Restricting cohomological representations of $SO_0(n, 1)$ and $SU(n, 1)$

Alain VALETTE

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To Slava Grigorchuk, on his 50th birthday

1 Introduction

Let π be a unitary representation of a locally compact group G. We shall denote by $Z^1(G, \pi)$ the space of (continuous) 1-cocycles on G with values in the Hilbert space of π, by $B^1(G, \pi)$ the subspace of 1-coboundaries, and by $H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$ the 1-cohomology of G with coefficients in π. We will also need the *reduced* 1-cohomology $\overline{H^1}(G, \pi) = Z^1(G, \pi)/\overline{B^1(G, \pi)}$, where the closure is taken in the topology of uniform convergence on compact subsets of G.

We say that π is *cohomological* if $H^1(G, \pi) \neq 0$. We will be mainly interested in irreducible cohomological representations. We will use the standard notation $\widehat{G}$ for the dual of G (i.e. the set of unitary irreducible representations of G, modulo unitary equivalence), and the standard abuse of notation $\pi \in \widehat{G}$ to mean that π is a unitary irreducible representation of G.

Recall that $SU(n, 1)$ denotes the group of isometries with determinant 1 of the hermitian form $x_1\overline{y_1} + \ldots x_n\overline{y_n} - x_{n+1}\overline{y_{n+1}}$ on $\mathbb{C}^{n+1}$, and that $SO_0(n, 1)$ denotes the connected component of identity of $SU(n, 1) \cap GL_{n+1}(\mathbb{R})$. Remember also that, up to a finite covering, $SO_0(n, 1)$ (resp. $SU(n, 1)$) is the group of orientation-preserving isometries of n-dimensional real hyperbolic space $\mathbb{H}^n(\mathbb{R})$ (resp. complex hyperbolic space $\mathbb{H}^n(\mathbb{C})$). Delorme has proved ([Del77], Théorème V.5) that, for $G = SO_0(n, 1)$ ($n \geq 3$), there exists, up to unitary equivalence, a unique irreducible unitary representation of G which is cohomological; while for $G = SU(n,1)$ $(n \geq 1)$ [1], there are exactly two inequivalent unitary irreducible representations of G which are cohomological; they are contragredient of each other.

This note is devoted to the properties of restrictions of irreducible cohomological representations of $SO_0(n,1)$ and $SU(n,1)$, to closed subgroups. Here is the first result.

Theorem 1 *Let G denote either $SO_0(n,1)$ or $SU(n,1)$. Let H be a closed subgroup of G, isomorphic either to $SO_0(m,1)$ or to $SU(m,1)$ for some $m \leq n$. Let $\pi_c \in \widehat{G}$ and $\sigma_c \in \widehat{H}$ be irreducible cohomological representations. Then the restriction $\pi_c|_H$ of π_c to H contains either σ_c or its contragredient $\overline{\sigma_c}$ as a sub-representation.*

In spring 2002, N. Bergeron was working on a vast generalization of Theorem 1 to restrictions of cohomological representations *in any cohomological degree* $\leq \frac{d_H}{2}$ where $d_H = m$ (resp. $d_H = 2m$) if $H = SO_0(m,1)$ (resp. $H = SU(m,1)$): see [Ber03], Theorem 3.4. His proof uses methods completely different from the ones of this paper. During a visit at Neuchtel, he asked me whether Theorem 1 could possibly admit a "soft" proof, based on general principles. After I produced the proof given below (already included in [Ber03] as fact 6.5), Bergeron used it to prove the following Lefschetz-type result ([Ber03], Theorem 6.4). Let X_G denote the Riemannian symmetric space associated to G (so that $X_G = \mathbb{H}^n(\mathbb{R})$ if $G = SO_0(n,1)$, and $X_G = \mathbb{H}^n(\mathbb{C})$ if $G = SU(n,1)$). Suppose that G and H above are given as algebraic $\mathbb{Q}$-groups [2]. Then the stable restriction map

$$\lim_{\Gamma} H^1(\Gamma\backslash X_G) \to \prod_{g\in G(\mathbb{Q})} \lim_{\Gamma} H^1((H(\mathbb{R}) \cap g^{-1}\Gamma g)\backslash X_H)$$

(where the inductive limit is taken over congruence subgroups Γ of $G(\mathbb{Z})$), is injective.

