Involutive isometries, eigenvalue bounds and a spectral property of Clifford tori *

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Abstract

In this paper we consider a compact Riemannian manifold or submanifold M, with an involutive isometry which has no fixed point, and we derive some spectral properties of this geometric situation. The aim is to give an upper bound for the gap $\lambda_2(D)-\lambda_1(D)$ of the first two eigenvalues of a Laplace type operator D acting on sections of a vector bundle over M.

In the first part, using the classical barycenter method, we derive sharp upper bounds for the gap of antipodal symmetric submanifold of Euclidean space. Moreover, if equality holds, we prove that the submanifold is minimal in a sphere. In particular, we give a spectral characterization of the Clifford torus $\mathbf{S}^p\left(\sqrt{\frac{p}{n}}\right)\times\mathbf{S}^{n-p}\left(\sqrt{\frac{n-p}{n}}\right)$ as the unique maximizer for the gap of the Hodge Laplacian on p-forms, among all antipodal symmetric hypersurfaces of the sphere $\mathbf{S}^{n+1}$.

In the second part we give upper bounds in the general case. The main point is that these bounds do not depend on the particular operator D we consider, but only on the natural intrinsic or extrinsic distance on M and on the displacement of the action of the isometry group considered.

1 Introduction

In this paper (M^n,g) denotes a closed, connected, orientable manifold with Riemannian metric g.

Assume that M admits an isometric involution γ such that the distance of any point to its image under γ is uniformly bounded below by a positive constant β. The main scope of this paper is to show that this simple situation has a rather strong influence on the spectrum of M, not only for the classical Laplacian or Schrödinger operators, but also for a large class of operators acting on sections of a vector bundle over M (the so-called Laplace-type operators) as long as they are γ-invariant. Namely, we prove that there exists a uniform upper bound for the gap between the first and the second eigenvalue of the operator, depending only on the displacement β and a rather weak metric invariant,

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called the *packing constant* of M (this invariant is itself independent on the operator and, a priori, on the curvature of (M, g)).

We illustrate this phenomenon by proving a result under the more general assumption that M admits a free, isometric action by a finite group (see Theorem 12); but perhaps the most interesting case is when the manifold is isometrically immersed in the unit sphere and is invariant under the antipodal action of the group $\mathbf{Z}_2$: in that case, in fact, we obtain sharp upper bounds only in terms of the dimension n of M, with equality cases which imply interesting rigidity results (see Theorems 1, 2, and 3).

Without the assumption of antipodal symmetry (or the existence of a group action with displacement bounded below) it is possible to construct families of examples with large gap: see the discussion in Remark 6.

In the rest of the introduction we state the main results of the paper.

Let $\pi : E \to M$ be a vector bundle over (M^n, g), and ∇^E a connection on E which is compatible with a given bundle metric (see [B] for details). An operator D acting on sections of the bundle is said to be of *Laplace-type* if it can be written

$$D = (\nabla^E)^\star \nabla^E + T,$$

where T is a self-adjoint endomorphism of the fiber. Then, D is self-adjoint and elliptic. We list its eigenvalues as $\lambda_1(D) \leq \lambda_2(D) \leq \cdots \leq \lambda_k(D) \leq \dots$

If a group G acts by isometries on (M^n, g), we suppose that there is an associated equivariant action on the sections of E. This means that, for $\gamma \in G$, we have $\gamma \cdot \nabla^E_X \psi = \nabla^E_{\gamma_\star X}(\gamma \cdot \psi)$ for all sections ψ of E and all tangent vectors X. We say that D is $G-$invariant if it commutes with any isometry of the group G. It is clear that this happens if and only if the endomorphism T commutes with any isometry of G, and in that case G preserves each eigenspace of D.

Important examples of Laplace-type operators are given by the Laplacian acting on differential forms and by the square of the Dirac operator, both of which are $G-$invariant; and by a Schrödinger operator acting on functions: $D = \Delta + q$, which is $G-$invariant if and only if the potential q is.

1.1 Sharp bounds for the antipodal action

Here we consider the special case in which M^n admits an isometric immersion into the unit sphere of dimension N and $\gamma : \mathbf{S}^N \to \mathbf{S}^N$ is the antipodal map. If M^n is invariant under γ, it inherits an isometric action by the group $G = \mathbf{Z}_2$ generated by γ.

Theorem 1. *Let $\phi : M^n \to \mathbf{S}^N$ be an isometric immersion, invariant under the antipodal map γ. Let D be any Laplace-type operator acting on sections of a vector bundle on M,* *which is also invariant under γ. Then:*

$$\lambda_2(D) - \lambda_1(D) \le n.$$

If equality holds, then M is minimal in $\mathbf{S}^N$ and any eigensection associated to $\lambda_1(D)$ is parallel.

For the proof we refer to Section 2, in which we consider, more generally, a centrally symmetric submanifold of $\mathbf{R}^m$ equipped with a radial conformal metric; we then obtain a bound in terms of the minimum distance of M from its center of symmetry.

It is perhaps surprising that the above bound holds for all Laplace-type operators, with no curvature assumptions. Note that equality implies minimality (which is a property depending only on the immersion) and also implies the existence of a parallel section, which of course depends on the bundle E we consider.

For example, if $D = \Delta + q$ is a Schrödinger operator acting on functions, equality implies that the immersion is minimal and that q is constant. When M^n is an antipodal symmetric minimal hypersurface of S^{n+1} and D is the stability operator, the inequality of Theorem 1 was obtained by O. Perdomo in [Pe], who used it to characterize Clifford tori as minimizers for the stability index of non equatorial minimal hypersurfaces with antipodal symmetry.

We then focus on the operator $D = \Delta_p$, the Laplacian acting on differential forms of arbitrary degree p. In that case we characterize the equality, and we show that the absolute maximum for the gap, when $p \neq n/2$, is reached at exactly one hypersurface: the unique (minimal) Clifford torus with $b_p(M) = 1$, where $b_p(M)$ is the $p-$th Betti number of M. Here is the precise statement.

