EIGENVALUE ESTIMATE FOR THE ROUGH LAPLACIAN ON DIFFERENTIAL FORMS

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ABSTRACT. In this article, we study the spectrum of the rough Laplacian acting on differential forms on a compact Riemannian manifold (M, g). We first construct on M metrics of volume 1 whose spectrum is as large as desired. Then, provided that the Ricci curvature of g is bounded below, we relate the spectrum of the rough Laplacian on 1–forms to the spectrum of the Laplacian on functions, and derive some upper bound in agreement with the asymptotic Weyl law.

1. INTRODUCTION

Let (M^n, g) be a compact and connected Riemannian manifold and ∇ the Levi-Civita connection associated to the metric g. We consider the rough Laplacian $\Delta = \nabla^*\nabla$ acting on differential p–forms. If $\partial M \neq \emptyset$, we add a boundary condition which is analogous to the Neumann boundary condition for functions, namely the normal derivative $\nabla_\nu \omega$ has to be zero at the boundary ∂M. In Section 4, we show in detail and in a more general context (i.e. on any vector bundle) that, with this boundary condition, the rough Laplacian is an order 2 elliptic operator and that its spectrum is an unbounded sequence of real numbers $(\lambda_k)_{k\in\mathbb{N}}$ which can be increasingly ordered

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots \nearrow +\infty \quad ,$$

where λ_0 denotes the zero eigenvalue (which exists only when (M^n, g) has non zero parallel p–forms) with its multiplicity ($=\dim \operatorname{Ker} \nabla$). In case where $\dim \operatorname{Ker} \nabla = 0$, the spectrum conventionally starts with $\lambda_1 > 0$.

Despite its great importance in mathematics and in mathematical physics via the Laplace type operators (Dirac Laplacian, Hodge Laplacian on p–forms, Schrödinger operators) and the associated Bochner–Weitzenböck formulas (cf. section 1.I. of [4] for instance), only a few general information are known about the rough Laplacian, see however [1, 3, 6, 11, 13, 17].

A first natural question so as to understand the spectrum of an operator is to see if it is possible to construct small or large eigenvalues under some usual geometric constraints, like fixed volume or bound on the curvature. Let us recall a few known facts in the classical cases of the Laplacian acting on functions of a closed Riemannian manifold.
What concern the upper bounds on the spectrum, in dimension higher than 2, a normalization of the volume is not enough to control the spectrum of the Laplacian acting on functions: indeed in virtue of the result of Colbois and Dodziuk [7], one can find a metric of given volume, with arbitrary large first non–zero eigenvalue. On the contrary, the knowledge of the volume is enough to give upper bound on the spectrum of a surface depending on its genus (see [16] and the introduction of [8] for other results of this type).
As regards the lower bounds on the spectrum, the classical example of the Cheeger dumbbell shows that restriction on the volume is not enough to get a lower bound on the first non–zero eigenvalue. If we add a lower bound on the Ricci curvature to the constraint on the volume, then there exists upper bound for the spectrum of functions with respect to the volume [5, 18] and some lower bound with respect to the diameter [18].

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The first main result of this article is that on any compact and connected manifold of dimension $n \geq 2$, it is possible to construct large eigenvalues for the rough Laplacian acting on differential p–forms ($1 \leq p \leq n-1$) for a metric of prescribed volume.

Theorem 1.1. *Every compact and connected manifold M (possibly with boundary) of dimension $n \geq 2$ admits metrics g of volume one with arbitrarily large first non–zero eigenvalue λ_1 for the rough Laplacian on p-forms, $1 \leq p \leq n-1$.*

In order to prove this result, we will construct a family of metrics $(\tilde{g}_N)_{N\in\mathbb{N}^*}$[1] of volume 1 that do not admit parallel p–forms (in other words the spectrum of these metrics starts with $\lambda_1(\tilde{g}_N) > 0$) and such that $\lim_{N\to+\infty} \lambda_1(\tilde{g}_N) = +\infty$. Another very interesting feature of this sequence of metrics is that the spectrum of the Laplacian on functions (and, as a consequence, on 1–forms) collapses (if $n \geq 3$) or is bounded (if $n = 2$) as $N \to +\infty$ (cf. Remark 2.2). In particular, this singles out an essential and maybe unexpected spectral difference between the two Laplacian type operators. It shows that, without assuming the Ricci curvature is bounded below, controlling the spectrum of the Laplace Beltrami operator is not enough to bound from above the spectrum of the rough Laplacian on 1–forms.
Almost the same arguments we use in the proof of Theorem 1.1 allow to construct small eigenvalues for 1–forms.