Recall that a representation of a semisimple Lie group S (with finite centre) is said to be *spherical* if it has a non-zero vector fixed under some maximal compact subgroup of S. To motivate our second result, recall another result of Delorme ([Del77], Proposition V.3): an irreducible, cohomological representation of S cannot be spherical.

[1] Remember that $SO_0(2,1)$ is locally isomorphic to $SU(1,1)$.

[2] Assume also here that G is not a $\mathbb{Q}$-isotropic form of $SO(3,1)$.

We will see that the cohomological irreducible representations of $SU(n,1)$ are non-spherical in a very strong sense: roughly speaking, they remain non-spherical after restricting to $SU(m,1)$ $(m < n)$.

Theorem 2 *Set $G = SU(n,1)$, and let π_c be an irreducible cohomological representation of G. Let H be a closed subgroup of G, isomorphic to $SU(m,1)$. Then $\pi_c|_H$ has no non-zero L-invariant vector, where L is a maximal compact subgroup of H.*

Conceivably, it is possible to prove Theorem 2 using the description of π_c in terms of Langlands parameter given in [BW80], 4 of Chapter VI, but we have not pursued this approach. Instead, we appeal to a geometric observation of Gromov [Gro03] on the growth of harmonic equivariant maps $\mathbb{H}^n(\mathbb{C}) \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space endowed with an affine isometric action of $SU(n,1)$. Note that Theorem 2 becomes false upon replacing $SU(n,1)$ by $SO_0(n,1)$, as we show in the final remark.

2 Proof of Theorem 1

We proceed in 3 steps.

• *First step:* we claim that $H^1(H, \pi_c|_H) \neq 0$. This follows immediately from a result of Shalom ([Sha00b], Theorem 3.4) who proved that, for every unitary representation ρ of G and any closed non-compact subgroup H of G, the restriction map $H^1(G,\rho) \to H^1(H,\rho|_H)$ is injective.

• *Second step:* we claim that $\pi_c|_H$ does not almost have invariant vectors. Indeed, let K be a maximal compact subgroup of G; by Theorem 5.4 in Chapter IV of [BW80], there exists some integer $N \in \mathbb{N}$ such that all K-finite matrix coefficients of the tensor power $\pi_c^{\otimes N}$ are in $L^2(G)$. It is known that this implies that $\pi_c^{\otimes N}$ is a subrepresentation of a direct sum of copies of the left regular representation of G (see Corollary 1.2.4 in Chapter V of [HT92]). Restricting to H, we see that $(\pi_c|_H)^{\otimes N}$ is a subrepresentation of a direct sum of copies of the left regular representation of H. Assume that $\pi_c|_H$ almost has invariant vectors. Then the same holds for $(\pi_c|_H)^{\otimes N}$. It follows that the left regular representation of H almost has invariant vectors. This contradicts the fact that H is not amenable.

• *Third step:* A result of Guichardet (Théorème 1 in [Gui72]) says that, for a unitary representation ρ of H without non-zero fixed vector, the space

$B^1(H,\rho)$ of coboundaries is closed in the space $Z^1(H,\rho)$ of cocycles if and only if ρ does not almost have invariant vectors. So, combining the first two steps, we have

$$\overline{H^1}(H,\pi_c|_H) = H^1(H,\pi_c|_H) \neq 0.$$

Now decompose $\pi_c|_H$ into a direct integral of irreducible representations of H :

$$\pi_c|_H = \int_{\widehat{H}} \sigma d\mu(\sigma).$$

Since $\overline{H^1}(H,\pi_c|_H) \neq 0$, there exists a Borel subset B of $\widehat{H}$ with $\mu(B) \neq 0$ such that $H^1(H,\sigma) \neq 0$ for every $\sigma \in B$ (see Proposition 4 in [Gui72]). On the other hand, by the result of Delorme mentioned above, there exists at most two irreducible representations of H with non-zero 1-cohomology. It follows that at least one of these representations, call it σ_c, must be an atom of μ, that is, $\mu\{\sigma_c\} \neq 0$. This means that σ_c is a subrepresentation of $\pi_c|_H$.