Theorem 2. *Let M^n be an immersed hypersurface of the sphere $\mathbf{S}^{n+1}$, invariant under the antipodal map. Let Δ_p be the Laplacian acting on p-forms. Then one has:*

$$\lambda_2(\Delta_p) - \lambda_1(\Delta_p) \le n$$

for all $p = 1, \dots, n-1$. Moreover:

a) *If $p = n/2$ equality never holds.*

b) *If $p \neq n/2$ equality holds if and only if M^n is isometric with the Clifford torus:*

$$\mathrm{CL}_{n,p} = \mathbf{S}^p\left(\sqrt{\frac{p}{n}}\right) \times \mathbf{S}^{n-p}\left(\sqrt{\frac{n-p}{n}}\right),$$

minimally embedded in $\mathbf{S}^{n+1}$.

To our knowledge, this is the first characterization of the Clifford tori $\mathrm{CL}_{n,p}$ by the spectrum of the Hodge Laplacian.

Remark. We can characterize equality because we will prove in Theorem 9 that the only minimal hypersurface of $\mathbf{S}^{n+1}$ supporting a parallel p-form in a degree $p \neq 0, n$ is the Clifford torus $\mathrm{CL}_{n,p}$. Consequently, if D is any Laplace-type operator acting on the bundle of p-forms and equality holds in Theorem 1, then M must be isometric to the Clifford torus $\mathrm{CL}_{n,p}$.

A little additional argument gives the following sharp upper bound for the first *positive* eigenvalue of the Laplacian acting on p-forms, which we denote by $\bar{\lambda}_1(\Delta_p)$.

Theorem 3. *Let M^n be an immersed hypersurface of $\mathbf{S}^{n+1}$, invariant under the antipodal map, and $\bar{\lambda}_1(\Delta_p)$ the first positive eigenvalue of Δ_p. We consider the following two cases:*

1) $p \neq n/2$ *and* $b_p(M) = 1$,

2) $p = n/2$, p *is odd and* $b_p(M) = 2$.

Then, under any of the above assumptions, one has:

$$\bar{\lambda}_1(\Delta_p) \leq n$$

with equality if and only if M is the Clifford torus $\mathrm{CL}_{n,p}$.

Note that, in part 2), we assume that $n = 4m + 2$ for some nonnegative integer m.

In dimension 2 the first positive eigenvalue of Δ_1 is equal to $\bar{\lambda}_1(M)$, the first positive eigenvalue of the Laplacian on functions. Taking $p = 1$, from the second part of the previous theorem we then obtain:

Corollary 4. *Let M^2 be an immersed surface of $\mathbf{S}^3$, invariant under the antipodal map. Assume that it has genus* 1. *Then*

$$\bar{\lambda}_1(M) \leq 2$$

with equality if and only if M is the minimal Clifford torus $\mathrm{CL}_{2,1} = \mathbf{S}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbf{S}^1\left(\frac{1}{\sqrt{2}}\right)$.

So, the Clifford torus $\mathrm{CL}_{2,1}$ is the unique maximizer for the first eigenvalue of the Laplacian on functions among all immersed, antipodal symmetric surfaces of $\mathbf{S}^3$ with genus 1.

Recall the following well-known conjecture, due to Yau:

− *Any embedded minimal hypersurface of $\mathbf{S}^{n+1}$ satisfies $\bar{\lambda}_1(M^n) = n$.*

On the other hand, a conjecture of Lawson states that:

− *The Clifford torus is the only embedded minimal surface of genus* 1 *in* $\mathbf{S}^3$.

Corollary 4 shows also that, in dimension 2 and in the antipodal symmetric case, the Yau conjecture implies the Lawson conjecture. We remark that this fact has been proven in full generality by Montiel and Ros in [MR].

1.2 Estimates for a general finite group action

The estimate of Theorem 1 is a special case of a much more general situation. Namely, assume that the finite group G of order p acts freely by isometries on a compact orientable Riemannian manifold (M^n, g). Then, we get a uniform upper bound for $\lambda_p(D) - \lambda_1(D)$ which holds for all $G-$invariant Laplace-type operators; that is, we control a number of eigenvalues corresponding to the order p of G.

The bound depends only on the metric structure d of M (through a constant $C_\infty(M, d)$ called the *packing constant of* M) and on the *displacement* of the action of the group:

$$\beta(G, d) = \inf_{x \in M} \{d(x, \gamma \cdot x) : \gamma \in G, \gamma \neq 1\}.$$

We then prove in Theorem 12 that:

$$\lambda_p(D) - \lambda_1(D) \leq \frac{16 C_\infty(M, d)}{\beta(G, d)^2} \tag{1}$$

For the definition of the constant $C_\infty(M, d)$, and for its estimates, see Theorem 12.

If M^n is isometrically immersed in a larger manifold $\bar{M}^N$ the same proof shows that, in the definitions of $C_\infty(M, d)$ and $\beta(G, d)$, one can replace the Riemannian intrinsic distance d by the extrinsic distance d_{ext}. Now, the packing constant of a submanifold of $\mathbf{R}^N$, for the extrinsic distance, is uniformly bounded above by 16^{2N}, and we see that the estimate (1) reduces to the following.

Theorem 5. *Let $\phi : M^n \to \mathbf{R}^N$ be an isometric immersion, invariant under the action of a group G of order p of isometries of $\mathbf{R}^N$. Let D be any Laplace-type operator acting on sections of a vector bundle on M, which is also invariant under G. Then:*

$$\lambda_p(D) - \lambda_1(D) \leq \frac{16^{2N+1}}{\beta(G, d_{\text{ext}})^2}$$

where d_{ext} is the extrinsic Euclidean distance on M^n.

When M^n is a submanifold of $\mathbf{S}^N$ invariant under the antipodal action by the group $\mathbf{Z}_2$, the displacement constant is simply $\beta(G, d_{\text{ext}}) = 2$ and the upper bound of Theorem 5 depends only on N: this explains the estimate of Theorem 1 although the constant that we get with this general approach is, quite naturally, not sharp.

Remark. Let us give an intuitive explanation of the bound (1) at least for the order $p = 2$. In [CS] it is shown that, for a general manifold with packing constant uniformly bounded above, the existence of a large gap $\lambda_{k+1}(D) - \lambda_1(D)$, for some k, implies that any unit L^2-norm eigensection ψ_1 associated to $\lambda_1(D)$ must concentrate its pointwise norm in a small neighborhood of a suitable set of (at most) k points of M. In particular, if $\lambda_2(D) - \lambda_1(D)$ is large, the measure $|\psi_1|^2 d\mathrm{vol}_g$ is close to a Dirac measure concentrated at one point. If there is an isometric action by the group $G = \mathbf{Z}_2$, with displacement bounded away from 0, this is impossible, simply because the norm of ψ_1 is $\mathbf{Z}_2-$invariant. Then, we must have a uniform upper bound of $\lambda_2(D) - \lambda_1(D)$.