Theorem 1.2. *Given $N \in \mathbb{N}^*$ and $\varepsilon > 0$, every compact and connected manifold M (possibly with boundary) of dimension $n \geq 2$ admits metrics $g_{N,\varepsilon}$ of volume one with $\lambda_N \leq \varepsilon$ for the rough Laplacian on 1-forms.*

The metrics we used for the proof of Theorem 1.1 have curvature *not* bounded below. This leads to the conclusion that a natural assumption in order to bound from above the spectrum of the rough Laplacian, is to suppose that the Ricci curvature is bounded below. Under this condition on the Ricci curvature, the spectrum of the Laplacian acting on functions, with Neumann boundary conditions if $\partial M \neq \emptyset$, that will be denoted by

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \leq \mu_{k+1} \leq \cdots \nearrow +\infty \quad ,$$

is known to be bounded from above, see [10]. The second main result of this note relates the spectrum of the rough Laplacian on 1-forms with the spectrum of the Laplacian on functions, and, as a consequence, allows to get upper bounds.

Theorem 1.3. *Let (M^n, g) be a compact and connected manifold with Ricci curvature bounded below $\mathrm{Ric}^g \geq -(n-1)a^2$, $a \geq 0$.*

(i) *If M is closed $(\partial M = \emptyset)$ then for any $k \in \mathbb{N}^*$ we have*

$$\lambda_k(M,g) \leq a^2(n-1) + \mu_{k+1}(M,g) \, .$$

(ii) *If M has a compact and convex boundary in the sense that ℓ the second fundamental form of ∂M is nonnegative, then for any $k \in \mathbb{N}^*$ we have*

$$\lambda_k(M,g) \leq a^2(n-1) + \mu_{k+1}(M,g) \, .$$

In both case there exist geometric constants $A_n, B_n > 0$ (only depending upon the dimension) such that for any $k \in \mathbb{N}^*$

$$\lambda_k(M,g) \leq \Big((n-1) + A_n\Big)a^2 + B_n\left(\frac{k+1}{V}\right)^{2/n} \quad ,$$

where $V = \mathrm{Vol}(M,g)$. In particular, we recover the Weyl asymptotic law.

The article is organized as follows: In Section 2, we give the proof of Theorem 1.1 and Theorem 1.2; Section 3 is devoted of the proof of Theorem 1.3; Finally the last section is an appendix which provides the detailed proof of the spectral theorem for the rough Laplacian acting on any vector bundle (also in case where the boundary $\partial M \neq \emptyset$).

[1] $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ in the whole article.

The first goal of this section is the construction of a metric of volume 1 with large first eigenvalue on any compact connected manifold. We do this in the same spirit of what Gentile and Pagliara did in [15], but with a slightly different deformation.
Before to begin with the construction, let us point out that we use regularly the same fact along the construction: if we take a domain U with smooth boundary in a manifold M, we can consider the boundary problem

$$\begin{cases} \nabla^*\nabla\omega = \lambda\omega & \text{on } U \\ \nabla_\nu\omega = 0 & \text{on } \partial U\,. \end{cases}$$

If we know a priori that 0 is not on the spectrum, this means that each p–form ω (which is not identically 0 on U) has a Rayleigh quotient

$$R(\omega) = \frac{\int_U |\nabla\omega|^2 \,\mathrm{dVol}_g}{\int_U |\omega|^2 \,\mathrm{dVol}_g} \geq \lambda_1(U) > 0\,.$$

A situation where we know a priori that 0 is not on the spectrum is if U is connected and has positive constant sectional curvature in an open subset. To see this, we use the classical Weizenböck formula (see for example [14], p.262)

$$\langle\Delta\omega,\omega\rangle = \frac{1}{2}\Delta(|\omega|^2) + |\nabla\omega|^2 + F(\omega)$$

where F denotes the curvature tensor acting on p–forms.

If 0 is on the spectrum of U, there exists a parallel p–form ω, that is $|\nabla\omega| = 0$. This implies that $|\omega|$ is constant, so that $\Delta(|\omega|) = 0$ and because of Lemma 6.8 in [14], $\Delta\omega = 0$. But this is impossible, because on the subset where the sectional curvature is positive and constant, the curvature tensor F is also positive, and we cannot have $F(\omega) = 0$.

