CONFORMAL UPPER BOUNDS FOR THE EIGENVALUES OF THE LAPLACIAN AND STEKLOV PROBLEM

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ABSTRACT. In this paper, we find upper bounds for the eigenvalues of the Laplacian in the conformal class of a compact Riemannian manifold (M, g). These upper bounds depend only on the dimension and a conformal invariant that we call "min-conformal volume". Asymptotically, these bounds are consistent with the Weyl law and improve previous results by Korevaar and Yang and Yau. The proof relies on the construction of a suitable family of disjoint domains providing supports for a family of test functions. This method is interesting for itself and powerful. As a further application of the method we obtain an upper bound for the eigenvalues of the Steklov problem in a domain with C^1 boundary in a complete Riemannian manifold in terms of the isoperimetric ratio of the domain and the conformal invariant that we introduce.

1. INTRODUCTION

Let (M, g) be a compact orientable m-dimensional Riemannian manifold. It is well known that the spectrum of the Laplace operator acting on functions is discrete and consists of a nondecreasing sequence $\{\lambda_k(M, g)\}_{k=1}^{\infty}$ of eigenvalues each occurring with finite multiplicity. If M has a smooth boundary then the same conclusion is valid for Dirichlet, Neumann or other reasonable boundary conditions. By Weyl's law, the asymptotic behavior of λ_k is given by (see e.g. [Be])

$$\lambda_k(M, g) \sim \alpha_m \left(\frac{k}{\mu_g(M)}\right)^{\frac{2}{m}}, \quad k \to \infty \tag{1}$$

where μ_g is the Riemannian measure associated with g, $\alpha_m = 4\pi^2 \omega_m^{-\frac{2}{m}}$ and ω_m is the volume of the unit ball in the standard $\mathbb{R}^m$.

A natural question suggested by this asymptotic formula is the following

Question 1. *Does there exists a constant C_m depending only on the dimension m such that we have*

$$\lambda_k(M, g)\mu_g(M)^{\frac{2}{m}} \leq C_m k^{\frac{2}{m}} \tag{2}$$

for every $k \in \mathbb{N}^*$?

An abundant literature has been devoted to this issue starting with Urakawa's paper [Ur]. It turns out that $\lambda_2(M, g)\mu_g(M)^{\frac{2}{m}}$ cannot be bounded above only in terms of m (see for example [BM], [CD], [CE], [L]). Consequently, the answer to Question 1 is negative.

In the particular case of the first positive eigenvalue, El Soufi and Ilias [EI] (see also [FN]) showed that an inequality like (2) holds with a constant $C_m([g])$ that depends on the conformal class $[g]$ of the metric g (namely, the conformal volume introduced by Li and Yau [LY] who proved the same but in dimension 2). In the case of surfaces, Yang and Yau [YY] proved inequality (2) with a constant that only depends on the genus of the surface. In 1993, Korevaar [Ko] generalized these results to higher order eigenvalues. More precisely, Korevaar obtained the following upper bounds:
(i) If (M^m, g) is a compact Riemannian manifold of dimension m, then for every $k \in N^*$,

$$\lambda_k(M, g)\mu_g(M)^{\frac{2}{m}} \leq c_m([g])k^{\frac{2}{m}}, \tag{3}$$

where $c_m([g])$ is a constant depending only on the conformal class $[g]$ of the metric g.
(ii) If (Σ_γ, g) is a compact orientable surface of genus γ, then for every $k \in N^*$,

$$\lambda_k(\Sigma_\gamma, g)\mu_g(\Sigma_\gamma) \leq C(\gamma + 1)k, \tag{4}$$

where C is a universal constant.

Notice that inequality (4) provides an affirmative answer to Yau's conjecture [Ya, page 19]. Korevaar's results have been discussed by Gromov [Gr] and revisited by Grigor'yan and Yau [GY] and Grigor'yan, Netrusov and Yau [GNY] who proposed different proofs.

Another important result in this direction was obtained by Buser [Bu] who proved that if (M^m, g) is a compact m-dimensional Riemannian manifold whose Ricci curvature satisfies $\mathrm{Ricci}_g \geq -(m-1)a^2$, then for every $k \in \mathbb{N}^*$,

$$\lambda_k(M, g) \leq \frac{(m-1)^2}{4}a^2 + \beta_m \left(\frac{k}{\mu_g(M)}\right)^{2/m}, \tag{5}$$

where β_m is a constant depending only on m.
Colbois and Maerten ([CM] Thm 1.3) proved a similar result for bounded domains in a complete manifold under Neumann boundary conditions.

In the same vein of the results of Korevaar and Buser, our aim in the present work is to understand how inequality (2) can be modified into a valid one. We obtain results that generalize those of Korevaar, Buser, and Colbois and Maerten mentioned above. The main feature of our approach is that the modification we propose consists in adding a term (depending on the conformal class $[g]$ or the genus γ) to the right hand side of (2), instead of letting the constant C_m depend on $[g]$ or γ as in Korevaar's inequalities (3) and (4). The principal advantage of our approach lies in the fact that it enables us to recover the inequality (2) for any integer k that exceeds a threshold depending only on $[g]$ or γ (see Corollary 1.3 below).

In order to state our main result we need to introduce the following conformal invariant. If (M,g) is a compact Riemannian manifold of dimension m, we define its **min-conformal** volume as follows :

$$V([g]) = \inf\{\mu_{g_0}(M) \ : \ g_0 \in [g],\ \mathrm{Ricci}_{g_0} \geq -(m-1)\}.$$

Denoting by $\rho^-(g)$ the smallest number $a \geq 0$ such that $\mathrm{Ricci}_g \geq -(m-1)a^2$, one can easily check that

$$\begin{aligned} V([g]) &= \inf\{\mu_{g'}(M)\rho^-(g')^{\frac{m}{2}} \ : \ g' \in [g]\} \\ &= \inf\{\rho^-(g')^{\frac{m}{2}} \ : \ g' \in [g]\,,\, \mu_{g'}(M) = 1\}. \end{aligned} \tag{6}$$

Theorem 1.1. *There exist, for each integer $m \geq 2$, two constants A_m and B_m such that, for every compact Riemannian manifold (M,g) of dimension m and every $k \in N^*$, we have*

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \leq A_m V([g])^{\frac{2}{m}} + B_m k^{\frac{2}{m}}. \tag{7}$$

It is important to notice that the constant B_m in inequality (7) cannot be equal to the constant α_m in the Weyl law. Indeed, it follows from [CE, Corollary 1] that such a B_m must satisfy : $B_m \geq m\omega_m^{\frac{2}{m}}$. On the other hand, inequality (7) also gives an upper bound on the conformal spectrum introduced by Colbois and El Soufi [CE] and shows that its asymptotic behavior obeys a Weyl type law.

