G-perfect lattices in dimension 8 over the Gauss and Eisenstein integers

François Sigrist

1 Introduction

This paper is an attempt to classify G-perfect quadratic forms up to dimension 8, using the relative version of Voronoï's algorithm [B-M-S]. The first case is G = cyclic group of order 4 $\{\pm 1, \pm i\}$ (forms over the Gaussian integers), the second G = cyclic group of order 6 $\{\pm 1, \pm\omega, \pm\omega^2\}$ (forms over the Eisenstein integers).

In dimension 4 and 6, the forms are well-known.

The determination of all faces of E_8 is practically out of reach in the Gauss case (the binomial coefficient $\binom{60}{14}$ is 53194089192720). However, as orbits of faces under the respective automorphism groups turn out to contain a fair amount of faces (warm thanks to Gabriele Nebe, who conducted the computations), the 970 faces produced after a few weeks of computations indicate that there are at least 20016 faces of E_8, and that the list is presumably complete.

On the other hand, the computation of the Eisenstein faces of E_8 is almost complete, but a computer crash stopped me short of the end. Presently, 56513 faces of E_8 have been listed, and the prediction (G. Nebe) from these data is 60300. It is therefore very unlikely that an orbit of neighbours of E_8 could escape.

The surprising, if not amazing, result, is that there are very few G-perfect forms in dimension 8. Over the Gaussian integers, E_8 and D_8 appear, as expected, but there seems to be no other G-perfect form. For the Eisenstein integers, one has 5 G-perfect forms, among them E_8, and 4 of them are perfect.

This contrasts with the 10000 known perfect lattices in dimension 8 (cf Jacques Martinet's homepage). From their list, on can see that ca 2/3 of them have a Bravais group of order 2, so one knows that the majority does not show any symmetry. But Gauss or Eisenstein structures appear only on the very few well-known lattices.

The seven lattices displayed below (GA, GB, EA, EB, EC, ED, EE) are strongly eutactic. They are perfect, except EE; their duals are strongly eutactic in cases GA, GB $\simeq$ EA (rootlattices), ED, EE (isodual).

The notations used below are customary:

For a lattice L, $\det(L)$ is the determinant of a Gram matrix of L, L^* is the dual lattice, norm(L) is the minimal norm, and s(L) the half kissing number.

carac(L)=[det(L),[s(L),norm(L)],[s(L^*),norm(L^*)],[elementary divisors of the quotient L^*/L]].

If, by Voronoï's algorithm, the quadratic form A' is the neighbouring form of a form A through the face F, then $A' = A + \rho F$. ρ is the *parameter* of the face F.

A separate text, written as a PARI-readable document, is available.

2 Gauss

GA=D_8 and GB=E_8 are the only Gauss-perfect forms. Both are perfect and strongly eutactic.

2.1 GA

Matrix is $\begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & -1 & 1 \\ 1 & 0 & 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 2 \end{pmatrix}$.

carac(GA)= $[4, [56, 2], [8, 2], [2, 2]]$;

GA has 256 faces,on one single orbit.

Orbit GABA contains 256 faces, each with s=42.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is GB. Parameter is 1.

2.2 GB

Matrix is $\begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 2 & 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 2 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$.

carac(GB)=$[1, [120, 2], [120, 2], [1]]$;

There are seven orbits of faces of GB, predicting a total number of $1920+5760+5760+3840+240+192+2304 = 20016$ faces.

Orbit GBAA contains 1920 faces, each with s=42.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is GA. Parameter is 1

.

Found by computer: 75 faces.

Orbit GBBA contains 5760 faces,each with s=34.

Matrix is $\begin{pmatrix} 2 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 0 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$. Neighbour is GB itself. Parameter is 1.

Found by computer: 84 faces.

Orbit GBBB contains 5760 faces,each with s=40.

Matrix is $\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 2 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is GB itself. Parameter is 1.

Found by computer: 250 faces.

Orbit GBBC contains 3840 faces,each with s=42.

Matrix is $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is GB itself. Parameter is 1.

Found by computer: 257 faces.

Orbit GBBD contains 240 faces, each with s=66.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$. Neighbour is GB itself. Parameter is 1.

Found by computer: 163 faces.

Orbit GBBE contains 192 faces, each with s=60.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 2 & 0 & 0 & 1 \\ 1 & -1 & 1 & 1 & 0 & 2 & -1 & 0 \\ -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}$. Neighbour is GB itself. Parameter is 1.

Found by computer: 98 faces.

Orbit GBBF contains 2304 faces, each with s=30.

