set:

$$\xi := \varepsilon^{\frac{1}{n+1}} \qquad \varepsilon_0 := \frac{\xi^n}{\varepsilon} f\Big(\frac{\xi}{\varepsilon}\Big)^{\frac{n}{2}}$$

where $f(r) = \frac{2}{1+r^2}$ is the function defined in the previous section. The test-spinor is then defined by:

$$\Phi_\varepsilon(x) = \begin{cases} f\big(\frac{x}{\varepsilon}\big)^{\frac{n}{2}}\big(1 - \frac{x}{\varepsilon}\big)\cdot\psi_p + \varepsilon_0\alpha_p(\psi_p) & \text{if } r \leq \xi \\ \\ \varepsilon_0\big(\omega_{n-1}G_D(x,p) - \eta(x)\theta_p(x)\big) + \eta(x) f\big(\frac{\xi}{\varepsilon}\big)^{\frac{n}{2}}\psi_p & \text{if } \xi \leq r \leq 2\xi \\ \\ \varepsilon_0\,\omega_{n-1}G_D(x,p) & \text{si } r \geq 2\xi \end{cases}$$

where η is a cut-off function such that:

$$\eta = \begin{cases} 1 & \text{on } B_p(\xi) \\ 0 & \text{on } M \setminus B_p(2\xi) \end{cases} \qquad \text{and} \qquad |\nabla\eta| \leq \frac{2}{\xi}$$

and $\theta_p(x) := v(x,p)\psi_p - \alpha_p(\psi_p)$ is a smooth spinor field (harmonic near p) which satisfies $|\theta_p| = O(r)$. Now an easy calculation shows that:

$$|D\Phi_\varepsilon|^{\frac{2n}{n+1}}(x) = \begin{cases} \big(\frac{n}{2}\big)^{\frac{2n}{n+1}}\varepsilon^{-\frac{2n}{n+1}} f\big(\frac{r}{\varepsilon}\big)^n & \text{if } r \leq \xi \\ \\ |\varepsilon_0\nabla\eta(x)\cdot\theta_p(x) - f\big(\frac{\xi}{\varepsilon}\big)^{\frac{n}{2}}\nabla\eta(x)\cdot\psi_p|^{\frac{2n}{n+1}} & \text{if } \xi \leq r \leq 2\xi \\ \\ 0 & \text{si } r \geq 2\xi. \end{cases}$$

On the other hand, since the function H satisfies the condition (34), we get that:

$$H(x) = H(p) + O(r^n)$$

that is:

$$H(x)^{-\frac{n-1}{n+1}} = H(p)^{-\frac{n-1}{n+1}}\big(1 + O(r^n)\big).$$

We can now give the estimate of the functional (24) (with $q = q_D$) evaluated at the spinor field Φ_ε. First, on $B_p(\xi)$ we have:

$$\int_{B_p(\xi)} H^{-\frac{n-1}{n+1}} |D\Phi_\varepsilon|^{\frac{2n}{n+1}} dx \leq \big(\frac{n}{2}\big)^{\frac{2n}{n+1}} \varepsilon^{-\frac{2n}{n+1}} H(p)^{-\frac{n-1}{n+1}} \Big(\int_{B_p(\xi)} f^n\big(\frac{x}{\varepsilon}\big) dx + C \int_{B_p(\xi)} r^n f^n\big(\frac{x}{\varepsilon}\big) dx \Big)$$

and we can easily compute that:

$$\begin{aligned} \int_{B_p(\xi)} f^n\big(\frac{x}{\varepsilon}\big) dx &\leq \varepsilon^n \int_{\mathbb{R}^n} f^n(x) dx \\ \int_{B_p(\xi)} r^n f^n\big(\frac{x}{\varepsilon}\big) dx &= o(\varepsilon^{2n-1}) \end{aligned}$$

which finally gives:

$$\int_{B_p(\xi)} H^{-\frac{n-1}{n+1}} |D\Phi_\varepsilon|^{\frac{2n}{n+1}} dx = \big(\frac{n}{2}\big)^{\frac{2n}{n+1}} \varepsilon^{\frac{n(n-1)}{n+1}} H(p)^{-\frac{n-1}{n+1}} I \big(1 + o(\varepsilon^{n-1})\big)$$