Now, if a metric g is conformally equivalent to a metric g_0 with $\mathrm{Ricci}_{g_0} \geq 0$, then $V([g]) = 0$ (see equality (6)). This leads to the following

Corollary 1.1. *(see* [Ko]*) If a compact Riemannian manifold (M,g) of dimension $m \geq 2$ is conformally equivalent to a Riemannian manifold with nonnegative Ricci curvature, then*

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \leq B_m k^{\frac{2}{m}}, \tag{8}$$

where B_m is a constant depending only on m.

In the case of a compact orientable surface Σ_γ of genus γ, the uniformization theorem tells us that any Riemannian metric g on Σ_γ is conformally equivalent to a metric of constant curvature. If $\gamma \geq 2$, then g is conformally equivalent to a hyperbolic metric g_γ. Thus, $V([g]) \leq \mu_{g_\gamma}(\Sigma_\gamma) = 4\pi(\gamma - 1)$, where the last equality follows from Gauss-Bonnet Theorem. If $\gamma = 0, 1$, then g is conformally equivalent to a positive constant curvature metric or a flat metric, respectively. Thus, $V([g]) = 0$ in the last two cases. Substituting in (7), one obtains the following improvement of Korevaar's inequality (4).

Corollary 1.2. *There exist two constants A and B such that, for every compact Riemannian surface (Σ_γ, g) of genus γ and every $k \in N^*$, we have*

$$\lambda_k(\Sigma_\gamma, g)\mu_g(\Sigma_\gamma) \leq A\gamma + Bk. \tag{9}$$

This result gives an upper bound to the topological spectrum introduced by Colbois and El Soufi [CE] and can be compared with the lower bound they obtained [CE, p. 341].

In relation with Question 1, we have the following corollary which is a direct consequence of inequalities (7) and (9).

Corollary 1.3. *There exist a constant $B' \in \mathbb{R}$ and, for each $m \geq 2$, a constant $B'_m \in \mathbb{R}$ such that the following properties hold.*
(1) For any compact Riemannian manifold (M, g) of dimension $m \geq 2$, there exists an integer $k_0([g])$ depending only on the conformal class of g, such that, for every $k \geq k_0([g])$,

$$\lambda_k(M, g)\mu_g(M)^{\frac{2}{m}} \leq B'_m \; k^{\frac{2}{m}};$$

(2) For any compact Riemannian surface (Σ_γ, g) of genus γ, there exists an integer $k_0(\gamma)$ depending only on γ, such that, for every $k \geq k_0(\gamma)$,

$$\lambda_k(\Sigma_\gamma, g)\mu_g(\Sigma_\gamma) \leq B'k.$$

For any relatively compact domain Ω with C^1 boundary in a Riemannian manifold (M, g), we denote by $\{\lambda_k(\Omega, g)\}_{k\geq 1}$ the nondecreasing sequence of eigenvalues of the Neumann realization of the Laplacian in Ω. The method we will use to prove Theorem 1.1 also allows us to obtain the following

Theorem 1.2. *Let (M, g_0) be a complete Riemannian manifold of dimension $m \geq 2$ with $\mathrm{Ricci}_{g_0}(M) \geq -(m-1)$. Let $\Omega \subset M$ be a relatively compact domain with C^1 boundary and g be any metric conformal to g_0. Then for every $k \in \mathbb{N}^*$, we have*

$$\lambda_k(\Omega, g)\mu_g(\Omega)^{\frac{2}{m}} \leq A'_m \mu_{g_0}(\Omega)^{\frac{2}{m}} + B'_m k^{\frac{2}{m}}, \tag{10}$$

where A'_m and B'_m are constants depending only on the dimension m.

It is easy to see that we can derive from Theorem 1.1 and Theorem 1.2, inequalities of type (5) as obtained by Buser [Bu] and Colbois and Maerten [CM] but with different constants.

The paper is organized as follows: In section 2 we introduce the main technical tool of the proof which consists in the construction of a suitable family of capacitors, using the methods of [GNY] and [CM]. The proofs of Theorem 1.1 and Theorem 1.2 are given in section 3. The last section is devoted to the Steklov eigenvalue problem. We prove that our method applies to this problem and give some upper bounds for the Steklov eigenvalues.

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2. Construction of families of capacitors in an m-m Space

In this section, we present the main technical tool of this paper. Let us start by recalling some definitions. Throughout this section, the notation (X, d, μ) will designate a complete and locally compact metric-measure space ($m - m$ *space*) with a metric d and a non-atomic finite Borel measure μ. Each pair (F, G) of Borel sets in X such that $F \subset G$ is called a *capacitor*.

Definition 2.1. *Given $\kappa > 1$ and $N \in \mathbb{N}^*$, we say that a metric space (X, d) satisfies the (κ, N)-covering property if each ball of radius $r > 0$ can be covered by N balls of radius $\frac{r}{\kappa}$.*

Similarly we define a local version of the covering property as follows:

Definition 2.2. *Given $\kappa > 1$, $\rho > 0$ and $N \in \mathbb{N}^*$, we say that a metric space (X, d) satisfies the $(\kappa, N; \rho)$-covering property if each ball of radius $0 < r \leq \rho$ can be covered by N balls of radius $\frac{r}{\kappa}$.*

Lemma 2.1. *If a metric space (X, d) satisfies the $(\kappa, N; \rho)$-covering property (the (κ, N)-covering property), then for any $\lambda > 1$, it satisfies the $(\lambda, K; \rho)$-covering property (the (λ, K)-covering property) for some $K = K(\lambda, \kappa, N)$ that does not depend on ρ.*

The proof of the lemma when (X, d) satisfies the (κ, N)-covering property is given in [GNY, Lemma 3.4]. For the $(\kappa, N; \rho)$-covering property, the same proof applies here verbatim.

Definition 2.3. *For any $x \in X$ and $0 \leq r \leq R$, we define the annulus $A(x, r, R)$ as*

$$A(x, r, R) := B(x, R) \setminus B(x, r) = \{y \in X : r \leq d(x, y) < R\}.$$

For any annulus $A(x, r, R)$ and $\lambda \geq 1$, set $\lambda A := A(x, \lambda^{-1} r, \lambda R)$. Similarly, for any ball $B = B(x, r)$ we set $\lambda B := B(x, \lambda r)$. If $F \subseteq X$ and $r > 0$, we denote the r-neighborhood of F by F^r, that is

$$F^r = \{x \in X : d(x, F) \leq r\}.$$

In the following lemmas we recall two methods for metric construction of disjoint domains.