Matrix is $\begin{pmatrix} 2 & 0 & 2 & 2 & 1 & 0 & 1 & 1 \\ 0 & 2 & -2 & 2 & 0 & 1 & -1 & 1 \\ 2 & -2 & 2 & 0 & 1 & -1 & 1 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$. Neighbour is GB itself. Parameter is 1.

Found by computer: 43 faces.

3 Eisenstein

There are 5 Eisenstein-perfect forms EA, EB, EC, ED and EE. EA=E_8. All are and strongly eutactic, and perfect, with the exception of EE, which has default of perfection 2, is strongly eutactic and isodual.

3.1 EA=E_8

Matrix is $\begin{pmatrix} 2 & -1 & 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 2 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & 2 \end{pmatrix}$.

carac(EA)= [1,[120, 2],[120, 2],[1]];

EA should have $5184 + 5184 + 2880 + 540 + 432 + 25920 + 8640 + 8640 + 2880 = 60300$ faces, on nine different orbits.

Found by computer: 56513 faces.

Orbit EAAA contains 5184 faces, each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 & 1 & -2 & 2 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & -2 \\ -1 & 1 & 1 & -2 & 4 & -2 & 0 & 3 \\ 0 & -1 & 1 & 1 & -2 & 4 & -3 & 0 \\ 0 & 1 & -2 & 0 & 0 & -3 & 2 & -1 \\ -1 & 0 & 2 & -2 & 3 & 0 & -1 & 2 \end{pmatrix}$. Neighbour is EA itself. Parameter is 1.

Found by computer: 4871 faces.

Orbit EAAB contains 5184 faces, each with s=45.

Matrix is $\begin{pmatrix} 2 & -1 & 1 & 0 & 0 & 2 & 1 & -1 \\ -1 & 2 & -1 & 1 & -2 & 0 & 0 & 1 \\ 1 & -1 & 2 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 2 & -1 & 0 & -1 \\ 2 & 0 & 1 & 1 & -1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$. Neighbour is EA itself. Parameter is 1.

Found by computer: 4776 faces.

Orbit EAAC contains 2880 faces,each with s=54.

Matrix is $\begin{pmatrix} 2 & -1 & 0 & -1 & 2 & -1 & 3 & -1 \\ -1 & 2 & 1 & 0 & -1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 2 & -1 & 0 & -1 & 2 & -1 & 3 & -1 \\ -1 & 2 & 1 & 0 & -1 & 2 & -2 & 3 \\ 3 & -2 & 0 & -1 & 3 & -2 & 4 & -2 \\ -1 & 3 & 1 & 0 & -1 & 3 & -2 & 4 \end{pmatrix}$. Neighbour is EA itself. Parameter is 1.

Found by computer: 2862 faces.

Orbit EAAD contains 540 faces,each with s=66.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & 2 \end{pmatrix}$. Neighbour is EA itself. Parameter is 1.

Found by computer: 540 faces.

Orbit EAAE contains 432 faces, each with s=60.

Matrix is $\begin{pmatrix} 2 & -1 & 1 & 0 & 1 & 0 & 2 & 0 \\ -1 & 2 & -1 & 1 & -1 & 1 & -2 & 2 \\ 1 & -1 & 2 & -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & -2 & 1 & -1 & 1 \\ 1 & -1 & 1 & -2 & 2 & -1 & 2 & -1 \\ 0 & 1 & 1 & 1 & -1 & 2 & -1 & 2 \\ 2 & -2 & 1 & -1 & 2 & -1 & 2 & -1 \\ 0 & 2 & 0 & 1 & -1 & 2 & -1 & 2 \end{pmatrix}$. Neighbour is EA itself. Parameter is 1.

Found by computer: 432 faces.

Orbit EABA contains 25920 faces,each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 & -1 \\ 1 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$. Neighbour is EB. Parameter is 1/2.

Found by computer: 23864 faces.

Orbit EABB contains 8640 faces,each with s=51.

Matrix is $\begin{pmatrix} 2 & -1 & 0 & 0 & 2 & -2 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 2 & -1 & 1 & 0 \\ -2 & 2 & 0 & -1 & -1 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$. Neighbour is EB. Parameter is 1/2.

Found by computer: 8622 faces.

Orbit EACA contains 8640 faces,each with s=45.

Matrix is $\begin{pmatrix} 4 & -2 & 1 & 0 & 1 & 1 & 4 & -3 \\ -2 & 4 & -1 & 1 & -2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 & -1 & 1 \\ 1 & -2 & 1 & -1 & 0 & 0 & 2 & -3 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 2 \\ 4 & -1 & 1 & -1 & 2 & 1 & 4 & -2 \\ -3 & 4 & 0 & 1 & -3 & 2 & -2 & 4 \end{pmatrix}$. Neighbour is EC. Parameter is 1/3.