where we let $I = \int_{\mathbb{R}^n} f^n(x) dx$. On $C_p(\xi) := B_p(2\xi) \setminus B_p(\xi)$, we have:

$$\begin{aligned} \int_{C_p(\xi)} H^{-\frac{n-1}{n+1}} |D\Phi_\varepsilon|^{\frac{2n}{n+1}} dx &\leq C \int_{C_p(\xi)} |\varepsilon_0 \nabla\eta \cdot \theta_p|^{\frac{2n}{n+1}} dx + C \int_{C_p(\xi)} |f\big(\frac{\xi}{\varepsilon}\big)^{\frac{n}{2}} \nabla\eta \cdot \psi_p|^{\frac{2n}{n+1}} dx \\ &+ C \int_{C_p(\xi)} r^n |\varepsilon_0 \nabla\eta \cdot \theta_p|^{\frac{2n}{n+1}} dx + C \int_{C_p(\xi)} r^n |f\big(\frac{\xi}{\varepsilon}\big)^{\frac{n}{2}} \nabla\eta \cdot \psi_p|^{\frac{2n}{n+1}} dx \end{aligned}$$

and since $\varepsilon_0 \leq C\varepsilon^{n-1}$, $|\nabla\eta| \leq 2\xi^{-1}$, $|\theta_p| = O(r)$ and $Vol\big(C_p(\xi)\big) \leq C\xi^n$, we get:

$$\int_{C_p(\xi)} H^{-\frac{n-1}{n+1}} |D\Phi_\varepsilon|^{\frac{2n}{n+1}} dx = o\big(\varepsilon^{\frac{(2n+1)(n-1)}{n+1}}\big).$$

To conclude the numerator in the estimate of (24) yields to:

$$\Big(\int_M H^{-\frac{n-1}{n+1}} |D\Phi_\varepsilon|^{\frac{2n}{n+1}} dv(g) \Big)^{\frac{n+1}{n}} = \big(\frac{n}{2}\big)^2 \varepsilon^{n-1} H(p)^{-\frac{n-1}{n}} I^{\frac{n+1}{n}} \big(1 + o(\varepsilon^{n-1})\big)$$

For the estimate of the denominator, we can compute that (see [AHM06]):

$$\int_M \langle D\Phi_\varepsilon, \Phi_\varepsilon\rangle dv(g) = \frac{n}{2} \varepsilon^{n-1} I \big(1 + J \langle \psi_p, \alpha_p(\psi_p)\rangle \varepsilon^{n-1} + o(\varepsilon^{n-1})\big)$$

where $J = \int_{\mathbb{R}^n} f(x)^{\frac{n}{2}+1} dx$. Now we choose $\psi_p \in \Sigma_p(M)$ as an eigenspinor for the mass endomorphism associated with a positive eigenvalue λ and we finally get:

$$\lambda_{\min} \leq \mathcal{F}_{q_D}(\Phi_\varepsilon) = K(n)^{-1} \big(\max_M H\big)^{-\frac{n-1}{n}} \big(1 - \lambda J \,\varepsilon^{n-1} + o(\varepsilon^{n-1})\big).$$

Now it is clear that for $\varepsilon > 0$ sufficiently small, (22) is fulfilled and thus Theorem 5 allows to conclude. □

Remark 7. *In dimension two, a Riemannian surface is always locally conformally flat and condition (34) is fulfilled for all $H \in C^\infty(M)$ however the mass endomorphism vanishes (see* [AHM06]*) and so Theorem 12 cannot be applied.*