Remark 6. In this remark we point out that, if one drops our assumption on the existence of an isometric action with displacement bounded below, then the gap may assume large values, in different contexts. Let us give a short and non-exhaustive list of examples, without detailed explanations: the interested reader can consult the given references.

For the Laplacian acting on functions, and for a given compact manifold M of dimension $n \geq 3$, it is possible to construct volume 1 metrics on M with arbitrarily large gap ([CD]), or even with a prescribed finite part of the spectrum ([Lo]). If $n \geq 4$, It is also possible to produce large gap for the Laplacian on p-forms, $2 \leq p \leq n-2$, ([GP]). Large gap also exists for the rough Laplacian on $\mathbb{C}$-line bundles, see [BCC].

There exist examples of submanifolds of Euclidean space having volume 1 and large gap for functions, even in codimension 1 ([CDE2], Theorem 1.4). Related to the situation of Theorem 1, we can also construct a family of submanifolds of the sphere S^N with volume uniformly bounded from below by a positive constant and arbitrarily large gap for functions (resp. for p-forms): it is enough to use the Nash-Kuiper theorem to embed the previous examples in a arbitrarily small ball of Euclidean space, and then argue up to quasi-isometry.

Finally, isometric actions were also considered in [AF] or [CDE1].

2 Sharp estimates in the conformal Euclidean case

Our Theorem 1 is a consequence of the following more general result. Consider an isometric immersion:

$$\phi : (M^n, g) \to (\mathbf{R}^N, h^2 g_{\mathrm{eucl}}),$$

where h is a radial, positive, non-decreasing smooth function on $\mathbf{R}^N$ and g_{eucl} is the canonical metric of $\mathbf{R}^N$. We assume that $\phi(M)$ is invariant under the antipodal map γ, and that the origin $O \notin \phi(M)$. Then γ induces an involutive isometry on (M, g).

Theorem 7. *In the above notation, let D be any Laplace-type operator acting on sections of a vector bundle on M, which is invariant under γ. Then:*

$$\lambda_2(D) - \lambda_1(D) \leq \frac{n}{d^2 h(d)^2},$$

where d is the minimum Euclidean distance of $\phi(M)$ to the origin.

a) *Assume that $N = n+1$ and that equality holds. Then M is isometric to the sphere of radius $dh(d)$, and any eigensection associated to $\lambda_1(D)$ is parallel.*

b) *Assume that $N \geq n+2$ and that equality holds. Then ϕ factors through a minimal immersion into the sphere $\mathbf{S}^{N-1}(dh(d))$, and any eigensection associated to $\lambda_1(D)$ is parallel.*

Clearly, if (M^n, g) admits an isometric immersion into $\mathbf{R}^N$ with the canonical metric, then we take $h = 1$ and obtain:

$$\lambda_2(D) - \lambda_1(D) \leq \frac{n}{d^2}.$$

If $\phi : M^n \to \mathbf{S}^N$ is an isometric immersion, invariant under the antipodal map, we compose ϕ with the canonical immersion $\mathbf{S}^N \to \mathbf{R}^{N+1}$, and apply Theorem 7 to this new immersion. We immediately obtain Theorem 1.

Now view the hyperbolic space as $\mathbf{H}^N = (B, h^2 g_{\text{eucl}})$ where B is the unit ball in $\mathbf{R}^N$ and $h = 4/(1-|x|^2)^2$. If $\phi : (M^n, g) \to \mathbf{H}^N$ is an isometric immersion satisfying the given assumptions then it is easily seen that

$$\lambda_2(D) - \lambda_1(D) \leq \frac{n}{\sinh^2 \tilde{d}},$$

where $\tilde{d}$ is the minimum hyperbolic distance of $\phi(M)$ to the center of symmetry O.

For the proof of Theorem 7 we need some preliminary notions. These are standard, but for completeness we decided to prove them explicitly.

2.1 Basic facts about immersions in $\mathbf{R}^N$ or $\mathbf{S}^N$

Consider a smooth immersion $\phi : M^n \to \mathbf{R}^N$. For simplicity of notation, we identify M with its image $\phi(M)$ and endow M with the metric, simply denoted by $\langle \cdot, \cdot \rangle$, induced by the Euclidean metric of $\mathbf{R}^N$.
Let $\nu = -x$ be the opposite of the position vector, so that $\nu = -\sum_{j=1}^N x_j \partial/\partial x_j$. We consider the family of functions given by the restriction to M of

$$f = \langle \bar{V}, \nu \rangle$$

where $\bar{V}$ is a parallel vector field on $\mathbf{R}^N$. These functions are simply the restriction to M of linear functions on $\mathbf{R}^N$. Let V be the vector field on M given by the orthogonal projection of $\bar{V}$ on M; then, we have the splitting

$$\bar{V} = V + V^\perp,$$

where $V \in TM$ and $V^\perp \in T^\perp M$. In what follows, $\bar\nabla$ is the Levi-Civita connection in $\mathbf{R}^N$ and ∇ the induced connection on M. For any tangent vector X to $\mathbf{R}^N$ one has $\bar\nabla_X \nu = -X$. As $\bar V$ is $\bar\nabla$-parallel, one gets immediately:

$$\nabla f = \nabla\langle \bar V, \nu\rangle = -V. \tag{2}$$

The family of unit length parallel vector fields on $\mathbf{R}^N$ can be identified with $\mathbf{S}^{N-1}$; in fact, writing $\bar V = \sum_{j=1}^N a_j \partial/\partial x_j$, we can identify $\bar V$ with $(a_1, \dots, a_N) \in \mathbf{S}^{N-1}$. Let $d\mu$ be the canonical measure of $\mathbf{S}^{N-1}$. Then:

$$\int_{\mathbf{S}^{N-1}} a_j^2 d\mu = \frac{\mathrm{Vol}(\mathbf{S}^{N-1})}{N} \quad \text{and} \quad \int_{\mathbf{S}^{N-1}} a_j a_k d\mu = 0 \text{ if } j \neq k.$$

Therefore, if we replace the measure $d\mu$ by

$$d\bar\mu = \frac{N}{\mathrm{Vol}(\mathbf{S}^{N-1})} d\mu, \tag{3}$$

we see that, at any point $p \in \mathbf{R}^N$ and for all tangent vectors X, Y at p one has:

$$\int_{\mathbf{S}^{N-1}} \langle \bar V, X\rangle \langle \bar V, Y\rangle d\bar\mu(\bar V) = \langle X, Y\rangle. \tag{4}$$

Let us now consider an isometric immersion $\phi : M^n \to \mathbf{S}^{N-1}$, where S^{N-1} is endowed with the canonical metric. If $\hat\nabla$ is the Levi-Civita connection in $\mathbf{S}^{N-1}$ then, for all vector fields X, Y tangent to M:

$$\hat\nabla_X Y = \nabla_X Y + L(X, Y), \tag{5}$$

where $L(X, Y) \in T\mathbf{S}^{N-1}$ is a vector normal to M. L is the second fundamental form of the immersion and the mean curvature vector H is defined as $nH = \mathrm{tr} L$.