Found by computer: 7899 faces.

Orbit EADA contains 2880 faces,each with s=45.

Matrix is $\begin{pmatrix} 4 & -2 & 1 & 0 & 1 & 0 & 0 & -1 \\ -2 & 4 & -1 & 1 & -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is ED. Parameter is 1/2.

Found by computer: 2647 faces.

3.2 EB

Matrix is $\begin{pmatrix} 4 & -2 & 2 & 0 & 2 & 1 & 1 & 1 \\ -2 & 4 & -2 & 2 & -3 & 2 & -2 & 1 \\ 2 & -2 & 4 & -2 & 3 & -1 & 2 & 0 \\ 0 & 2 & -2 & 4 & -2 & 3 & -2 & 2 \\ 2 & -3 & 3 & -2 & 6 & -3 & 2 & 1 \\ 1 & 2 & -1 & 3 & -3 & 6 & -3 & 2 \\ 1 & -2 & 2 & -2 & 2 & -3 & 4 & -2 \\ 1 & 1 & 0 & 2 & 1 & 2 & -2 & 4 \end{pmatrix}$.

carac(EB)=[729,[54,4],[3,18],[27,9,3]];

EB has 70 faces,on eight different orbits.

Orbit EBAA contains 18 faces,each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -2 & 1 & -1 & 0 & -2 & 1 & -2 & -1 \\ 1 & -2 & 1 & -1 & 1 & -2 & 3 & -2 \\ 1 & 2 & 0 & 2 & -2 & 3 & 0 & 0 \\ -3 & 1 & -2 & 0 & -1 & -2 & 0 & 0 \end{pmatrix}$. Neighbour is EA. Parameter is 1.

Orbit EBAB contains 6 faces,each with s=51.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & -2 & 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & -1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -2 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 & 1 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$. Neighbour is A1. Parameter is 1.

Orbit EBBA contains 18 faces, each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 & -2 & 1 & -1 & -1 \\ 0 & -1 & 1 & -1 & 1 & -2 & 2 & -1 \\ 1 & 0 & 0 & 1 & -1 & 2 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is EB itself. Parameter is 1.

Orbit EBBB contains 9 faces,each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & -2 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 & -2 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is EB itself. Parameter is 1.

Orbit EBBC contains 6 faces,each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$. Neighbour is EB itself. Parameter is 1.

Orbit EBCA contains 9 faces, each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & -2 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 3 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & -3 & 2 & 0 \\ -2 & 3 & -3 & 2 & -6 & 3 & -2 & -1 \\ -1 & -2 & 1 & -3 & 3 & -6 & 3 & -2 \\ 1 & 0 & 0 & 2 & -2 & 3 & 0 & 0 \\ -1 & 1 & -2 & 0 & -1 & -2 & 0 & 0 \end{pmatrix}$. Neighbour is EC. Parameter is 1/3.

Orbit EBDA contains 3 faces, each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & -2 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is ED. Parameter is 1.

Orbit EBEA contains 1 face,with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ -2 & 1 & -1 & 2 & -2 & 1 & -2 & 1 \\ 1 & -2 & -1 & -1 & 1 & -2 & 1 & -2 \\ -1 & 2 & 0 & 0 & -2 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & -2 & 0 & 0 \end{pmatrix}$. Neighbour is EE. Parameter is 1/3.

3.3 EC

Matrix is $\begin{pmatrix} 6 & -3 & 3 & 0 & 2 & 1 & 2 & 1 \\ -3 & 6 & -3 & 3 & -3 & 2 & -3 & 2 \\ 3 & -3 & 6 & -3 & 3 & -1 & 3 & -1 \\ 0 & 3 & -3 & 6 & -2 & 3 & -2 & 3 \\ 2 & -3 & 3 & -2 & 6 & -3 & 2 & 1 \\ 1 & 2 & -1 & 3 & -3 & 6 & -3 & 2 \\ 2 & -3 & 3 & -2 & 2 & -3 & 6 & -3 \\ 1 & 2 & -1 & 3 & 1 & 2 & -3 & 6 \end{pmatrix}$.

carac(EC)=[20736,[48,6],[9,16],[36,12,6,2,2,2]];

ES3 has 16 faces,on three different orbits.