3 Proof of Theorem 2

We fix several notations. Let K be a maximal compact subgroup of G. Let $G = KAK$ be a Cartan decomposition of G. We will use the fact that any inclusion of $H = SU(m,1)$ into $G = SU(n,1)$ is induced by an inclusion of $\mathbb{H}^m(\mathbb{C})$ into $\mathbb{H}^n(\mathbb{C})$ as a totally geodesic submanifold (see [Ber03], Proposition 6.3). Therefore we may assume that $L = K \cap H$ and $A \subset H$, so that $H = LAL$ is a Cartan decomposition of H. Since $\dim A = 1$, we have $A = \{\exp tY : t \in \mathbb{R}\}$ for a unit vector Y in the Lie algebra $\mathfrak{a}$ of A. Denote by $\mathcal{H}$ the Hilbert space of π_c.

Let $\widehat{H}_{\mathrm{s}}$ be the spherical dual of H, that is, the set of all spherical irreducible unitary representations of H. Set

$$\widehat{H}_{\mathrm{ns}} = \widehat{H} \setminus \widehat{H}_{\mathrm{s}}.$$

We have a direct integral decomposition

$$\begin{aligned} \pi|_H &= \int_{\widehat{H}} \sigma d\mu(\sigma) = \int_{\widehat{H}_{\mathrm{s}}} \sigma d\mu(\sigma) \oplus \int_{\widehat{H}_{\mathrm{ns}}} \sigma d\mu(\sigma) \\ &= \rho_{\mathrm{s}} \oplus \rho_{\mathrm{ns}}. \end{aligned}$$

If $b \in Z^1(G,\pi_c)$, set

$$\beta(Y) = \frac{d}{dt} b(\exp tY)|_{t=0}.$$

Write

$$b = b_{\mathrm{s}} \oplus b_{\mathrm{ns}} \quad and \quad \beta(Y) = \beta(Y)_{\mathrm{s}} \oplus \beta(Y)_{\mathrm{ns}}$$

in the decomposition $\pi|_H = \rho_{\mathrm{s}} \oplus \rho_{\mathrm{ns}}$.

Let $\mathcal{H}_{\mathrm{s}}, \mathcal{H}_{\mathrm{ns}}$ be the subspaces defined by the representations $\rho_{\mathrm{s}}, \rho_{\mathrm{ns}}$ respectively. We have to show that the subspace $\mathcal{H}_{\mathrm{s}}$ is zero. We start with a weaker statement:

Lemma 1 *Let $b \in Z^1(G, \pi_c)$ be a cocycle which is not a coboundary and such that $b|_K = 0$. Then $\beta(Y)_{\mathrm{s}} = 0$*

Proof of the lemma: Endow the Hilbert space $\mathcal{H}$ with the affine isometric action of G given by

$$\alpha(g)v = \pi_c(g)v + b(g)$$

($v \in \mathcal{H}$, $g \in G$). Then the map $b : G \to \mathcal{H}$ factors through a G-equivariant mapping $F : G/K \simeq \mathbb{H}^n(\mathbb{C}) \to \mathcal{H}$ with $F(x_0) = 0$, where $x_0 = K$. By an unpublished result of Shalom (for a proof, see either the preprint version of [Sha00a], or Proposition 3.3.15 in [BdlHV]), the map F is harmonic (in the sense that $\Delta F = 0$, where Δ is the Laplace operator on $\mathbb{H}^n(\mathbb{C})$).