Lemma 2.2. [GNY, Corollary 3.12] *Let (X, d, μ) be an $m - m$ space satisfying the $(2, N)$-covering property. Then for every $n \in \mathbb{N}^*$, there exists a family $\mathcal{A} = \{(A_i, B_i)\}_{i=1}^n$ of capacitors in X such that*

(a) For each i, A_i is an annulus and $\mu(A_i) \geq \frac{\mu(X)}{cn}$,
(b) $\{B_i\}_{i=1}^n$ are mutually disjoint and $B_i = 2A_i$,

where c is a positive constant depending only on N (in fact one can take $c = 2 + 4K(1600, 2, N)$, where K is the function defined in Lemma 2.1).

Lemma 2.3. ([CM, Corollary 2.3] and [CEG1, Lemma 2.1]) *Let (X, d, μ) be an $m - m$ space satisfying the $(2, N; 1)$-covering property. For every $n \in \mathbb{N}^*$, let $r > 0$ be such that for each $x \in X$, $\mu(B(x, r)) \leq \frac{\mu(X)}{4\tilde{N}^2 n}$, where $\tilde{N} = K(4, 2, N)$. Then there exists a family $\mathcal{A} = \{(A_i, A_i^r)\}_{i=1}^n$ of capacitors in X such that*
(a) for each i, $\mu(A_i) \geq \frac{\mu(X)}{2\tilde{N}n}$, and
(b) the subsets $\{A_i^r\}_{i=1}^n$ are mutually disjoint.

In the original statement of Lemma 2.3, (X, d) is supposed to have the $(4, \tilde{N}; 1)$-covering property. According to Lemma 2.1, one can replace the $(4, \tilde{N}; 1)$-covering property by the $(2, N; 1)$-covering property.
The main construction given in the following theorem results from a merging of the two previous lemmas. It consists in constructing a disjoint family of capacitors.

Theorem 2.1. *Let (X, d, μ) be an $m - m$ space satisfying the $(2, N; 1)$-covering property. Then for every $n \in \mathbb{N}^*$, there exists a family of capacitors $\mathcal{A} = \{(F_i, G_i)\}_{i=1}^n$ with the following properties:*

- *(i) $\mu(F_i) \geq \nu := \frac{\mu(X)}{8c^2 n}$, where c is as in Lemma 2.2 ;*
- *(ii) the G_i's are mutually disjoint ;*
- *(iii) the family $\mathcal{A}$ is such that either*
  - (a) *all the F_i's are annuli and $G_i = 2F_i$, with outer radii smaller than one, or*
  - (b) *all the F_i's are domains in X and $G_i = F_i^{r_0}$, with $r_0 = \frac{1}{1600}$.*

Proof of Theorem 2.1. In order to find a desired family of capacitors, we start with the method used by Grigor'yan, Netrusov and Yau [GNY, proof of Theorem 3.5]. We will call their method *GNY-construction*. However we do not have the $(2, N)$-covering property in order to apply directly the GNY-construction. Roughly speaking, we will see that when an $m - m$ space X has the local covering property (i.e. $(2, N; 1)$-covering property), the GNY-construction is applicable to the "*massive part*" of X (i.e. where balls of radii r_0 have measure greater than ν). If the number of capacitors built using the GNY-construction on the massive part is not equal to n, then we introduce a new measure on X. The support of this new measure is a subset of the complement of the massive part. We shall see that in this case the method of Colbois and Maerten (Lemma 2.3) that we will call *CM-construction*, is applicable.
Let us define

$$\tau_1 := \sup\{r : \mu(B(x, r)) \leq \nu \quad \forall x \in X\}.$$

If $\tau_1 \leq r_0$ then we follow the step 1 (see below). Otherwise we move on to the step 2 in order to apply the CM-construction.

Step 1. Applying GNY-construction. Assume $\tau_1 \leq r_0$.
We essentially follow the steps of the GNY-construction. However, it is necessary to make some adaptations since our covering property is of local nature. We use the same formalism and notations that is used in the GNY-construction (see [GNY, page 172]). Our goal is to construct by induction two sequences $\{\mathcal{A}_i\}$ and $\{\mathcal{B}_i\}$ where $\mathcal{A}_i$ is a family of annuli in X, and $\mathcal{B}_i$ is a family of balls that cover $\mathcal{A}_i$. These two families satisfy the following properties :

(i) for each $a \in \mathcal{A}_i$ we have

$$\mu(a) \geq \nu \ ;$$

(ii) the annuli $\{2a\}_{a\in\mathcal{A}_i}$ are disjoint ;
(iii) for each $a \in \mathcal{A}_i$, the outer radius of a is smaller than one ;
(iv) the following inclusions hold

$$\bigcup_{a\in\mathcal{A}_i} 2a \subset \bigcup_{b\in\mathcal{B}_i} \frac{1}{4} b \ ;$$

(v) we have the inequality

$$\mu(\bigcup_{b\in\mathcal{B}_i} b) \leq c\nu i \ ;$$

(vi) $|\mathcal{A}_1| = |\mathcal{B}_1| = 1$ and if $i > 1$ then
- either $|\mathcal{A}_i| = |\mathcal{A}_{i-1}| + 1$ and $|\mathcal{B}_i| \leq |\mathcal{B}_{i-1}| + 1$,
- or $|\mathcal{A}_i| = |\mathcal{A}_{i-1}|$ and $|\mathcal{B}_i| \leq |\mathcal{B}_{i-1}| - 1$,

where $|\mathcal{A}|$ denote the cardinal of the family $\mathcal{A}$;
(vii) if $i > 1$, then $\mathcal{A}_{i-1} \subseteq \mathcal{A}_i$;
$(viii)$ if $i > 1$, then $\bigcup_{b\in\mathcal{B}_{i-1}} b \subseteq \bigcup_{b\in\mathcal{B}_i} b$.

Observe that by (vi) the sequence of $\{2|\mathcal{A}_i| - |\mathcal{B}_i|\}$ is strictly increasing with respect to i and, since $2|\mathcal{A}_1| - |\mathcal{B}_1| = 1$, one has

$$2|\mathcal{A}_i| \geq i.$$

Notice that if we can continue the inductive process till $i = 2n$, then we get a family $\mathcal{A} = \mathcal{A}_{2n}$ of at least n capacitors satisfying the desired properties $(i), (ii)$ and $(iii)(a)$ of Theorem 2.1. However here we only have a local covering property rather than a global one. In order to perform the induction, we will need to fix an upper bound on the radii of balls in $\mathcal{B}_i$ (this restriction is crucial to have property (v)). This restriction does not always allow us to continue the inductive process till $i = 2n$.