We can now begin with our construction for the proof of Theorem 1.1.

Proof of Theorem 1.1. First, we consider a piece of the type $C_0 = [0,5] \times \mathbb{S}^{n-1}$ with an adapted warped metric g_0. We will then glue $N \in \mathbb{N}^*$ of these pieces together to get a long periodic cylinder. We will close it as in [15] with an hemisphere and consider a connected sum of it with M.
More explicitly, if we denote by (r,x) a point of C_0, the metric g_0 is such that for $r \in [0,1]$ or $r \in [4,5]$, we get the Riemannian product of the interval with the sphere of radius 1 in $\mathbb{R}^n$. This implies that the boundary of (C_0, g_0) is totally geodesic.
For $r \in [2,3] \times \mathbb{S}^{n-1}$, the metric g_0 is such that we get a part isometric to

$$\left\{ (r,y) \in [2,3] \times \mathbb{R}^n \,\middle|\, \left(r - \frac{5}{2}\right)^2 + \|y\|^2 = 1 \right\}\,,$$

namely to a part of the round sphere of radius 1 in $\mathbb{R}^{n+1}$. In this part, the sectional curvature of (C_0, g_0) is 1, and this will be useful in the sequel.

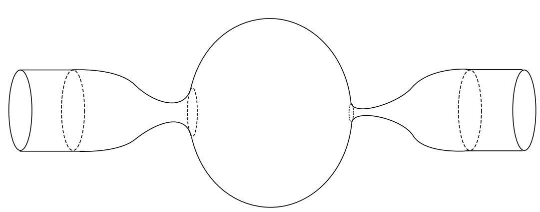

FIGURE 1. The piece (C_0, g_0).

For $r \in [1,2]$ or $r \in [3,4]$, we just choose the metric g_0 to be globally smooth on C_0.
Then, we glue in an obvious way N copies of (C_0, g_0) and we get a cylinder with a periodic metric denoted by C_N. The volume of C_N is proportional to N. We close C_N in one side with an hemisphere H of a round sphere of radius 1 and get a Riemannian manifold denoted by D_N, which is topologically a ball

of dimension n. We make a connected sum with the manifold M we are interested in, and get a resulting manifold M_N diffeomorphic to M with a submanifold Ω naturally identified with D_N. We fix a Riemannian metric g_N on M_N which coincides on Ω with the metric already constructed on D_N, and which is fixed on $M \setminus \Omega$ (and so it is not depending on N). Moreover, on a fixed open set of $M \setminus \Omega$, we compel the metric g_N to have sectional curvature 1.
Clearly, the volume of $M_N := (M, g_N)$ grows proportionally to N, and to conclude, it remains to show that $\lambda_1(M_N)$ is uniformly bounded below by a positive constant A. After renormalization to have a volume 1 metric, the first non–zero eigenvalue will increase to ∞ with N. The idea is the following: we see M_N as the union of its two extremities $M \setminus \Omega$ and H, with the N fundamental pieces (C_0, g_0). We show that any smooth p–form ω on M_N has a Rayleigh quotient $R(\omega) = \frac{\int_M |\nabla\omega|^2 \,\mathrm{dVol}_{g_N}}{\int_M |\omega|^2 \,\mathrm{dVol}_{g_N}}$ uniformly (that is to say independently of N) bounded below on each of these parts, which implies a uniform lower bound on the whole $M_N = (M, g_N)$.

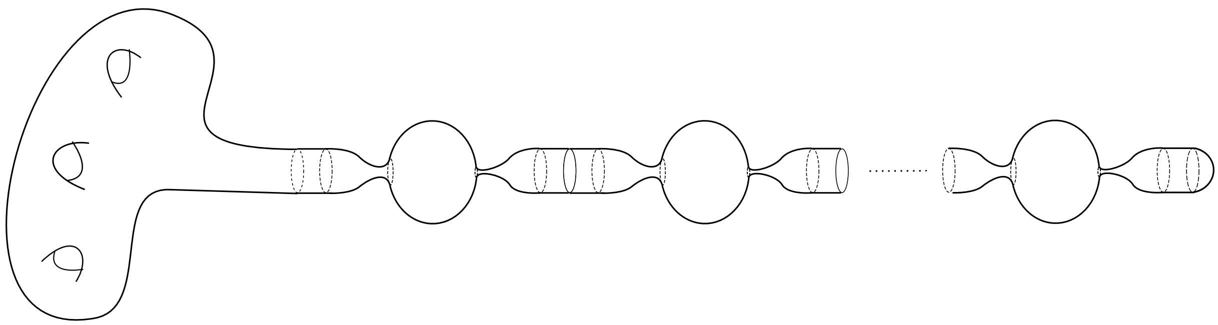

FIGURE 2. The manifold M_N.