Orbit ECAA contains 6 faces,each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 1 & -1 \\ -1 & 1 & -2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}$. Neighbour is EA. Parameter is 1.

Orbit ECBA contains 9 faces, each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 \\ -1 & 0 & 0 & -2 & 0 & 0 & -1 & -2 \\ 1 & -1 & 2 & 0 & 0 & 0 & 3 & -1 \\ 2 & 0 & 0 & 1 & -1 & 3 & 0 & 0 \\ -2 & 2 & -1 & 0 & -2 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is EB. Parameter is 1/2.

Orbit ECDA contains 1 face, with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & -3 & 3 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -3 & 0 & -1 & 0 & -1 \\ -3 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 3 & -3 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is ED. Parameter is 1/2.

3.4 ED=L_8^4 (Barnes' lattice)

Matrix is $\begin{pmatrix} 4 & -2 & 2 & -1 & 2 & -1 & 3 & 0 \\ -2 & 4 & -1 & 2 & -1 & 2 & -3 & 3 \\ 2 & -1 & 4 & -2 & 2 & -1 & 3 & 0 \\ -1 & 2 & -2 & 4 & -1 & 2 & -3 & 3 \\ 2 & -1 & 2 & -1 & 4 & -2 & 3 & 0 \\ -1 & 2 & -1 & 2 & -2 & 4 & -3 & 3 \\ 3 & -3 & 3 & -3 & 3 & -3 & 6 & -3 \\ 0 & 3 & 0 & 3 & 0 & 3 & -3 & 6 \end{pmatrix}$.

carac(ED)=[729,[54,4],[12,6],[9,3,3,3,3]];

ED has 216 faces, on three different orbits.

Orbit EDAA contains 72 faces,each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & -1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & -1 \\ 0 & 1 & 0 & 0 & -2 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & -2 & 1 & -1 \\ -2 & 1 & -2 & 1 & 0 & 0 & -1 & 0 \\ 1 & -2 & 1 & -2 & 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -2 & 1 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 & -2 \end{pmatrix}$. Neighbour is EA. Parameter is 1.

Orbit EDBA contains 108 faces, each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is EB. Parameter is 1.

Orbit EDCA contains 36 faces,each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & -2 & 1 & -2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & -2 & 1 & -1 \\ -2 & 1 & 0 & 0 & -2 & 1 & -1 & 0 \\ 1 & -2 & 0 & 0 & 1 & -2 & 1 & -1 \\ -2 & 1 & -2 & 1 & 0 & 0 & -1 & 0 \\ 1 & -2 & 1 & -2 & 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -2 & 1 \\ 0 & -1 & 0 & -1 & 0 & -1 & 1 & -2 \end{pmatrix}$. Neighbour is EC. Parameter is 1/3.

3.5 EE

Matrix is $\begin{pmatrix} 6 & -3 & 3 & 0 & 2 & 2 & 1 & 1 \\ -3 & 6 & -3 & 3 & -4 & 2 & -2 & 1 \\ 3 & -3 & 6 & -3 & 4 & -2 & 3 & 0 \\ 0 & 3 & -3 & 6 & -2 & 4 & -3 & 3 \\ 2 & -4 & 4 & -2 & 8 & -4 & 2 & 2 \\ 2 & 2 & -2 & 4 & -4 & 8 & -4 & 2 \\ 1 & -2 & 3 & -3 & 2 & -4 & 6 & -3 \\ 1 & 1 & 0 & 3 & 2 & 2 & -3 & 6 \end{pmatrix}$.

carac(EE)=[20736,[48,6],[48,6],[12,12,6,6,2,2]];

EE is 12-modular, isodual, and has perfection rank 34 (default 2).

EE has 16 faces,on one single orbit, each with s=45.

Matrix is $\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & -2 & 3 & 0 \\ -1 & 2 & -2 & 1 & -4 & 2 & -1 & -1 \\ -1 & -1 & 1 & -2 & 2 & -4 & 2 & -1 \\ 1 & 1 & 0 & 3 & -1 & 2 & 0 & 0 \\ -2 & 1 & -3 & 0 & -1 & -1 & 0 & 0 \end{pmatrix}$. Neighbour is EE. Parameter is 1/2.

References

[B-M-S] Anne-Marie Bergé, Jacques Martinet, François Sigrist : *Une généralisation de l'algorithme de Voronoï pour les formes quadratiques.* Astérisque **209** (1992), 137-158.

4 Address of author

François Sigrist
Institut de mathématiques
Université de Neuchâtel
Rue Emile Argand 11
CH-2007 Neuchâtel
Switzerland
francois.sigrist@unine.ch