To start the induction, take $r \in (\tau_1, 2\tau_1]$. Then there exists a point $x_0 \in X$ such that

$$\mu(B(x_0, r)) \geq \nu.$$

We define $\mathcal{A}_1 = \{B(x_0, r)\}$ and $\mathcal{B}_1 = \{B(x_0, 8r)\}$. It is easy to see that properties $(i), (ii), (iii), (iv), (vi), (vii)$ and $(viii)$ are satisfied. Let us verify property (v). Since $8r \le 16\tau_1 < 1$, by Lemma 2.1, one can cover $B(x_0, 8r)$ by $K(16, 2, N)$ balls of radii $r/2 < \tau_1$. Therefore,

$$\mu(B(x_0, 8r)) \le K(16, 2, N)\nu < c\nu,$$

which proves the property (v).
Assume now we have constructed $\mathcal{A}_1, \dots, \mathcal{A}_i$ and $\mathcal{B}_1, \dots, \mathcal{B}_i$ for some $i < 2n$. It follows from the property (iv) for the family $\mathcal{B}_i$ that

$$\begin{aligned} \mu(X \setminus \bigcup_{b \in \mathcal{B}_i} b) &> \mu(X) - ic\nu > \mu(X) - 2nc\nu = \mu(X) - 2nc\frac{\mu(X)}{8c^2 n} \\ &= (1 - \frac{1}{4c})\mu(X) > \frac{\mu(X)}{2} > \nu, \end{aligned} \tag{11}$$

because $c > 1$. Hence, there exists $x_i \in X$ such that

$$\mu\big(B(x_i, r) \setminus \bigcup_{b \in \mathcal{B}_i} b\big) > \nu. \tag{12}$$

We define

$$\tau_{i+1} := \sup\{r : \mu\big(B(x, r) \setminus \bigcup_{b \in \mathcal{B}_i} b\big) \le \nu \quad \forall x \in X\}.$$

At this stage the continuation of the construction process depends on the size of τ_{i+1}.

- If $\tau_{i+1} > r_0$, we move on to the step 2.
- If $\tau_{i+1} \le r_0$, we construct families $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ as follows.

We can assume that $r \in (\tau_{i+1}, 2\tau_{i+1}]$ in (12). We denote κ the cardinal of:

$$B := \{b \in \mathcal{B}_i : \ B(x_i, 7 \times 4r) \cap \frac{1}{2}b \neq \emptyset\}.$$

Following the GNY-construction, we define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ according to the following alternatives (for more details see [GNY, pp. 174–178]):

Case $\kappa = 0$: We define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ by

$$\mathcal{A}_{i+1} = \mathcal{A}_i \cup \{B(x_i, r)\}, \quad \text{and} \quad \mathcal{B}_{i+1} = \mathcal{B}_i \cup \{B(x_i, 8r)\}.$$

Case $\kappa \ge 2$: We define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ by

$$\mathcal{A}_{i+1} = \mathcal{A}_i, \quad \text{and} \quad \mathcal{B}_{i+1} = (\mathcal{B}_i \setminus \{\text{all balls in the set } B\}) \cup \{B(x_i, 98 \times 8r)\}.$$

Note that the ball $B(x_i, 98 \times 8r)$ contains all balls in B (see [GNY, p. 175]).

Case $\kappa = 1$: If there exists a ball $b = B(y, s) \in B$ such that

$$B(x_i, 2r) \cap \frac{1}{2}b \neq \emptyset,$$

then we define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ by

$$\mathcal{A}_{i+1} = \mathcal{A}_i \cup \{A(y, \frac{1}{2}s, 8r)\} \quad \text{and} \quad \mathcal{B}_{i+1} = \mathcal{B}_i \cup \{B(x_i, 14 \times 8r)\}.$$

Notice that $A(y, \frac{1}{2}s, 8r) \subset B(x_i, 14 \times 8r)$ (see [GNY, p. 177]).
Otherwise we define $\mathcal{A}_{i+1}$ and $\mathcal{B}_{i+1}$ like in the case $\kappa = 0$.

Now let us prove that these two families have the properties $(i) - (viii)$. The properties $(vi), (vii)$ and $(viii)$ are clearly satisfied in each of the three cases. To check the conditions (i), (ii) and (iv), we can use word-for-word the arguments given in [GNY, pp. 173–178]. Indeed, this part of their proof is independent of covering properties.

Let us verify the condition (v). In each of the three cases, we see that $|\mathcal{B}_{i+1} \setminus \mathcal{B}_i| = 1$. Let us denote by b_{i+1} the unique ball in $\mathcal{B}_{i+1} \setminus \mathcal{B}_i$. According to the three cases, the radius r_{i+1} of b_{i+1} is at most $98 \times 8r$. Since $r \in (\tau_{i+1}, 2\tau_{i+1}]$, we have

$$r_{i+1} \leq 98 \times 8 \times 2\tau_{i+1} < 1600\tau_{i+1} \leq 1, \tag{13}$$

where the last inequality follows from the assumption $\tau_{i+1} \leq r_0$. By Lemma 2.1, the ball b_{i+1} can be covered by $K(1600, 2, N) < c$ balls with radii $\frac{r_{i+1}}{1600} \leq \tau_{i+1}$. Therefore

$$\begin{aligned} \mu\Big(\bigcup_{b \in \mathcal{B}_{i+1}} b\Big) &= \mu\Big(\bigcup_{b \in \mathcal{B}_i} b\Big) + \mu\Big(b_{i+1} \setminus \bigcup_{b \in \mathcal{B}_i} b\Big) \leq c\nu i + \mu\Big(b_{i+1} \setminus \bigcup_{b \in \mathcal{B}_{i+1}} b\Big) \\ &\leq c\nu i + K(1600, 2, N)\nu \leq c\nu i + c\nu \leq c\nu(i+1), \end{aligned}$$

which proves the condition (v).
It remains to check the condition (iii). For this, it is enough to verify that the outer radius of the annulus $a \in \mathcal{A}_{i+1} \setminus \mathcal{A}_i$ is smaller that one. One can see in each of the three cases, $\mathcal{A}_{i+1} \setminus \mathcal{A}_i \subset \mathcal{B}_{i+1} \setminus \mathcal{B}_i = \{b_{i+1}\}$. By inequality (13), the radius of b_{i+1} is smaller than one and proves the condition (iii) for $\mathcal{A}_{i+1}$.