We now need a lemma:

Lemma 2.1. *There exists a positive constant A_1 such that for any smooth p–form ω on (C_0, g_0), we have*

$$\int_{C_0} |\nabla\omega|^2 \,\mathrm{dVol}_{g_0} \geq A_1 \int_{C_0} |\omega|^2 \,\mathrm{dVol}_{g_0} \tag{2.1}$$

Proof. On (C_0, g_0), we consider the rough Laplacian $\Delta = \nabla^*\nabla$ acting on differential p–forms, with the Neumann condition boundary condition.
As observed, the presence of a portion of (C_0, g_0) with constant curvature 1 insures the first eigenvalue of Δ is strictly positive, say greater or equal to some constant $A_1 > 0$.
The form ω we consider may be seen as a test form for the Rayleigh quotient, and $R(\omega)$ has to be greater than A_1. ∎

Now, we can conclude the proof Theorem 1.1: let ω be any smooth p–form on M_N. We have

$$\int_{C_N} |\nabla\omega|^2 \,\mathrm{dVol}_{g_N} \geq A_1 \int_{C_N} |\omega|^2 \,\mathrm{dVol}_{g_N}, \tag{2.2}$$

because this is true by Lemma 2.1 for all components C_0 of C_N.
Moreover, the same is true, with a positive constant A_2 on the hemisphere H which is of constant curvature 1, and it is also true on $(M \setminus \Omega, g)$ with a constant $A_3 > 0$: the reason is that g_N is fixed on $M \setminus \Omega$ and has a sectional curvature 1 on an open set, so that parallel p–forms cannot exist.
We conclude by choosing $A = \min(A_1, A_2, A_3) > 0$: we have shown that for each smooth ω we have $R(\omega) \geq A$ which implies for any $N \in \mathbb{N}^*$, $\lambda_1(M_N, g) \geq A > 0$ by Theorem 4.5. ∎

Remark 2.2. *Note that the family of metrics $(M, g_N)_{N\in\mathbb{N}^*}$ has small eigenvalues for functions (and so for the Laplace-Beltrami operator on 1–forms) if $n \geq 3$ and that the eigenvalues have uniform upper bounds if $n = 2$. Let us take some $k \in \mathbb{N}^*$ and consider the family of metrics $(M, g_{kN})_{N\in\mathbb{N}^*}$. We have* $\mathrm{Vol}(M, g_{kN}) = V_0 + kN\,\mathrm{Vol}(C_0, g_0)$

for a certain positive V_0*. We denote by* f_k *the plateau function which is* 0 *outside the* k*–th piece* C_N *and goes linearly to* 1 *inside the* k*–th* C_N*, and we use the family* $(f_k)_k$ *as test functions for the Rayleigh quotient:*

$$R(f_k) \leq \frac{2\,\mathrm{Vol}(\mathbb{S}^{n-1})}{N\,\mathrm{Vol}(C_0, g_0)} ,$$

and since $(f_k)_k$ *is an* H^1*–orthogonal family (as functions with disjoint support), the MinMax principle asserts that*

$$\mu_k(g_{kN}) \leq \frac{2\,\mathrm{Vol}(\mathbb{S}^{n-1})}{N\,\mathrm{Vol}(C_0, g_0)} .$$

Now if we denote by $(M, \tilde{g}_{kN})_{N\in\mathbb{N}^*}$ *the family of normalized metrics with volume* 1*, it follows that*

$$\mu_k(\tilde{g}_{kN}) \leq \frac{2\,\mathrm{Vol}(\mathbb{S}^{n-1})\Big(V_0 + kN\,\mathrm{Vol}(C_0, g_0)\Big)^{\frac{2}{n}}}{N\,\mathrm{Vol}(C_0, g_0)} \underset{N\to\infty}{\sim} 2\,\mathrm{Vol}(\mathbb{S}^{n-1})\,\mathrm{Vol}(C_0, g_0)^{\frac{2}{n}-1} k^{\frac{2}{n}} N^{\frac{2}{n}-1} ,$$

which implies $\lim_{N\to\infty} \mu_k(\tilde{g}_{kN}) = 0$ *for any* $k \in \mathbb{N}^*$ *and for any dimension* $n \geq 3$*. In dimension* 2*, this limit is a priori only bounded in* N*. This behavior points out the essential difference between the spectrum of the rough Laplacian and the spectrum of the Laplacian on functions, since we have proved above that* $\lim_{N\to\infty} \lambda_k(\tilde{g}_{kN}) = +\infty$ *for any* $k \in \mathbb{N}^*$ *and for any dimension* $n \geq 2$*.*