Now note that the vector field ν is normal to $\mathbf{S}^{N-1}$; as $\bar\nabla_X \nu = -X$, we see that the second fundamental form of the canonical immersion $\mathbf{S}^{N-1} \to \mathbf{R}^N$ is simply the identity. Therefore, for all tangent vector fields X, Y to $\mathbf{S}^{N-1}$ one has: $\bar\nabla_X Y = \hat\nabla_X Y + \langle X, Y\rangle \nu$, and from (5) we see that, for all tangent vector fields $X, Y \in TM$:

$$\bar\nabla_X Y = \nabla_X Y + \langle X, Y\rangle \nu + L(X, Y), \tag{6}$$

where L is the (vector-valued) second fundamental form of the immersion $\phi : M^n \to \mathbf{S}^{N-1}$.

Lemma 8. *Let $\phi : M^n \to \mathbf{S}^{N-1}$ be an isometric immersion and $\bar V$ a parallel vector field on $\mathbf{R}^N$. Let V be the tangential component of $\bar V$. If X, Y are vectors tangent to M one has:*

$$\langle \nabla_X V, Y\rangle = \langle \bar V, \nu\rangle \langle X, Y\rangle + \langle \bar V, L(X, Y)\rangle,$$

where L is the second fundamental form of ϕ.
Moreover, if $f = \langle \bar{V}, \nu \rangle$ then:

$$\Delta f = nf + n\langle \bar{V}, H \rangle,$$

where H is the mean curvature vector of ϕ.

Proof. We first write

$$\begin{aligned}\langle \nabla_X V, Y \rangle &= X \cdot \langle V, Y \rangle - \langle V, \nabla_X Y \rangle \\ &= X \cdot \langle \bar{V}, Y \rangle - \langle \bar{V}, \nabla_X Y \rangle\end{aligned}$$

since Y and $\nabla_X Y$ are both tangent to M. As $\bar{V}$ is $\bar{\nabla}$-parallel, we obtain:

$$X \cdot \langle \bar{V}, Y \rangle = \langle \bar{V}, \bar{\nabla}_X Y \rangle,$$

and the first formula in the lemma now follows from (6). If $f = \langle \bar{V}, \nu \rangle$ then $\nabla f = -V$ by (2) and so

$$\Delta f = -\mathrm{tr} \nabla^2 f = \mathrm{tr} \nabla V = n\langle \bar{V}, \nu \rangle + n\langle \bar{V}, H \rangle,$$

as asserted. □

2.2 Proof of Theorem 7

The assumption here is that $\phi : (M^n, g) \to (\mathbf{R}^N, h^2 g_{\mathrm{eucl}})$ is an isometric immersion, invariant under the antipodal map $\gamma(x) = -x$. The conformal factor h is assumed to be radial and non-decreasing. As h is radial, γ induces an isometry of (M^n, g). Let D be a Laplace-type operator acting on the sections of a vector bundle on (M^n, g), also invariant under γ. We have to show:

$$\lambda_2(D) - \lambda_1(D) \leq \frac{n}{d^2 h(d)^2} \tag{7}$$

where d is the minimum Euclidean distance of M to the origin, center of symmetry of M.
In this proof, for simplicity of notation, we denote by ∇^g the Levi-Civita connection (on the tangent bundle TM) in the given metric $g = \langle \cdot, \cdot \rangle_g$, and by ∇ the Levi-Civita connection in the metric $\langle \cdot, \cdot \rangle$ induced from the Euclidean metric on $\mathbf{R}^N$. In all formulas below, integration is taken with respect to the Riemannian measure dvol_g.
Let ψ be a smooth section of the bundle and $\mathcal{Q}$ the quadratic form associated to D:

$$\mathcal{Q}(\psi) = \int_M |\nabla^E \psi|^2 + \langle T\psi, \psi \rangle, \tag{8}$$

If f is a Lipschitz function on M, an easy integration by parts (see Lemma 8 in [CS]) gives:

$$\mathcal{Q}(f\psi) = \int_M f^2 \langle D\psi, \psi\rangle + |\nabla^g f|_g^2 |\psi|^2. \tag{9}$$

The theorem is significant only when the first eigenspace of D is simple, and is trivially true otherwise. So we can assume that $V(\lambda_1)$ has dimension 1. As D commutes with γ , we see that γ preserves $V(\lambda_1)$ and acts as an isometry on it.

Fix an eigensection ψ associated to $\lambda_1(D)$. Then ψ is either even or odd with respect to γ; in both cases, $|\psi|^2$ is an even function. Fix a parallel vector field $\bar{V}$ on $\mathbf{R}^N$ and consider the function $f = \langle \bar{V}, \nu\rangle$. As f is odd:

$$\int_M \langle f\psi, \psi\rangle = \int_M f|\psi|^2 = 0,$$

because $f|\psi|^2$ is odd. So the section $f\psi$ is orthogonal to ψ for all $\bar{V}$, and we can use it as a test-section for $\lambda_2(D)$. By the min-max principle:

$$\lambda_2(D) \int_M f^2 |\psi|^2 \le \mathcal{Q}(f\psi),$$

where $\mathcal{Q}$ is the quadratic form defined in (8). Since

$$|\nabla^g f|_g^2 = \frac{1}{h^2} |\nabla f|^2 = \frac{1}{h^2} |V|^2$$

we have, by (9):

$$(\lambda_2(D) - \lambda_1(D)) \int_M \langle \bar{V}, \nu\rangle^2 |\psi|^2 \le \int_M \frac{1}{h^2} |V|^2 |\psi|^2,$$

for all $\bar{V}$. We now integrate the above inequality with respect to $\bar{V}$ on the unit sphere of $\mathbf{R}^N$. At any point $x \in M$ and for the measure $d\bar{\mu}$ defined in (3) one has, by (4):

$$\int_{\mathbf{S}^{N-1}} \langle \bar{V}, \nu\rangle^2 d\bar{\mu}(\bar{V}) = |\nu|^2 = \rho(x)^2,$$

where $\rho(x)$ is Euclidean distance to the origin. Moreover $\int_{\mathbf{S}^{N-1}} |V|^2 d\bar{\mu}(\bar{V}) = n$. Therefore, by the Fubini theorem:

$$(\lambda_2(D) - \lambda_1(D)) \int_M \rho^2 |\psi|^2 \le n \int_M \frac{1}{h^2} |\psi|^2,$$

and as $\rho \ge d$ and $h \ge h(d)$ we get inequality (7).