5. A remark on manifolds with boundary

In this last section, we briefly study the case of manifolds with boundary. Since the calculations are quite closed from these of the boundaryless case, we only point out arguments which need some explanations.
Indeed, let (M^n, g) be a n-dimensional compact Riemannian spin manifold with smooth boundary equipped with a chirality operator γ, that is an endomorphism of the spinor bundle which satisfies:

$$\begin{aligned} \gamma^2 = Id, &\qquad \langle \gamma\psi, \gamma\varphi\rangle = \langle \psi, \varphi\rangle \\ \nabla_X(\gamma\psi) = \gamma(\nabla_X \psi), &\qquad X \cdot \gamma\psi = -\gamma(X \cdot \psi), \end{aligned}$$

for all $X \in \Gamma(TM)$ and for all spinor fields $\psi, \varphi \in \Gamma\big(\Sigma_g(M)\big)$. Thus the orthogonal projection:

$$B_g^{\pm} := \frac{1}{2}(Id \pm \nu_g \cdot \gamma),$$

where ν_g denotes the inner unit vector fields normal to ∂M, defines a (local) elliptic boundary condition (called the chiral bag boundary condition or (CHI) boundary condition) for the Dirac operator D_g of (M, g). Moreover, under this boundary condition, the spectrum of the Dirac operator consists of entirely isolated real eigenvalues with finite multiplicity. In [Rau] (see also [Rau06a]), we define a spin conformal invariant similar to (6) using this boundary condition. More precisely, if $\lambda_1^{\pm}(\overline{g})$ stands for the first eigenvalue of the Dirac operator $D_{\overline{g}}$ under the chiral bag boundary condition $B_{\overline{g}}^{\pm}$ then the chiral bag invariant is defined by:

$$\lambda_{\min}(M, \partial M) := \inf_{\overline{g}\in[g]} |\lambda_1^{\pm}(\overline{g})| \mathrm{Vol}(M, \overline{g})^{\frac{1}{n}}$$

and one can check that:

$$\lambda_{\min}(M, \partial M) = \inf_{\varphi} \Big\{ \frac{\big(\int_M |D_g\varphi|^{\frac{2n}{n+1}} dv(g)\big)^{\frac{n+1}{n}}}{\big|\int_M \langle D_g\varphi, \varphi\rangle dv(g)\big|} \Big\} \tag{35}$$

where the infimum is taken for all spinor fields $\varphi \in H_1^{q_D}$ such that $B_g^{\pm}\varphi_{|\partial M} = 0$. On the round hemisphere $(\mathbb{S}^n_+, g_{\mathrm{st}})$, we can compute that:

$$\lambda_{\min}(\mathbb{S}^n_+, \partial\mathbb{S}^n_+) = \frac{n}{2}\Big(\frac{\omega_n}{2}\Big)^{\frac{1}{n}} = 2^{-(1/n)} K(n)^{-1}, \tag{36}$$

and thus using the conformal covariance of (35) and the fact that the hemisphere is conformally isometric to the half Euclidean space $(\mathbb{R}^n_+, \xi)$, we conclude that:

$$\Big| \int_{\mathbb{R}^n_+} \langle D_\xi\psi, \psi\rangle dx \Big| \leq 2^{\frac{1}{n}} K(n) \Big(\int_{\mathbb{R}^n_+} |D_\xi\psi|^{\frac{2n}{n+1}} dx \Big)^{\frac{n+1}{n}} \tag{37}$$

for all $\psi \in \Gamma_c\big(\Sigma_\xi(\mathbb{R}^n_+)\big)$ where $\Gamma_c\big(\Sigma_\xi(\mathbb{R}^n_+)\big)$ denotes the space of smooth spinor fields over $(\mathbb{R}^n_+, \xi)$ with compact support. In order to prove a Sobolev-type inequality for manifolds with boundary, we give a result similar to Lemma 2in this context:

Lemma 13. *If the Dirac operator is invertible under the chiral bag boundary condition then there exists a constant $C>0$ such that:*

$$||\varphi||_{p_D}\leq C||D_g\varphi||_{q_D}$$

for all $\varphi\in H_1^{q_D}$ such that $B_g^{\pm}\varphi_{|\partial\mathrm{M}}=0$.

