Free groups and reduced 1-cohomology of unitary representations

Florian Martin*, Alain Valette

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To Alain Connes, with admiration

Abstract

Guichardet [Gui72] showed that every unitary representation of the free group $\mathbb{F}_n$ $(2 \leq n < \infty)$ has non-zero 1-cohomology. We construct a continuum of pairwise inequivalent, irreducible, unitary representations of $\mathbb{F}_n$, with vanishing reduced 1-cohomology and such that the C^*-algebra generated by each representation is the unitized algebra of the compact operators.

1 Introduction

If G is a countable discrete group and π a unitary representation of G, we denote by $H^1(G, \pi)$ the first cohomology of G with coefficients in π, i.e. the quotient of the space $Z^1(G, \pi)$ of 1-cocycles by the space $B^1(G, \pi)$ of 1-coboundaries. Endowed with the topology of pointwise convergence, $Z^1(G, \pi)$ becomes a Fréchet space, and the *reduced* 1-cohomology $\overline{H^1}(G, \pi)$ is defined as the quotient of $Z^1(G, \pi)$ by the closure of the space of 1-coboundaries.

Reduced 1-cohomology was first considered by Guichardet [Gui72] and its relevance to questions of rigidity and geometric group theory was emphasized more recently in papers of Shalom (see [Sha00], [Sha04]).

This paper focuses on the free group on n generators $G = \mathbb{F}_n$ $(2 \leq n \leq \infty)$: its 1-cohomology has the following interesting property established

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by Guichardet (Example 1 in [Gui72]): $H^1(\mathbb{F}_2, \pi) \neq 0$ for every unitary representation π of $\mathbb{F}_2$. Using the dictionary between 1-cohomology and affine isometric actions on Hilbert spaces (see e.g. [BHV08], p.73), the geometric equivalent of this observation is: every unitary representation of $\mathbb{F}_2$ is the linear part of some affine isometric action without globally fixed point.

We illustrate the difference between reduced and ordinary 1-cohomology by establishing:

Theorem 1.1 *Fix $n \in \mathbb{N} \cup \{\infty\}$ ($n \geq 2$). There exists a continuum of pairwise inequivalent, unitary, irreducible representations σ of $\mathbb{F}_n$ such that*

1) $\overline{H^1}(\mathbb{F}_n, \sigma) = 0$.

2) The C^*-algebra generated by $\sigma(\mathbb{F}_n)$ is $\tilde{\mathcal{K}}$, the unitized C^*-algebra of the algebra $\mathcal{K}$ of compact operators on an infinite-dimensional separable Hilbert space.

There are other instances of the fact that, for a given group, the vanishing of the 1-cohomology is not equivalent to the vanishing of its reduced counterpart: for example, let λ_G be the left regular representation of a countably infinite amenable group G: then $H^1(G, \lambda_G) \neq 0$, by Théorème 1 in [Gui72], while $\overline{H^1}(G, \lambda_G) = 0$ by [MV07]. Let us mention however a remarkable result by Shalom [Sha00]: for a *compactly generated* locally compact group, the vanishing of reduced 1-cohomology for all unitary representations, is equivalent to the vanishing of 1-cohomology for all unitary representations (the latter being equivalent to Kazhdan's property (T), by the Delorme-Guichardet theorem, see Chapter 2 in [BHV08]).

2 Proof of Theorem 1.1

Let us denote by $Im\, T$ the range of the linear operator T.

Lemma 2.1 *Fix an integer $n \geq 2$. Let $U_1, ..., U_n$ be unitary operators on a Hilbert space such that:*

1) 1 is not an eigenvalue of U_i, for $1 \leq i \leq n$;

2) for $2 \leq j \leq n$:

$$Im(U_j - 1) \cap (\sum_{i=1}^{j-1} Im(U_i - 1)) = \{0\}.$$

Then the assignment $\pi(x_i) = U_i^*$ defines a unitary representation π of the free group $\mathbb{F}_n$ on n generators $x_1, ..., x_n$, such that $\overline{H^1}(\mathbb{F}_n, \pi) = 0$.