We now take care of the equality case.

Observe that, if equality holds, we must have in particular that

$$\int_M (\rho^2 - d^2)|\psi|^2 = 0.$$

Since ψ is a solution of an elliptic equation, the function $|\psi|^2$ cannot vanish on any open set, and this forces $\rho = d$ on M. Then $\phi(M)$ is contained in the sphere $\mathbf{S}^{N-1}(d)$, that is, ϕ factors through an isometric immersion $M \to \mathbf{S}^{N-1}(d)$; as h is radial, the metric g is homothetic to the metric induced from the canonical metric of the sphere.

In what follows we then assume (without loss of generality) that $d = 1$ and $h(r) = 1$.

We first prove b). By assumption:

$$\lambda_2(D) - \lambda_1(D) = n.$$

The equality assumption also implies that the section $f\psi$, where $f = \langle \bar{V}, \nu \rangle$, is an eigensection associated to $\lambda_2(D)$ for all $\bar{V}$. For any Laplace-type operator D, and any smooth function f, a straightforward calculation shows that

$$D(f\psi) = \Delta f \cdot \psi + fD\psi - 2\nabla^E_{\nabla f}\psi.$$

As $D(f\psi) = \lambda_2 f\psi$ and $D\psi = \lambda_1\psi$ we then obtain

$$\Delta f \cdot \psi - 2\nabla^E_{\nabla f}\psi = nf\psi.$$

By Lemma 8, $\Delta f = nf + n\langle \bar{V}, H \rangle$ and $\nabla f = -V$. So:

$$n\langle \bar{V}, H \rangle \psi = 2\nabla^E_V \psi. \tag{10}$$

This equation holds everywhere on M, for all choices of $\bar{V}$. Fix a point $x \in M$ and any tangent vector $\xi(x) \in T_xM$. Choose $\bar{V}$ so that $\bar{V}(x) = \xi(x)$. Then $\langle \bar{V}(x), H(x) \rangle = 0$ because H is normal to M, and therefore

$$\nabla^E_{\xi(x)}\psi = 0.$$

As $\xi(x)$ was arbitrary, ψ must be parallel at x. As x was also arbitrary, ψ is parallel on M (in particular, it never vanishes). Then, by (10):

$$\langle \bar{V}, H \rangle = 0$$

for all $\bar{V}$. But this clearly implies that $H = 0$ on M, hence M is minimal as asserted.

We finally prove a). Assume that $N = n + 1$ and that equality holds. Then we have seen that ϕ factors through an immersion $M^n \to \mathbf{S}^n$. Since M^n has empty boundary we must have $M^n = \mathbf{S}^n$ and ϕ is just the identity, hence it is totally geodesic. It remains to show that the eigensection ψ is parallel: but this is a special case of the previous argument. This ends the proof of Theorem 7.

2.3 A rigidity theorem

We observe the following rigidity result, which is perhaps of independent interest.

Theorem 9. *Let M^n be a (not necessarily compact) hypersurface of the unit sphere $\mathbf{S}^{n+1}$, and assume that M supports a non-trivial parallel p-form for some $p = 1, \dots, n-1$. Then at each point $x \in M$ there are only two principal curvatures, precisely*

$$\begin{cases} k(x) & \text{with multiplicity } p \\ -\dfrac{1}{k(x)} & \text{with multiplicity } n-p \end{cases}$$

for some $k(x) \neq 0$.

Proof. Let N be unit normal vector and let $L(X, Y)$ be the *scalar* second fundamental form of the immersion, defined by:

$$\hat{\nabla}_X Y = \nabla_X Y + L(X, Y)N,$$

for all vectors tangent to M. The shape operator S is the endomorphism of TM defined by $\langle S(X), Y\rangle = L(X, Y)$. Let $R(X, Y) = [\nabla_Y, \nabla_X] + \nabla_{[X,Y]}$ be the endomorphism of TM given by the Riemann tensor of M. The Gauss formula says that:

$$R(X, Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X + L(X, Z)S(Y) - L(Y, Z)S(X),$$

from which we get, if θ is a 1-form:

$$R(X, Y)\theta = \theta(X)Y^\star - \theta(Y)X^\star + \theta(S(X))S(Y)^\star - \theta(S(Y))S(X)^\star,$$

where $\star$ denotes the dual 1-form of the given tangent vector.
Now $R(X, Y)$ acts on $\Lambda^p(TM)$ as a derivation. We extend the above formula by derivation and obtain, for any p-form ω:

$$R(X, Y)\omega = Y^\star \wedge i_X\omega - X^\star \wedge i_Y\omega + S(Y)^\star \wedge i_{S(X)}\omega - S(X)^\star \wedge i_{S(Y)}\omega. \tag{11}$$

We now fix the point $x \in M$ and let $(E_1, \dots, E_n)$ be an orthonormal basis of principal directions of M at x. This means that $S(E_j) = \eta_j E_j$ for all j, where η_j is the associated principal curvature. Assume that ω is a parallel p-form. Then we have, for all i, j:

$$R(E_i, E_j)\omega = 0. \tag{12}$$

On the other hand, by (11):

$$R(E_i, E_j)\omega = (1 + \eta_i\eta_j)\Phi_{ij}, \tag{13}$$

where Φ_{ij} is the p-form:

$$\Phi_{ij} = E_j^\star \wedge i_{E_i}\omega - E_i^\star \wedge i_{E_j}\omega.$$

By (12) and (13) we see that:

$$(1+\eta_i\eta_j)\Phi_{ij} = 0 \tag{14}$$

for all $i, j = 1, \dots, n$.