Step 2. Applying CM-construction. Assume $\tau_i > r_0$ for some $1 \leq i \leq 2n$. It means that

- if $i = 1$, then $\mu(B(x, r_0)) \leq \nu$, for all $x \in X$;
- if $1 < i \leq 2n$, then $\mu(B(x, r_0) \setminus \bigcup_{b \in \mathcal{B}_{i-1}} b) \leq \nu$, for all $x \in X$.

We consider the $m - m$ space $(X, d, \tilde{\mu}_i)$ where

- $\tilde{\mu}_i := \mu$ if $i = 1$;
- $\tilde{\mu}_i(A) := \mu(A \setminus \bigcup_{b \in \mathcal{B}_{i-1}} b)$ if $1 < i \leq 2n$.

It follows from inequality (11) and the above inequalities that

$$\tilde{\mu}_i(X) > \frac{\mu(X)}{2},$$

and

$$\tilde{\mu}_i(B(x, r_0)) \leq \frac{\mu(X)}{8c^2 n} \leq \frac{\tilde{\mu}_i(X)}{4\tilde{N}^2 n}.$$

Consequently, that the $m - m$ space $(X, d, \tilde{\mu}_i)$ satisfies the assumptions of Lemma 2.3. Therefore, there exists a family $\{(A_j, A_j^{r_0})\}$ of n capacitors in

X such that the $A_j^{r_0}$'s are mutually disjoint and

$$\tilde{\mu}_i(A_j) \geq \frac{\tilde{\mu}_i(X)}{2\tilde{N}n} \geq \frac{\mu(X)}{8c^2 n}.$$

Since $\mu(A_j) \geq \tilde{\mu}_i(A_j)$, this family of capacitors satisfies the conditions $(i), (ii)$ and $(iii)(b)$ of Theorem 2.1. □

The following proposition shows that for a sufficiently large integer n, it is always possible to apply the GNY-construction to obtain a family of n capacitors satisfying the properties $(i), (ii)$ and $(iii)(a)$ of Theorem 2.1. The application of this observation to the eigenvalue problem is discussed in Remark 3.2 of the next section.

Proposition 2.1. *Let (X, d, μ) be a compact $m-m$ space satisfying the $(2, N; 1)$-covering property. Then there exists a positive integer k_X such that for every $n > k_X$, there exists a family $\mathcal{A}$ of n mutually disjoint capacitors in X that satisfies the properties $(i), (ii)$ and $(iii)(a)$ of Theorem 2.1.*

Proof. Since X is compact, we can cover X by T balls of radii $r_0 = \frac{1}{1600}$. Set

$$k_X = \frac{T}{4c^2}.$$

It is enough to show that for every $n > k_X$ and $1 \leq i \leq 2n$, we have $\tau_i \leq r_0$. Indeed, suppose that there exists an integer $j \leq 2n$ such that $\tau_j > r_0$. Then by the definition of τ_j, we have the following inequality

$$\tilde{\mu}_j(B(x, r_0)) \leq \nu = \frac{\mu(X)}{8c^2 n}. \tag{14}$$

It follows from the above inequality that

$$\frac{\mu(X)}{2} \leq \tilde{\mu}_j(X) \leq \mu\left(\bigcup_{x_i \in X, 1 \leq i \leq T} B(x_i, r_0)\right) \leq T\frac{\mu(X)}{8c^2 n}.$$

Hence n should be smaller than $\frac{T}{4c^2}$. Therefore, $\tau_j \leq r_0$ for every $j \leq 2n$. It follows that at the step $i = 2n$ of the inductive process (see the proof of Theorem 2.1 step 1), we have a family of n mutually disjoint capacitors satisfying the proposition, which completes the proof. □

3. Eigenvalues estimates on Riemannian Manifolds

In this section we apply Theorem 2.1 to a special case of $m-m$ spaces which are Riemannian manifolds, in order to prove Theorem 1.1 and Theorem 1.2. The arguments we use to prove these two theorems are similar. We start by giving in details the proof of Theorem 1.2.

Definition 3.1. *Let (M^m, g) be a Riemannian manifold of dimension m. The capacity of a capacitor (F, G) in M is defined by*

$$\operatorname{cap}_g(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^2 d\mu_g,$$

where $\mathcal{T} = \mathcal{T}(F, G)$ is the set of all compactly supported Lipschitz functions on M such that $\mathrm{supp}\ \varphi \subset \mathrm{G}^\circ = \mathrm{G} \setminus \partial\mathrm{G}$ *and $\varphi \equiv 1$ in a neighborhood of F. If $\mathcal{T}(F, G)$ is empty, then $cap_g(F, G) = +\infty$. Similarly, we can define the m-capacity as*

$$\mathrm{cap}^{(m)}_{[g]}(F, G) = \inf_{\varphi \in \mathcal{T}} \int_M |\nabla_g \varphi|^m d\mu_g.$$

Since m is the dimension of M, it is clear that the m-capacity depends only on the conformal class $[g]$ of the metric g.

Proposition 3.1. *Under the assumptions of Theorem 1.2, take the $m - m$ space (Ω, d_{g_0}, μ), where d_{g_0} is the Riemannian distance corresponding to the metric g_0 and μ is a non-atomic finite measure on Ω. Then for every $n \in \mathbb{N}^*$, there exists a family of capacitors $\mathcal{A} = \{(F_i, G_i)\}_{i=1}^n$ with the following properties:*

(i) $\mu(F_i) \geq \frac{\mu(\Omega)}{8c_m^2 n}$ *;*
(ii) the G_i's are mutually disjoint ;
(iii) the family $\mathcal{A}$ is such that either
 (a) *all the F_i's are annuli, $G_i = 2F_i$ and $cap^{(m)}_{[g_0]}(F_i, 2F_i) \leq Q_m$, or*
 (b) *all the F_i's are domains in Ω and $G_i = F_i^{r_0}$,*

where $r_0 = \frac{1}{1600}$ and, c_m and Q_m are constants depending only on the dimension.