Proof of Theorem 1.2. As in Theorem 1.1, we will consider the connected sum of a ball with the given manifold M endowed with a metric of volume $\frac{1}{2}$, but the construction is much simple. We just have to consider a Riemannian product which is a cylinder $C_L = [0, L] \times \mathbb{S}^{n-1}_{1/L}$, where $\mathbb{S}^{n-1}_{1/L}$ is the sphere of radius $1/L$ in $\mathbb{R}^{n-1}$ that we close at one side with an hemisphere of the round sphere $\mathbb{S}^n_{1/L}$ of radius $1/L$. As L is large, the volume of C_L goes to zero, and so is less that $\frac{1}{2}$, so that we get a family of manifolds $(M_L)_L$ with volume less than 1.

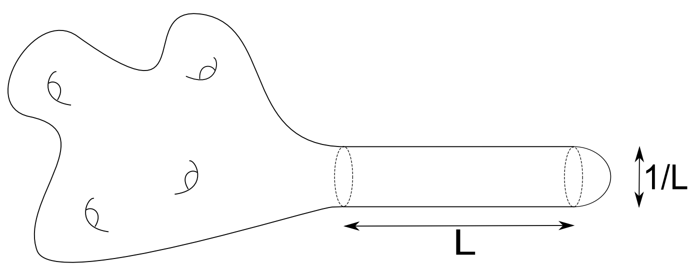

FIGURE 3. M_L the connected sum of M with a slim long nose.

To construct small eigenvalues on M_L, it suffices to construct $N \in \mathbb{N}^*$ disjointly supported test 1–form denoted by $\omega_1, ..., \omega_N$ with Rayleigh quotient $R(\omega_i) \leq \varepsilon$ where $\varepsilon > 0$ is a small fixed positive real. So let us fix $N \in \mathbb{N}^*$ and $\varepsilon > 0$.

In order to construct our test 1–forms, we denote by $g_L = \mathrm{d}r^2 + g_{\mathbb{S}^{n-1}_{1/L}}$ the product metric on the cylinder $C_L = [0, L] \times \mathbb{S}^{n-1}_{1/L} \ni (r, x)$, and we consider the regular subdivision of step $\frac{L}{N}$ of $[0, L]$ by setting $I_k = \big[(k-1)\frac{L}{N}, k\frac{L}{N}\big]$ for any $k \in \{1, 2, \cdots, N\}$. Now we set for any k

$$f_k = \begin{cases} \sin\left(\frac{2\pi N}{L}\left[r - (k-1)\frac{L}{N}\right]\right) & \text{on } I_k \times \mathbb{S}^n_{1/L} \\ 0 & \text{on } M_L \smallsetminus (I_k \times \mathbb{S}^n_{1/L}) \end{cases} ,$$

and also

$$\omega_k := \begin{cases} f_k \mathrm{d}r & \text{on } I_k \times \mathbb{S}^n_{1/L} \\ 0 & \text{on } M_L \smallsetminus (I_k \times \mathbb{S}^n_{1/L}) \end{cases} .$$

The family of 1–forms $\omega_1, ..., \omega_N$ is disjointly supported (and so is an H^1–orthogonal family) and satisfies (since $\mathrm{d}r$ is a parallel 1–form)

$$\nabla\omega_k = \begin{cases} \frac{2\pi N}{L}\cos\left(\frac{2\pi N}{L}\left[r - (k-1)\frac{L}{N}\right]\right)\mathrm{d}r^2 & \text{on } I_k \times \mathbb{S}^n_{1/L} \\ 0 & \text{on } M_L \smallsetminus (I_k \times \mathbb{S}^n_{1/L}) \end{cases} .$$

Since our test forms only depend upon r, their Rayleigh quotient is nothing but

$$R(\omega_k) = \frac{\int_{I_k} |\mathrm{d} f_k|^2}{\int_{I_k} f_k^2} = \left(\frac{2\pi N}{L}\right)^2 \frac{\int_0^{2\pi} \cos^2(t)\,\mathrm{d}t}{\int_0^{2\pi} \sin^2(t)\,\mathrm{d}t} = \left(\frac{2\pi N}{L}\right)^2 .$$