We want to show that, if $i \leq p$ and $j \geq p+1$, then Φ_{ij} is a non-zero p-form at x.

In fact, as ω is parallel, and non-trivial, it is non-zero at x. By re-ordering, if necessary, we can assume that $\omega(E_1, \dots, E_p) \neq 0$. If $i \leq p$ and $j \geq p+1$, one has:

$$\begin{cases} E_j^\star \wedge i_{E_i}\omega(E_1, \dots, \hat{E}_i, \dots, E_p, E_j) = \pm\omega(E_1, \dots, E_p) \\ E_i^\star \wedge i_{E_j}\omega(E_1, \dots, \hat{E}_i, \dots, E_p, E_j) = 0 \end{cases}$$

so that:

$$\Phi_{ij}(E_1, \dots, \hat{E}_i, \dots, E_p, E_j) = \pm\omega(E_1, \dots, E_p) \neq 0,$$

as asserted.
We can now prove the final assertion. For all $i \leq p$ and $j \geq p+1$ the form Φ_{ij} is non-zero and then, by (14):

$$1 + \eta_i\eta_j = 0;$$

setting $\eta_1 = k(x)$ we see that $\eta_j = -\frac{1}{k(x)}$ for all $j \geq p+1$; this in turn implies that $\eta_i = k(x)$ for all $i \leq p$. $\square$

Corollary 10. *Let M be a compact, minimal hypersurface of the unit sphere $\mathbf{S}^{n+1}$ supporting a parallel p-form for some $p = 1, \dots, n-1$. Then M is the Clifford torus $\mathrm{CL}_{n,p}$.*

Proof. At each point the minimality gives:

$$pk(x) - \frac{n-p}{k(x)} = 0,$$

and then $k(x) = \pm\sqrt{\frac{n-p}{p}}$ for all x. If $|S|^2$ denotes the squared norm of the second fundamental form, we then have $|S|^2 = n$ at all points. By a well-known result of Chern, do Carmo and Kobayashi [CCK] we get that M is a Clifford torus. But the only Clifford torus supporting a parallel (hence harmonic) p-form is $\mathrm{CL}_{n,p}$ and the assertion follows. $\square$

2.4 Proof of Theorem 2

The inequality follows immediately from Theorem 1, so we only need to prove the equality case. Assume $p = n/2$. If equality holds, then M is minimal in $\mathbf{S}^{n+1}$ and has a non-trivial parallel (hence harmonic) p-form. In particular, $\lambda_1(\Delta_p) = 0$ and $\lambda_2(\Delta_p) = n$. By Corollary 10, $M = \mathrm{CL}_{n,p}$. But it is immediate to see that, for $p = n/2$, $\mathrm{CL}_{n,p}$ has p-th Betti number equal to 2, hence $\lambda_2(\Delta_p) = 0$, which is a contradiction. Then equality never holds.

If $p \neq n/2$ and equality holds, then again M must be isometric to $\mathrm{CL}_{n,p}$. Conversely, if $M = \mathrm{CL}_{n,p}$, a direct calculation using the Künneth formula shows that the first positive eigenvalue satisfies $\bar{\lambda}_1(\Delta_p) = n$ (see Section 4); as M has $p-$th Betti number equal to 1, we have $\lambda_1(\Delta_p) = 0$ and so $\lambda_2(\Delta_p) - \lambda_1(\Delta_p) = \bar{\lambda}_1(\Delta_p) = n$ and the gap is indeed equal to n. Hence equality is attained if and only if $M = \mathrm{CL}_{n,p}$.

2.5 Proof of Theorem 3

The case $p \neq n/2$ is a particular case of Theorem 2. In fact, if $b_p(M) = 1$, then

$$\bar{\lambda}_1(\Delta_p) = \lambda_2(\Delta_p) - \lambda_1(\Delta_p) \leq n,$$

with equality if and only if M is the Clifford torus $\mathrm{CL}_{n,p}$.

Now assume $p = n/2$, p is odd and $b_p(M) = 2$. The space $H^p(M)$ of harmonic $p-$forms is then $\gamma-$invariant and two-dimensional. As $\gamma^2 = I$, a standard averaging argument shows that there exists a non-trivial harmonic $p-$form ξ which is either even or odd with respect to γ: note that then $|\xi|^2$ will be an even function on M. The Hodge $\star$ operator acts isometrically on $H^p(M)$ and, since p is odd, it satisfies $\star^2 = -I$; so, $\star\xi$ is pointwise orthogonal to ξ and $(\xi, \star\xi)$ is an orthonormal basis of $H^p(M)$.

For any parallel vector field $\bar{V}$ on $\mathbf{R}^{n+2}$, we consider the function $\langle \bar{V}, \nu \rangle$ and the test $p-$form

$$\phi_{\bar{V}} = \langle \bar{V}, \nu \rangle \xi.$$

By our assumptions, $\langle \bar{V}, \nu \rangle |\xi|^2$ is an odd function on M, hence $\phi_{\bar{V}}$ is L^2-orthogonal to ξ. On the other hand, $\phi_{\bar{V}}$ is pointwise (hence L^2) orthogonal to $\star\xi$. So, $\phi_{\bar{V}}$ is orthogonal to $H^p(M)$ for all $\bar{V}$. We apply the min-max principle to $\phi_{\bar{V}}$ and, proceeding as in the proof of Theorem 1, we obtain

$$\bar{\lambda}_1(\Delta_p) \int_M \langle \bar{V}, \nu \rangle^2 |\xi|^2 \leq \int_M |V|^2 |\xi|^2,$$

for all $\bar{V}$. After integrating with respect to $\bar{V}$, we arrive at

$$\bar{\lambda}_1(\Delta_p) \leq n,$$

and, if equality holds, then M is minimal and the form ξ must be parallel. By Corollary 10, equality implies that M must be the Clifford torus $\mathrm{CL}_{n,p}$. On the other hand, the calculation in the Appendix proves that $\mathrm{CL}_{n,p}$ does satisfy $\bar{\lambda}_1(\Delta_p) = n$. The proof is now complete.