Proof of the lemma: We start the same way as Guichardet in Example 1 in [Gui72], in his proof of $H^1(\mathbb{F}_2, \sigma) \neq 0$ for every unitary representation σ of $\mathbb{F}_2$. For a unitary representation π of $\mathbb{F}_n$ on a Hilbert space V, the map

$$Z^1(\mathbb{F}_n, \pi) \to V^n : b \mapsto (b(x_1), ..., b(x_n))$$

is a topological isomorphism (surjectivity follows from the freeness of the group: a 1-cocycle can be defined arbitrarily on generators). In that isomorphism, $B^1(\mathbb{F}_n, \pi)$ corresponds to the image of the map

$$\psi : V \to V^n : v \mapsto ((\pi(x_1) - 1)v, ..., (\pi(x_n) - 1)v)$$

So $\overline{H^1}(\mathbb{F}_n, \pi) = 0$ if and only if ψ has dense image, if and only if $\psi^* : V^n \to V$ is injective. But

$$\psi^*(v_1, ..., v_n) = \sum_{i=1}^{n} (\pi(x_i)^* - 1)v_i.$$

With $U_i = \pi(x_i)^*$, we see that our assumptions on $U_1, ..., U_n$ exactly mean that ψ^* is injective. □

We now come to a problem in operator theory, namely construct families of unitary operators satisfying the conditions in Lemma 2.1. We will elaborate on Dixmier's elegant construction [Dix49] (see also Theorem 3.6 in [FW71]) to answer that question.

Proof of Theorem 1.1: On $V = L^2[0, 2\pi]$ with the trigonometric orthonormal system $(e_k)_{k\in\mathbb{Z}}$, let us construct unitary operators U_n $(n \geq 1)$ such that $U_n - 1$ is trace-class, 1 is not an eigenvalue of U_n and $Im(U_n - 1) \cap (\sum_{m=1}^{n-1} Im(U_m - 1)) = \{0\}$ for $n \geq 2$. Moreover U_1, U_2 will be shown to act together irreducibly on V. Taking into account the fact that every irreducible C^*-algebra intersecting $\mathcal{K}$ non-trivially, must contain it (see [Dix77], Corollary 4.1.10), we get the second statement in the Theorem. For $n < \infty$, the first statement (vanishing of $\overline{H^1}$) will follow straight from Lemma 2.1. For $n = \infty$, we observe that if a group Γ is the increasing union of subgroups Γ_n with $\overline{H^1}(\Gamma_n, \sigma|_{\Gamma_n}) = 0$, then clearly $\overline{H^1}(\Gamma, \sigma) = 0$.

To construct a continuum of such families of unitary operators, fix a real-valued rapidly decreasing sequence $a = (a_k)_{k\in\mathbb{Z}}$, such that $0 \neq a_k \neq a_m$ for $k \neq m \in \mathbb{Z}$, and define a diagonal, trace-class operator $A^{(a)}$ on V by

$$A^{(a)}e_k = a_k e_k \quad (k \in \mathbb{Z}).$$

Note that $A^{(a)}$ is injective with all eigenvalues of multiplicity 1, by our choice of a. Now let $(x_n)_{n>0}$ be a strictly increasing sequence in $[0,2\pi[$, with $x_1 = 0$ and $\frac{x_2}{\pi}$ irrational. Define a function ξ_n on $[0,2\pi[$ by

$$\xi_n(x) = \begin{cases} -1 & if \quad 0 \leq x < x_n \\ 1 & if \quad x_n \leq x < 2\pi \end{cases}$$

Let M_n be the operator of multiplication by ξ_n: this is a self-adjoint unitary operator on V; note that $M_1 = 1$. Set $A_n^{(a)} = M_n A^{(a)} M_n^*$; the following holds:

Claim: For $n \geq 2$:

$$Im\, A_n^{(a)} \cap (\sum_{i=1}^{n-1} Im\, A_i^{(a)}) = \{0\}.$$

To prove the claim, observe that , because a is rapidly decreasing, $Im\, A^{(a)}$ is contained in the space of restrictions of real-analytic functions to $[0,2\pi[$. Then $\sum_{i=1}^{n-1} Im\, A_i^{(a)}$ is contained in the space of functions on $[0,2\pi[$ whose restrictions to all intervals $[x_1,x_2[$, $[x_2,x_3[$, ..., $[x_{n-1},2\pi[$ are real analytic. On the other hand non-zero functions in $Im\, A_n^{(a)}$ are not analytic in the neighborhood of $x_n \in]x_{n-1},2\pi[$, proving the claim.