Using the irreducibility of the isotropy representation of K on the tangent space $T_{x_0}(G/K)$, it is easy to see that there exists $\lambda > 0$ such that $\lambda \|dF_x(Z)\| = \|Z\|$ (for every $x \in G/K$, $Z \in T_x(G/K)$; for details, see Proposition 3.3.17 in [BdlHV]). So, replacing b by λb and F by λF, we may assume that F is a local isometry. In particular $\|\beta(Y)\| = \|dF_{x_0}(Y)\| = \|Y\| = 1$.

By the second step in the proof of Theorem 1, $\pi_c|_H$ and hence ρ_{s}, do not almost have invariant vectors. On the other hand, spherical representations have trivial cohomology, as was already mentioned. Therefore,

$$H^1(H, \rho_{\mathrm{s}}) = \overline{H^1}(H, \rho_{\mathrm{s}}) = 0.$$

Since $H^1(H, \pi|_H) \neq 0$, it follows that b_{ns} is not a coboundary, hence $\beta(Y)_{\mathrm{ns}} \neq 0$.

For $x \in \mathbb{H}^m(\mathbb{C})$, define

$$F_{\mathrm{ns}}(x) = \frac{b_{\mathrm{ns}}(h)}{\|\beta(Y)_{\mathrm{ns}}\|},$$

where $h \in H$ is such that $hx_0 = x$. The mapping $F_{\mathrm{ns}} : \mathbb{H}^m(\mathbb{C}) \to \mathcal{H}_{\mathrm{ns}}$ is well-defined, since $b|_L = 0$. Moreover, F_{ns} is H-equivariant with respect to the affine action of H on $\mathcal{H}_{\mathrm{ns}}$:

$$\alpha_{\mathrm{ns}}(h)\xi = \rho_{\mathrm{ns}}(h)\xi + \frac{b_{\mathrm{ns}}(h)}{\|\beta(Y)_{\mathrm{ns}}\|}, \qquad h \in H,\ \xi \in \mathcal{H}_{\mathrm{ns}};$$

F_{ns} satisfies $F_{\mathrm{ns}}(x_0) = 0$, and it is a local isometry, since $(dF_{\mathrm{ns}})_{x_0}(Y) = \beta(Y)_{\mathrm{ns}}/\|\beta(Y)_{\mathrm{ns}}\|$.

Claim: F_{ns} is harmonic. Indeed, by the computation in the proof of Lemma 3.3.20 in [BdlHV], we have

$$\Delta F_{\mathrm{ns}}(x) = -\dim \mathbb{H}^m(\mathbb{C}) \int_L \rho_{\mathrm{ns}}(h^{-1}k)\rho_{\mathrm{ns}}(Y)\frac{\beta(Y)_{\mathrm{ns}}}{\|\beta(Y)_{\mathrm{ns}}\|}dk,$$

for $x \in \mathbb{H}^m(\mathbb{C})$ and $h \in H$ such that $hx_0 = x$ (here dk denotes normalized Haar measure on the compact group L). This integral is zero, since ρ_{ns} has no non-zero L-invariant vectors. This proves the claim.

We have

$$\|F(\exp tYx_0)\|^2 = \|b_{\mathrm{s}}(\exp tY)\|^2 + \|\beta(Y)_{\mathrm{ns}}\|^2\|F_{\mathrm{ns}}(\exp tYx_0)\|^2, \qquad t \in \mathbb{R}.$$

Since, as seen above, $H^1(H, \rho_{\mathrm{s}}) = 0$, the function

$$t \mapsto \|b_{\mathrm{s}}(\exp tY)\|^2$$

is bounded. On the other hand, it is an observation of Gromov (Example (b) on p. 111 in [Gro03]; see also Proposition 3.3.21 in [BdlHV]) that the growth rate of a harmonic, locally isometric, equivariant mapping on $\mathbb{H}^n(\mathbb{C})$ is independent of n. Hence, by the Claim, F and F_{ns} have the same growth rate:

$$\|F(\exp tYx_0)\|^2 = 2t + o(t) = \|F_{\mathrm{ns}}(\exp tYx_0)\|^2 \ \ as \ \ t \to \infty.$$

This implies $\|\beta(Y)_{\mathrm{ns}}\|^2 = 1$, that is, $\beta(Y)_{\mathrm{s}} = 0$. This concludes the proof of the Lemma.