3 Estimates in the general case

Let $M = (M, g)$ be a Riemannian manifold with a free isometric action by the finite group G. We choose a distance function d on M, assumed invariant under the action of G. The natural choice for d is the usual *intrinsic distance* d_{int} coming from the Riemannian structure. However, when M is a submanifold isometrically immersed in a larger manifold $\bar{M}$, we will consider also the *extrinsic distance* d_{ext} on M inherited from the Riemannian distance on $\bar{M}$. Let us be a bit more precise about that. If $\phi : M \to \bar{M}$ is an isometric immersion, define, for $x, y \in M$:

$$d_{\text{ext}}(x, y) = d_{\text{int}}(\phi(x), \phi(y)).$$

If ϕ is an embedding then d_{ext} is indeed a distance. In general, d_{ext} will only be a *pseudo-distance*, that is, d_{ext} is symmetric and satisfies the triangle inequality (it might happen in fact that $d_{\text{ext}}(x, y) = 0$ for $x \neq y$). However d_{ext} is dominated by the intrinsic distance on M:

$$d_{\text{ext}}(x, y) \leq d_{\text{int}}(x, y)$$

for all $x, y \in M$. This fact and the triangle inequality imply that, for any $x \in M$, the function $y \to d_{\text{ext}}(x, y)$ is locally Lipschitz and $|\nabla d_{\text{ext}}(x, \cdot)| \leq 1$.

In both cases, then, the (pseudo)-distance chosen is locally Lipschitz, and has Lipschitz constant bounded by 1. This is what we need in the proof of our result.

With the chosen (intrinsic or extrinsic) distance d, we consider the triple (M, G, d) and let

$$\beta(G, d) = \inf_{x \in M} \{d(x, \gamma \cdot x) : \gamma \in G, \gamma \neq 1\},$$

The invariant β will be called the *smallest displacement* of the group action.

Our results will be stated in terms of two metric invariants depending only on d.

Definition 11. *For $r > 0$, define $C_r(M, d)$ to be the minimal number of balls of radius r in (M, d) needed to cover a ball of radius $2r$. Then $C_r(M, d)$ is finite for all r. We will set:*

$$C_\infty(M, d) = \sup_{r > 0} C_r(M, d), \tag{15}$$

assuming that it is finite, and call it the packing constant *of the pair* (M,d)*. We can also define a* local packing constant, *as follows:*

$$C_1(M,d) = \sup_{0<r\leq 1} C_r(M,d), \tag{16}$$

With these notations, we have

Theorem 12. *Assume that* (M,g) *admits a free isometric action by the finite group* G *and that the distance* d *is* G*-invariant. Let* D *be any* G*-invariant Laplace-type operator on* M*. Then:*

$$\lambda_p(D) - \lambda_1(D) \leq \frac{16C_\infty(M,d)}{\beta(G,d)^2}$$

and also

$$\lambda_p(D) - \lambda_1(D) \leq \frac{16C_1(M,d)}{\min\{\beta(G,d)^2, 1\}}.$$

Observe that the upper bound depends only on the smallest displacement of the action of G and the packing constant, both of which are purely metric invariants. Then, the upper bound is purely metric: it does not depend on the curvatures of (M,g), or the vector bundle E defining the operator D; in particular, it does not depend on the potential T of D, as long as it is $G-$invariant.

Before proving the theorem, we give a few estimates of the packing constants.

- Assume that $\mathrm{Ric}_{\mathrm{M^n}} \geq 0$. Then $C_\infty(M, d_{\mathrm{int}}) \leq c(n)$, a constant depending only on n. It turns out that we can take $c(n) = 16^n$ (basically, the packing constant of $\mathbf{R}^n$: see for example [Zu] Lemma 3.6 p. 230). Consequently Theorem 12 gives the estimate:

$$\lambda_p(D) - \lambda_1(D) \leq \frac{16^{n+1}}{\beta(G,d_{\mathrm{int}})^2}.$$

- If we only assume $\mathrm{Ric}_{M^n} \geq -(n-1)K$, then we have: $C_1(M, d_{\mathrm{int}}) \leq c(n,K)$, where $c(n,K)$ is the packing constant of the hyperbolic space n-space with curvature $-K$, which can be computed in terms of the hyperbolic functions. Up to a numerical constant, $c(n,K)$ behaves like $2^{2(n-1)}e^{(n+1)\frac{7K}{4}}$ (see [Zu]). Therefore:

$$\lambda_p(D) - \lambda_1(D) \leq \frac{16c(n,K)}{\max\{\beta(G,d_{\mathrm{int}})^2, 1\}}.$$

- If M^n is a submanifold of $\bar{M}^N$, and one chooses the extrinsic distance, using the triangle inequality one shows easily that $C_\infty(M, d_{\mathrm{ext}}) \leq C_\infty(\bar{M}, d_{\mathrm{int}})^2$. In particular, if the ambient manifold is $\mathbf{R}^N$ with its Euclidean distance, then we have $C_\infty(M, d_{\mathrm{ext}}) \leq 16^{2N}$ and we obtain

$$\lambda_p(D) - \lambda_1(D) \leq \frac{16^{2N+1}}{\beta(G,d_{\mathrm{ext}})^2}.$$

3.1 Proof of Theorem 12

We prove the theorem for the packing constant $C = C_\infty(M, d)$; the proof for the other constant $C_1(M, d)$ is basically the same, and we omit it.
We fix a unit L^2-norm eigensection ψ associated to the first eigenvalue $\lambda_1(D)$. If f is a Lipschitz function then, by (9), the Rayleigh quotient of the section $f\psi$ is

$$R(f\psi) = \lambda_1(D) + \frac{\int_M |\nabla f|^2 |\psi|^2}{\int_M f^2 |\psi|^2}. \tag{17}$$

So, we can control the gap $\lambda_p(D) - \lambda_1(D)$ using test-sections of type $f\psi$.