Let then $U_n^{(a)}$ be the Cayley transform of $A_n^{(a)}$:

$$U_n^{(a)} = (1 - iA_n^{(a)})(1 + iA_n^{(a)})^{-1}.$$

Then $U_n^{(a)}$ is unitary, 1 is not an eigenvalue of $U_n^{(a)}$, and $U_n^{(a)}$ is diagonal in the basis $(M_n e_k)_{k\in\mathbb{Z}}$, with all eigenvalues of multiplicity 1. Moreover

$$U_n^{(a)} - 1 = -2iA_n^{(a)}(1 + iA_n^{(a)})^{-1}$$

so that $U_n^{(a)} - 1$ is trace-class, and $Im(U_n^{(a)} - 1) \cap (\sum_{m=1}^{n-1} Im(U_m^{(a)} - 1)) = \{0\}$ by the claim.

To prove that $U_1^{(a)}$, $U_2^{(a)}$ together act irreducibly on V, let S be an operator on V which commutes both with $U_1^{(a)}$ and $U_2^{(a)}$. Since $U_1^{(a)}$, $U_2^{(a)}$ have all eigenvalues of multiplicity 1, the operator S must be diagonal both in the bases $(e_k)_{k\in\mathbb{Z}}$ and $(M_2 e_k)_{k\in\mathbb{Z}}$. From the Fourier series expansion of $M_2 e_k$:

$$M_2 e_k = (1 - \frac{x_2}{\pi})e_k + \sum_{m\neq k} \frac{i}{\pi(m-k)}[1 - e^{i(k-m)x_2}]e_m$$

and the fact that all Fourier coefficients of $M_2 e_k$ are non-zero (because $\frac{x_2}{\pi}$ is irrational), it follows that S must be scalar. Irreducibility then follows from Schur's lemma.

Finally, to get a continuum of pairwise inequivalent representations, we notice that, since $U_1^{(a)} - 1$ is trace-class, the complex number

$$Tr(U_1^{(a)} - 1) = -2i\, Tr\, A^{(a)}(1 + iA^{(a)})^{-1} = -2i \sum_{k\in\mathbb{Z}} a_k(1 + ia_k)^{-1}$$

is an invariant of unitary equivalence of the associated representation. So varying a in the space of real-valued rapidly decreasing sequences satisfying $0 \neq a_k \neq a_m$ for $k \neq m \in \mathbb{Z}$, we get the desired continuum. □

Theorem 1.1 motivates:

Question 1 *Is there a countable group G such that $\overline{H^1}(G, \pi) \neq 0$ for every unitary representation π of G?*

Note that such a group, if it exists, must be non-amenable: indeed, by Corollary 5.2 in [MV07], a countable amenable group G has $\overline{H^1}(G, \lambda_G) = 0$, where λ_G is the left regular representation.

3 A remark

The irreducible representations σ constructed in Theorem 2.1 satisfy $\sigma(g) - 1 \in \mathcal{K}$ for every $g \in \mathbb{F}_n$. We observe that this fact alone is responsible for the non-vanishing of H^1.

Proposition 3.1 *Let G be a discrete group. Assume that there exists a unitary irreducible representation π of G with the property that $1 - \pi(g)$ is a compact operator for every $g \in G$. Then $B^1(G, \pi)$ is not closed in $Z^1(G, \pi)$; in particular $H^1(G, \pi) \neq 0$.*

Proof: Observe that by irreducibility the C^*-algebra generated by $\pi(G)$ is $\tilde{\mathcal{K}}$, and consider the short exact sequence

$$0 \to \mathcal{K} \to \tilde{\mathcal{K}} \xrightarrow{q} \mathbb{C} \to 0.$$

For $f \in \ell^1(G)$, we have $q(\pi(f)) = \sum_{g\in G} f(g)q(\pi(g)) = \sum_{g\in G} f(g)$ for every $f \in \ell^1(G)$, so that

$$|\sum_{g\in G} f(g)| \leq \|\pi(f)\|.$$

By Theorem 3.4.4 in [Dix77], this means exactly that π weakly contains the trivial representation of G. We conclude by applying another result by Guichardet ([Gui72], Théorème 1): for a unitary representation without non-zero fixed vectors, the space of 1-coboundaries is not closed in the space of 1-cocycles if and only if the representation weakly contains the trivial representation. □

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Authors'addresses:

Philip Morris International
Quai Jeanrenaud 56
CH-2000 Neuchâtel

SWITZERLAND
florian.martin@pmintl.com

Institut de Mathématiques
Rue Emile Argand 11-BP158
CH-2009 Neuchâtel
SWITZERLAND
alain.valette@unine.ch