Proof of Theorem 2: Assume, by contradiction, that $\mathcal{H}_{\mathrm{s}} \neq 0$, and let ξ be a unit vector in $\mathcal{H}_{\mathrm{s}}$.

Claim: $\rho_{\mathrm{s}}(Y)\xi \neq 0$. Indeed, otherwise, $\rho_{\mathrm{s}}(\exp tY)\xi = \xi$ for every $t \in \mathbb{R}$. Since $H = LAL$, it would follow that ξ is $\rho_{\mathrm{s}}(H)$-fixed. Hence, the matrix coefficient $g \mapsto \langle \pi_c(g)\xi|\xi\rangle$ would be 1 on the non-compact closed subgroup H, and this would contradict the Howe-Moore theorem [HM79] on the vanishing of coefficients at infinity of G. This establishes the claim.

Let $b \in Z^1(G, \pi_c)$ be a cocycle which is not a coboundary. Since $H^1(K, \pi_c|_K) = 0$, up to adding a coboundary, we may assume that $b|_K = 0$. By the lemma: $\beta(Y)_{\mathrm{s}} = 0$. Replace now the cocycle b by the cohomologous cocycle

$$b' : g \mapsto b(g) + \pi_c(g)\xi - \xi.$$

For the corresponding vector

$$\beta(Y)' = \frac{d}{dt} b'(\exp tY)|_{t=0},$$

we have $\beta(Y)' = \beta(Y) + \rho_{\mathrm{s}}(Y)\xi$. The Lemma, applied now to $\beta(Y)'$, shows that $\beta(Y)'_{\mathrm{s}} = 0$. This is a contradiction, since $\beta(Y)'_{\mathrm{s}} = \rho_{\mathrm{s}}(Y)\xi \neq 0$. This concludes the proof of Theorem 2.

Remark: We conclude by explaining why Theorem 2 fails when replacing $SU(n,1)$ by $SO_0(n,1)$. Indeed, set $G = SO_0(3,1)$ and $H = SO_0(2,1)$.

Let $G = KAN$ be the Iwasawa decomposition of G (with $K = SO(3)$), and let $P = MAN$ be the standard minimal parabolic subgroup of G (with $M = SO(2)$); write an element of M as $r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, and define a character $\chi : M \to U(1)$ by $\chi(r_\theta) = e^{i\theta}$; extend χ to a character $\tilde{\chi}$ of P by $\tilde{\chi}(r_\theta an) = \chi(r_\theta)$.

It is known (see [Del77], Proposition V.6) that the unique irreducible cohomological representation of G is the principal series representation $\pi_c = Ind_P^G\tilde{\chi}$. The K-types are easily determined: indeed it is well-known (see e.g. [Lip74], Example (4) on p. 48) that $\pi_c|_K$ is unitarily equivalent to $Ind_M^K\chi$. Let σ_n denote the unique irreducible representation of $K = SO(3)$ in degree $2n+1$. By Frobenius reciprocity, σ_n appears in $\pi_c|_K$ if and only if χ appears in $\sigma_n|_M$, and this happens exactly for $n \geq 1$.

Let us now restrict to H, whose maximal compact subgroup is $L \simeq SO(2)$. Since, for every $n \geq 1$, the restriction $\sigma_n|_L$ has non-zero fixed vectors, we see that $\pi_c|_L$ has an infinite-dimensional subspace of fixed vectors.

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Author's address:

Institut de Mathématiques - Université de Neuchâtel
Rue Emile Argand 11
CH-2007 Neuchâtel - SWITZERLAND
alain.valette@unine.ch