Proof. Let us start with the observation that the metric space (Ω, d_{g_0}) satisfies the $(2, N; 1)$-covering property. For each ball $B(x, r)$ with center in Ω and radius smaller than 1, take a maximal family $\{B(x_i, r/4)\}$ of disjoint balls with centers in $B(x, r)$. Let κ be the cardinal of that family. The family of balls $\{B(x_i, r/2)\}$ covers $B(x, r)$. Hence

$$\kappa \min_i \mu_{g_0}(B(x_i, r/4)) \leq \sum_i \mu_{g_0}(B(x_i, r/4)) \leq \mu_{g_0}(B(x, r + r/4)).$$

Take x_{i_0} such that $\mu_{g_0}(B(x_{i_0}, r/4)) = \min_i \mu_{g_0}(B(x_i, r/4))$. We have

$$\kappa \leq \frac{\mu_{g_0}(B(x, r + r/4))}{\min_i \mu_{g_0}(B(x_i, r/4))} \leq \frac{\mu_{g_0}(B(x, 2r))}{\mu_{g_0}(B(x_{i_0}, r/4))} \leq \frac{\mu_{g_0}(B(x_{i_0}, 4r))}{\mu_{g_0}(B(x_{i_0}, r/4))}.$$

Since $\mathrm{Ricci}_{g_0}(\Omega) \geq -(m - 1)$, thanks to the Bishop-Gromov volume comparison Theorem, we have $\forall\ 0 < s < r$,

$$\frac{\mu_{g_0}(B(x, r))}{\mu_{g_0}(B(x, s))} \leq \frac{\int_0^r \sinh^{m-1} t\ dt}{\int_0^s \sinh^{m-1} t\ dt}.$$

Since for every positive t one has $t \leq \sinh t \leq te^t$, we get

$$\frac{\mu_{g_0}(B(x, r))}{\mu_{g_0}(B(x, s))} \leq \left(\frac{r}{s}\right)^m e^{(m-1)r}.$$

In particular, we have

$$\mu_{g_0}(B(x, r)) \leq r^m e^{(m-1)r} \tag{15}$$

and, $\forall r < 1$,

$$\kappa \leq \frac{\mu_{g_0}(B(x_{i_0}, 4r))}{\mu_{g_0}(B(x_{i_0}, r/4))} \leq 2^{4m} e^{4(m-1)r} =: C(r) \leq C(1). \tag{16}$$

One can take $N = C(1)$ and deduce that (Ω, d_{g_0}) has the $(2, N; 1)$ covering property where N depends only on the dimension.
Now the proof of Proposition 3.1 is a straightforward consequence of Theorem 2.1. Recall that in the statement of Theorem 2.1, the constant c depends only on N. Therefore, in our case c depends only on the dimension. It remains to verify that in the case of annuli, there exists a constant Q_m depending only on the dimension such that for each i, we have

$$cap^{(m)}_{[g_0]}(F_i, 2F_i) \leq Q_m.$$

According to Theorem 2.1, the outer radii of the annuli we consider are smaller than one. It is enough to show that for each point $x \in \Omega$ and $0 \leq r < R \leq 1/2$, we have

$$cap^{(m)}_{[g_0]}(A, 2A) \leq Q_m, \tag{17}$$

where $A = A(x, r, R)$. Set

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in A(x, r, R) \\ \frac{2d_{g_0}(x, B(x, r/2))}{r} & \text{if} \quad x \in A(x, r/2, r) = B(x, r) \setminus B(x, r/2) \\ 1 - \frac{d_{g_0}(x, B(x, R))}{R} & \text{if} \quad x \in A(x, R, 2R) = B(x, 2R) \setminus B(x, R) \\ 0 & \text{if} \quad x \in M \setminus A(x, r/2, 2R) \end{cases}.$$

It is clear that $f \in \mathcal{T}(A, 2A)$ and

$$|\nabla_{g_0} f(x)| \leq \frac{2}{r}, \quad \text{on } B(x, r) \setminus B(x, r/2),$$
$$|\nabla_{g_0} f| \leq \frac{1}{R}, \quad x \in B(x, 2R) \setminus B(x, R).$$

Therefore

$$\begin{aligned} cap^{(m)}_{[g_0]}(A, 2A) &\leq \int_M |\nabla_{g_0} f|^m d\mu_{g_0} \\ &\leq \left(\frac{2}{r}\right)^m \mu_{g_0}(A(x, r/2, r)) + \left(\frac{1}{R}\right)^m \mu_{g_0}(A(x, R, 2R)) \\ &\leq \left(\frac{2}{r}\right)^m \mu_{g_0}(B(x, r)) + \left(\frac{1}{R}\right)^m \mu_{g_0}(B(x, 2R)). \end{aligned}$$

Now since $r, 2R \in (0, 1]$, Using inequality (15), one can control the last inequality by a constant Q_m depending only on the dimension which completes the proof of inequality (17). □

Remark 3.1. *Since $C(r)$ defined in (16) is a strictly increasing function of r, it follows that (Ω, d_{g_0}) does not necessarily satisfy the $(2, N)$-covering property for some N depending only on the dimension.*

Now we show how Theorem 1.2 follows from Proposition 3.1.

Proof of Theorem 1.2. Take the $m-m$ space $(\Omega, d_{g_0}, \mu_\Omega)$, where $\mu_\Omega = \mu_g|_\Omega$. According to Proposition 3.1, there exists a family $\{(F_i, G_i)\}$ of $3k$ capacitors which satisfies the properties $(i), (ii)$ and either $(iii)(a)$ or $(iii)(b)$ of the proposition. We consider each case separately.

Case 1. If $\{(F_i, G_i)\}_{i=1}^{3k}$ is a family with the properties $(i), (ii)$ and $(iii)(a)$ of Proposition 3.1, then

$$\lambda_k(\Omega, g) \le A'_m \left(\frac{k}{\mu_g(\Omega)}\right)^{\frac{2}{m}}, \tag{18}$$

where $A'_m = 24c_m^2(2Q_m)^{\frac{2}{m}}$.

Indeed, we begin by choosing a family of $3k$ test functions $\{f_i : f_i \in \mathcal{T}(F_i, G_i)\}_{i=1}^{3k}$ such that

$$\int_M |\nabla_{g_0} f_i|^m d\mu_{g_0} \le cap_{[g_0]}^{(m)}(F_i, G_i) + \epsilon.$$

Therefore,

$$\begin{aligned} R(f_i) &= \frac{\int_\Omega |\nabla_g f_i|^2 d\mu_g}{\int_\Omega |f_i|^2 d\mu_g} \le \frac{\left(\int_\Omega |\nabla_{g_0} f_i|^m d\mu_{g_0}\right)^{\frac{2}{m}} \left(\int_\Omega 1_{\mathrm{supp} f_i} d\mu_g\right)^{1-\frac{2}{m}}}{\int_\Omega |f_i|^2 d\mu_g} \\ &\le \frac{\left(cap_{[g_0]}^{(m)}(F_i, G_i) + \epsilon\right)^{\frac{2}{m}} \left(\mu_\Omega(G_i)\right)^{1-\frac{2}{m}}}{\mu_\Omega(F_i)}. \end{aligned} \tag{19}$$

The first inequality follows from Hölder inequality and, because of the conformal invariance of $\int |\nabla_g f_i|^m d\mu_g$, we have replaced g by g_0. Since the G_i's are disjoint domains and $\sum_{i=1}^{3k} \mu_\Omega(G_i) \le \mu_g(\Omega)$, at least k of them have measure smaller than $\frac{\mu_g(\Omega)}{k}$. Up to re-ordering, we assume that for the first k of the G_i's we have

$$\mu_\Omega(G_i) \le \frac{\mu_g(\Omega)}{k}. \tag{20}$$

Now, we can take $\epsilon = Q_m$. Using Proposition 3.1 (i) and (iii)(a) and inequality (20), we get from inequality (19)

$$R(f_i) \le A'_m \frac{\left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\mu_g(\Omega)}{k}} = A'_m \left(\frac{k}{\mu_g(\Omega)}\right)^{\frac{2}{m}},$$

with $A'_m = 24c_m^2(2Q_m)^{\frac{2}{m}}$, which completes the proof of Case 1.