We proceed as follows. Consider the measure $\mu = |\psi|^2 d\mathrm{vol}_g$, so that

$$\mu(A) = \int_A |\psi|^2.$$

Let $\beta = \beta(G, d)$ be the smallest displacement of the $G-$action and set

$$\alpha = \sup_{x \in M} \left\{ \mu(B(x, \frac{\beta}{4})) \right\}.$$

Fix a positive number $\epsilon < \alpha$. Then, there exists $x_1 \in M$ such that

$$\mu(B(x_1, \frac{\beta}{4})) \geq \alpha - \epsilon \geq \mu(B(x, \frac{\beta}{4})) - \epsilon$$

for all $x \in M$. By the definition of the packing constant C, any ball of radius $\beta/2$ is covered by C balls of radius $\beta/4$. By the property of x_1 we then get

$$\mu(B(x_1, \frac{\beta}{2})) \leq C\mu(B(x_1, \frac{\beta}{4})) + C\epsilon \tag{18}$$

Define the plateau function, depending only on the distance to x_1:

$$f_1 = \begin{cases} 1 \quad \text{on } B(x_1, \frac{\beta}{4}), \\ 0 \quad \text{on the complement of } B(x_1, \frac{\beta}{2}), \\ \text{linear on the annulus } B(x_1, \frac{\beta}{2}) \setminus B(x_1, \frac{\beta}{4}). \end{cases}$$

As the distance function d is Lipschitz, f_1 is Lipschitz; moreover f_1 is supported on $B(x_1, \frac{\beta}{2})$ and $|\nabla f_1|$ is bounded above by $4/\beta$ because $|\nabla d| \leq 1$. Then, by (18):

$$\int_M |\nabla f_1|^2 |\psi|^2 \leq \frac{16}{\beta^2} \mu(B(x_1, \frac{\beta}{2})) \leq \frac{16C}{\beta^2} \mu(B(x_1, \frac{\beta}{4})) + \frac{16C\epsilon}{\beta^2}$$

On the other hand, one has immediately $\int_M {f_1}^2|\psi|^2 \geq \mu(B(x_1, \frac{\beta}{4}))$. From (17) we get

$$R(f_1\psi) \leq \lambda_1(D) + \frac{16C}{\beta^2} + \frac{16C\epsilon}{\beta^2(\alpha - \epsilon)}$$

Write $G = \{\gamma_1 = 1, \gamma_2, ..., \gamma_p\}$ and consider the sections:

$$\phi_1 = f_1\psi_1,\ \phi_2 = \gamma_2 \cdot \phi_1, \ldots, \ \phi_p = \gamma_p \cdot \phi_1.$$

By the definition of β, they are disjointly supported because the distance is $G-$invariant. As G acts by isometries on the vector bundle, they have the same Rayleigh quotient, hence

$$R(\phi_j) = R(\phi_1) \leq \lambda_1(D) + \frac{16C}{\beta^2} + \frac{16C\epsilon}{\beta^2(\alpha - \epsilon)}$$

for all $j = 1, \ldots, p$. A standard min-max argument now shows that

$$\lambda_p(D) \leq \lambda_1(D) + \frac{16C}{\beta^2} + \frac{16C\epsilon}{\beta^2(\alpha - \epsilon)}.$$

We let $\epsilon \to 0$ and get the assertion.

4 Appendix: spectrum of Clifford tori

Proposition 13. *Let* $M = \mathbf{S}^p\left(\sqrt{\frac{p}{n}}\right), N = \mathbf{S}^{n-p}\left(\sqrt{\frac{n-p}{n}}\right)$ *and let* $\bar{\lambda}_1(\Delta_p)$ *be the first positive eigenvalue of the Laplacian acting on* $p-$*forms of the Riemannian product* $M \times N = \mathrm{CL}_{n,p}$. *Then*

$$\bar{\lambda}_1(\Delta_p) = n.$$

For the proof, we first assume $p \neq n/2$.

Let Spec_j denote the spectrum of the Laplacian acting on j-forms. The Künneth formula says that:

$$\mathrm{Spec}_p(M \times N) = \cup_{j=0}^{p} \left(\mathrm{Spec}_j(M) + \mathrm{Spec}_{p-j}(N)\right).$$

By Hodge duality we can assume that $p \leq n - p$; since we are in the case $p \neq n/2$ we can then assume $p < n - p$. Then

$$\mathrm{Spec}_p(M \times N) = A \cup B \cup C, \tag{19}$$

where

$$\begin{cases} A = \mathrm{Spec}_0(M) + \mathrm{Spec}_p(N) \\ B = \mathrm{Spec}_p(M) + \mathrm{Spec}_0(N) \\ C = \cup_{j=1}^{p-1} \left(\mathrm{Spec}_j(M) + \mathrm{Spec}_{p-j}(N)\right) \end{cases} \tag{20}$$

Now

$$\mathrm{Spec}_0\left(\mathbf{S}^m(a)\right) = \frac{1}{a^2}\mathrm{Spec}_0(\mathbf{S}^m) = \frac{1}{a^2}\{0, m, \dots\}.$$

Since $M = \mathbf{S}^p\left(\sqrt{\frac{p}{n}}\right)$ is $p-$dimensional we have

$$\mathrm{Spec}_0(M) = \mathrm{Spec}_p(M) = \{0, n, \dots\}.$$

Similarly one gets $\mathrm{Spec}_0(N) = \{0, n, \dots\}$ and then, if λ_j denotes the $j-$th eigenvalue of the indicated set:

$$\lambda_1(B) = 0, \quad \lambda_2(B) = n. \tag{21}$$

Now, it is known that, if $j \neq 0, m$, then the first eigenvalue of $\mathrm{Spec}_j(\mathbf{S}^m)$ is (see for example [IT]):

$$\lambda_1(\mathrm{Spec}_j(\mathbf{S}^m)) = \min\{j(m-j+1), (j+1)(m-j)\}.$$

An easy argument shows that the right-hand side is bounded below by m for all $j = 1, \dots, m-1$. We then have, in that range of j:

$$\lambda_1\left(\mathrm{Spec}_j(\mathbf{S}^m(a))\right) \geq m/a^2. \tag{22}$$

By the definition of N and (22) one has $\lambda_1(\mathrm{Spec}_p(N)) \geq n$ which implies that

$$\lambda_1(A) \geq n. \tag{23}$$

Finally, if $j < p$ we get $\lambda_1(\mathrm{Spec}_j(M)) \geq n$; since $p < n-p$ we also have $p-j < n-p$ and so $\lambda_1(\mathrm{Spec}_{p-j}(N)) \geq n$. Therefore

$$\lambda_1(C) \geq 2n. \tag{24}$$

Collecting (19), (21), (23), (24) we obtain

$$\lambda_1(\mathrm{Spec}_p(M \times N)) = 0, \quad \lambda_2(\mathrm{Spec}_p(M \times N)) = n$$

and the first positive eigenvalue is indeed equal to n.

We now assume $p = n/2$. In that case we have $M = N$; the first three eigenvalues of $A \cup B$ are $0, 0, n$, and the eigenvalues of C are bounded below by $2n$. The assertion follows as well.

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