Proof: Since the Dirac operator is assumed to be invertible and since the Fredholm property of D_g does not depend on the choice of the Sobolev spaces (see [Sch95]), we have that:

$$D_g:\mathcal{H}_{\pm}^{q_D}:=\big\{\varphi\in H_1^{q_D}\;/\;B_g^{\pm}\varphi_{|\partial\mathrm{M}}=0\big\}\longrightarrow L^{q_D}$$

defines a continous bijection. Thus using the open mapping theorem, the inverse map is also continous and then we get the existence of a constant $C>0$ such that:

$$||\varphi||_{H_1^{q_D}}=||D_g^{-1}(D_g\varphi)||_{H_1^{q_D}}\leq C||D_g\varphi||_{q_D},$$

for all $\varphi\in\mathcal{H}_{\pm}^{q_D}$. On the other hand, using the Sobolev embedding Theorem, the map $H_1^{q_D}\hookrightarrow L^{p_D}$ is continous, so there exists a constant $C>0$ such that:

$$||\varphi||_{p_D}\leq C||\varphi||_{H_1^{q_D}},$$

for all $\varphi\in H_1^{q_D}$ and this conclude the proof. □

Remark 8. *Lemma 13 gives a similar result of Lott (see* [Lot86]*) for the Dirac operator on manifolds with boundary. More precisely, we have that if D_g is invertible under the chiral bag boundary condition, then:*

$$\lambda_{\min}(M,\partial M)>0. \tag{38}$$

Indeed, the Hölder inequality gives:

$$\Big|\int_M\langle D_g\varphi,\varphi\rangle dv(g)\Big|\leq||\varphi||_{p_D}||D_g\varphi||_{q_D},$$

and then Lemma 2 yields to:

$$\frac{||D_g\varphi||_{q_D}^2}{\Big|\int_M\langle D_g\varphi,\varphi\rangle dv(g)\Big|}\geq\frac{||D\varphi||_{q_D}}{||\varphi||_{p_D}}\geq C,$$

for all $\varphi\in\mathcal{H}_{\pm}^{q_D}$. Using the variational characterisation (35) of $\lambda_{\min}(M,\partial M)$ leads to the result. In [Rau06b]*, we give an explicit lower bound for the chiral bag invariant given by:*

$$\lambda_{\min}(M,\partial M)^2\geq\frac{n}{4(n-1)}\mu_{[g]}(M,\partial M). \tag{39}$$

The number $\mu_{[g]}(M,\partial M)$ is a conformal invariant of the manifold (called the Yamabe invariant) introduced by Escobar in [Esc92] *to study the Yamabe problem on manifolds with boundary and defined by:*

$$\mu_{[g]}(M,\partial M)=\inf_{u\in C^1(M),u\neq0}\frac{\int_{\mathrm{M}}\big(4\frac{n-1}{n-2}|\nabla u|^2+R_gu^2\big)dv(g)+2(n-1)\int_{\partial M}h_gu^2ds(g)}{\Big(\int_M u^{\frac{2n}{n-2}}ds(g)\Big)^{\frac{n-2}{n}}}.$$

Here h_g denotes the mean curvature of the boundary of $(\partial M, g)$ in (M, g). Inequality (39) is significant only if the Yamabe invariant is positive and in this case, the Dirac operator under the chiral bag boundary condition is invertible. So Inequality (38) is more general (39) however it doesn't give an explicit lower bound.

We can now argue like in the proof of Theorem 3 and state a Sobolev-like inequality on manifolds with boundary:

Theorem 14. *Let (M^n, g, σ) be an n-dimensional compact spin manifold with non empty smooth boundary and equipped with a chirality operator. Moreover, we assume that the Dirac operator under the chiral bag boundary condition is invertible. Then for all $\varepsilon > 0$, there exists a constant B_ε such that:*

$$\Big|\int_M \langle D_g\varphi, \varphi\rangle dv(g)\Big| \leq \Big(2^{1/n}K(n) + \varepsilon\Big)\Big(\int_M |D_g\varphi|^{q_D} dv(g)\Big)^{2/q_D} + B_\varepsilon\Big(\int_M |\varphi|^{q_D} dv(g)\Big)^{2/q_D},$$

for all $\varphi \in H_1^{q_D}$ such that $B_g^{\pm}\varphi_{|\partial\mathrm{M}} = 0$.

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