Case 2. If $\{(F_i, G_i)\}_{i=1}^{3k}$ is a family with the properties $(i), (ii)$ and $(iii)(b)$ of Proposition 3.1, then

$$\lambda_k(\Omega, g) \le B'_m \left(\frac{\mu_{g_0}(\Omega)}{\mu_g(\Omega)}\right)^{\frac{2}{m}}, \tag{21}$$

where $B'_m = \frac{24c_m^2}{r_0^2}$.

Indeed, we define the test functions f_i as follows

$$f_i(x) = \begin{cases} 1 & \text{if} \quad x \in F_i \\ 1 - \frac{d_{g_0}(x,F_i)}{r_0} & \text{if} \quad x \in (G_i \setminus F_i) \\ 0 & \text{if} \quad x \in G_i^c \end{cases} .$$

We have $|\nabla_{g_0} f_i| \leq \frac{1}{r_0}$. Therefore,

$$\begin{aligned} R(f_i) &= \frac{\int_\Omega |\nabla_g f_i|^2 d\mu_g}{\int_M |f_i|^2 d\mu_g} \leq \frac{\left(\int_\Omega |\nabla_{g_0} f_i|^m d\mu_{g_0}\right)^{\frac{2}{m}} \left(\int_\Omega 1_{\mathrm{supp} f_i} d\mu_g\right)^{1-\frac{2}{m}}}{\int_\Omega |f_i|^2 d\mu_g} \\ &\leq \frac{\frac{1}{r_0^2} \left(\mu_{g_0}(G_i \cap \Omega)\right)^{\frac{2}{m}} \left(\mu_\Omega(G_i)\right)^{1-\frac{2}{m}}}{\mu_\Omega(F_i)}. \end{aligned} \tag{22}$$

Since the G_i's are disjoint, we have

$$\sum_{i=1}^{3k} \mu_{g_0}(G_i \cap \Omega) \leq \mu_{g_0}(\Omega) \quad \text{and} \quad \sum_{i=1}^{3k} \mu_\Omega(G_i) \leq \mu_g(\Omega).$$

Hence, there exist at least $2k$ sets among $G_1, \ldots, G_{3k}$ such that $\mu_{g_0}(G_i) \leq \frac{\mu_{g_0}(\Omega)}{k}$. Similarly, there exist at least $2k$ sets (not necessarily the same ones) such that $\mu_g(G_i) \leq \frac{\mu_g(\Omega)}{k}$. Therefore, up to re-ordering, we assume that the first k of the G_i's satisfy both of the two following inequalities

$$\mu_\Omega(G_i) \leq \frac{\mu_g(\Omega)}{k} \quad \text{and} \quad \mu_{g_0}(G_i \cap \Omega) \leq \frac{\mu_{g_0}(\Omega)}{k}. \tag{23}$$

Using Proposition 3.1 (i) and inequalities (23), we get from inequality (22)

$$\begin{aligned} R(f_i) &\leq B'_m \frac{\left(\frac{\mu_{g_0}(\Omega)}{k}\right)^{\frac{2}{m}} \left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\mu_g(\Omega)}{k}} \\ &\leq B'_m \left(\frac{\mu_{g_0}(\Omega)}{\mu_g(\Omega)}\right)^{\frac{2}{m}} \end{aligned}$$

with $B'_m = \frac{24c_m^2}{r_0^2}$, which completes the proof of Case 2.

In both cases, $\lambda_k(\Omega, g)$ is bounded above by the sum of the right-hand sides of (18) and (21), which completes the proof. □

Remark 3.2. To avoid a possible confusion, it is judicious to examine the proof of Theorem 1.2. In the proof, we begin with the GNY-construction but the method may break down for some $j < 2n$ in the sense that we may not be able to find j (or more) disjoint small annuli. In such a case, inequality (14) holds. The validity of inequality (14) implies that the CM-construction is applicable with $r = r_0$ which gives an estimate for λ_k of the form given in inequality (21). This may appear to be unreasonable since the right hand side is independent of k. However, as pointed out in Proposition 2.1, the GNY-construction for a given compact Riemannian manifold is applicable for all n sufficiently large, but we have no control over the constants and how large n should be. The method described above enables one to establish the validity of the estimate for those finite number of k's for which the GNY-construction is not applicable.

Proof of Theorem 1.1. Consider the $m-m$ space (M, d_{g_0}, μ_g), where d_{g_0} is the distance associated with the metric g_0 and μ_g is the measure associated with the metric g. We easily see that we can follow the same arguments as in the proof of Theorem 1.2 to derive the following inequality

$$\lambda_k(M,g)\mu_g(M)^{\frac{2}{m}} \leq A_m \mu_{g_0}(M)^{\frac{2}{m}} + B_m k^{\frac{2}{m}}. \tag{24}$$

The left hand side does not depend on g_0. Hence, we can take the infimum with respect to $g_0 \in [g]$ such that $\mathrm{Ricci}_{g_0} \geq -(m-1)$, which leads to the desired conclusion. □

4. Steklov Eigenvalues

It is worth pointing out that Theorem 2.1 is formalized in a general setting and is applicable to other eigenvalue problems. In this section we present an application of this theorem to the Steklov eigenvalue problem.

Steklov problem. Let Ω be a bounded subdomain of a complete m-dimensional Riemannian manifold (M,g) and assume that Ω has nonempty smooth boundary $\partial\Omega$. Given a function $u \in H^{\frac{1}{2}}(\partial\Omega)$, we denote by $\bar{u}$ the unique harmonic extension of u to Ω, that is

$$\begin{cases} \Delta_g \bar{u} = 0 & \text{in } \Omega \\ \bar{u} = u & \text{on } \partial\Omega \end{cases}.$$

Let ν be the outward unit normal vector along $\partial\Omega$. The Steklov operator is the map

$$\begin{array}{rcl} L : H^{\frac{1}{2}}(\partial\Omega) & \to & H^{-\frac{1}{2}}(\partial\Omega) \\ u & \mapsto & \dfrac{\partial \bar{u}}{\partial \nu}. \end{array}$$

The operator L is an elliptic pseudo differential operator (see [Ta, pages 37-38]) which admits a discrete spectrum tending to infinity denoted by

$$0 = \sigma_1 \leq \sigma_2 \leq \sigma_3 \ldots \nearrow \infty$$

The eigenvalue σ_k of L can be characterized variationally as follows (see [CEG2]):

$$\sigma_k(\Omega) = \inf_{V_k} \sup\{\frac{\int_\Omega |\nabla_g \bar{u}|^2 d\mu_g}{\int_{\partial\Omega} |\bar{u}|^2 d\bar{\mu}_g} : 0 \neq \bar{u} \in V_k\}, \tag{25}$$

where V_k is a k-dimensional linear subspace of $H^1(\Omega)$ and $\bar{\mu}_g$ is the Riemannian measure associated to g on the boundary.

The relationships between the geometry of the domain and the spectrum of the corresponding Steklov operator have been investigated by several authors (see for example [CEG2], [FS] and [GP]). Recently, Fraser and Schoen [FS, Theorem 2.3] proved the following inequality for the Steklov eigenvalues of a compact Riemannian surface (Σ_γ, g) of genus γ and κ boundary components:

$$\sigma_2(\Sigma_\gamma)\ell_g(\partial\Sigma_\gamma) \leq 2(\gamma+\kappa)\pi,$$

where $\ell_g(\partial\Sigma)$ is the length of the boundary. This result was generalized to higher eigenvalues by Colbois, El Soufi and Girouard [CEG2, Theorem 1.5]. Indeed, the authors proved the following inequality for every $k \in \mathbb{N}^*$

$$\sigma_k(\Sigma_\gamma)\ell_g(\partial\Sigma_\gamma) \leq C(\gamma+1)k, \tag{26}$$

where C is a universal constant.
For a domain in a higher dimensional manifold, the authors [CEG2, Theorem 1.3] also obtained an upper bound for σ_k depending on the isoperimetric ratio of the domain. More precisely, if (M, g) is conformally equivalent to a complete manifold with non-negative Ricci curvature, then for every bounded domain Ω of M and every $k \in \mathbb{N}^*$,

$$\sigma_k(\Omega)\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} \leq C_m \frac{k^{\frac{2}{m}}}{I_g(\Omega)^{1-\frac{1}{m-1}}}, \tag{27}$$

where $I_g(\Omega)$ is the isoperimetric ratio ($I_g(\Omega) = \frac{\bar{\mu}_g(\partial\Omega)}{\mu_g(\Omega)^{\frac{m-1}{m}}}$) and C_m is a constant depending only on m.

The theorem below is motivated by the work of [CEG2], and we obtain an improvement of inequalities (26) and (27) using Proposition 3.1.

Theorem 4.1. *Let (M, g_0) be a complete Riemannian manifold of dimension $m \geq 2$ with $\mathrm{Ricci}_{g_0}(M) \geq -(m-1)$. Let $\Omega \subset M$ be a relatively compact domain with C^1 boundary and g be any metric conformal to g_0. Then we have*

$$\sigma_k(\Omega)\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} \leq \frac{A_m\mu_{g_0}(\Omega)^{\frac{2}{m}} + B_m k^{\frac{2}{m}}}{I_g(\Omega)^{1-\frac{1}{m-1}}}, \tag{28}$$

where A_m and B_m are constants depending only on m.

As an immediate consequence we get the following inequality in the case of Riemann surfaces:

Corollary 4.1. *Let (Σ_γ, g) be a compact oriented Riemannian surface with genus γ, and Ω be a subdomain of Σ_γ. Then*

$$\sigma_k(\Omega)\ell_g(\partial\Omega) \leq A\gamma + Bk, \tag{29}$$

where A and B are constants.

Proof of Theorem 4.1. We consider the $m-m$ space $(\Omega, d_{g_0}, \bar{\mu})$, where $\bar{\mu}(A) := \bar{\mu}_g(A\cap\partial\Omega)$. We apply again Proposition 3.1. Therefore, there exist a family of $3k$ capacitors $\{(F_i, G_i)\}$ satisfying properties $(i), (ii)$ and either $(iii)(a)$, or $(iii)(b)$ of Proposition 3.1. We proceed analogously to the proof of Theorem 1.2. Using the variational characterization of σ_k, we construct a family of test functions as in Case 1 and Case 2 of the proof of Theorem 1.2. In both cases, we have

$$\sigma_k(\Omega) \leq \frac{\int_\Omega |\nabla_g f_i|^2 d\mu_g}{\int_{\partial\Omega} |f_i|^2 d\bar{\mu}_g} \leq \frac{\left(\int_\Omega |\nabla_{g_0} f_i|^m d\mu_{g_0}\right)^{\frac{2}{m}} \mu_g(G_i)^{1-\frac{2}{m}}}{\bar{\mu}(F_i)}.$$

If the family $\{(F_i, G_i)\}$ satisfies the properties $(i), (ii)$ and $(iii)(a)$ of Proposition 3.1, then

$$\sigma_k(\Omega) \leq A_m \frac{\left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\bar{\mu}_g(\partial\Omega)}{k}} \leq A_m \frac{k^{\frac{2}{m}}}{\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} I_g(\Omega)^{1-\frac{1}{m-1}}}. \tag{30}$$

If on the other hand, the family $\{(F_i, G_i)\}$ satisfies the properties $(i), (ii)$ and $(iii)(b)$ of Proposition 3.1, then

$$\sigma_k(\Omega) \leq B_m \frac{\left(\frac{\mu_{g_0}(\Omega)}{k}\right)^{\frac{2}{m}} \left(\frac{\mu_g(\Omega)}{k}\right)^{1-\frac{2}{m}}}{\frac{\bar{\mu}_g(\partial\Omega)}{k}} \leq B_m \frac{\mu_{g_0}(\Omega)^{\frac{2}{m}}}{\bar{\mu}_g(\partial\Omega)^{\frac{1}{m-1}} I_g(\Omega)^{1-\frac{1}{m-1}}}, \tag{31}$$

where the constant coefficients A_m and B_m are the same as A'_m and B'_m in Theorem 1.2.

The proof of inequalities (30) and (31) are along the same lines as Theorem 1.2. In both cases, $\sigma_k(\Omega)$ is bounded above by the sum on the right-hand sides of (30) and (31), and it completes the proof. □

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