Using Theorem 4.5, we deduce $\lambda_N \leq \left(\frac{2\pi N}{L}\right)^2 \underset{L\to\infty}{\longrightarrow} 0$ and consequently $\lambda_N \leq \varepsilon$ for L large enough. ■

3. Ricci curvature and upper bound for the spectrum

In this section, we study the spectrum of the rough Laplacian on differential 1–forms (or equivalently on vector fields) on a compact manifold (M^n, g) with Ricci curvature bounded below $\mathrm{Ric}^g \geq -(n-1)a^2$, $a \geq 0$. In the following, $\mu_k(M,g)$ (or μ_k in short, $k \geq 1$) denotes the k–th eigenvalue of the closed or Neumann problem on M (depending on whether the boundary ∂M is empty or not).

Theorem 3.1. *Let (M^n, g) be a compact manifold with Ricci curvature bounded below $\mathrm{Ric}^g \geq -(n-1)a^2$, $a \geq 0$.*

(i) *If M is closed ($\partial M = \emptyset$) then for any $k \in \mathbb{N}^*$ we have*

$$\lambda_k(M,g) \leq a^2(n-1) + \mu_{k+1}(M,g) .$$

(ii) *If M has a compact and convex boundary in the sense that ℓ the second fundamental form of ∂M is nonnegative, then for any $k \in \mathbb{N}^*$ we have*

$$\lambda_k(M,g) \leq a^2(n-1) + \mu_{k+1}(M,g) .$$

In both case there exist geometric constants $A_n, B_n > 0$ (only depending upon the dimension) such that for any $k \in \mathbb{N}^*$

$$\lambda_k(M,g) \leq \Big((n-1) + A_n\Big)a^2 + B_n \left(\frac{k+1}{V}\right)^{2/n} ,$$

where $V = \mathrm{Vol}(M,g)$. In particular, we recover the Weyl asymptotic law.

Proof. We make the proof when the boundary is not empty (if it is not the case, just drop the boundary integrals in the integration by parts we will write below). Let us denote by $(f_i, \mu_i)_{i=1}^{k+1}$ the $(k+1)$ first eigenfunctions for the Neumann problem with their respective eigenvalues. We normalize the family of exact 1–forms $(\alpha_i := \mathrm{d} f_i)_{i=2}^{k+1}$ such that it is L^2–orthonormal (notice that $\alpha_1 \equiv 0$) i.e.

$$\forall i,j \in \{2,3,\cdots,k+1\} \qquad \int_M \langle \alpha_i, \alpha_j \rangle \,\mathrm{dVol}_g = \int_M \langle \nabla f_i, \nabla f_j \rangle \,\mathrm{dVol}_g = \delta_{ij} .$$

We apply the Bochner formula $\nabla^*\nabla \mathrm{d} f = \mathrm{d}(\Delta f) - \mathrm{Ric}^g(\mathrm{d} f)$ (cf. the proof of Proposition 4.15 in [12]) to a linear combination of the f_i that is for $f = \sum_{i=1}^{k+1} \beta_i f_i$ (or if you prefer to a linear combination of the α_i). On the one hand, it comes out

$$\nabla^*\nabla \mathrm{d} f = \mathrm{d}\left(\Delta\left(\sum_{i=1}^{k+1} \beta_i f_i\right)\right) - \mathrm{Ric}^g(\mathrm{d} f) = \sum_{i=2}^{k+1} \beta_i \mu_i \mathrm{d} f_i - \mathrm{Ric}^g(\mathrm{d} f) ,$$

which leads thanks to integration to the estimate

$$\begin{aligned}
\int_M \langle \nabla^*\nabla \mathrm{d} f, \mathrm{d} f\rangle \,\mathrm{dVol}_g &= \sum_{i,j=2}^{k+1} \mu_i \beta_i \beta_j \int_M \langle \nabla f_i, \nabla f_j \rangle \,\mathrm{dVol}_g - \int_M \mathrm{Ric}^g(\mathrm{d} f, \mathrm{d} f)\,\mathrm{dVol}_g \\
&= \sum_{i=2}^{k+1} \mu_i \beta_i^2 - \int_M \mathrm{Ric}^g(\mathrm{d} f, \mathrm{d} f)\,\mathrm{dVol}_g \\
&\leq \mu_{k+1}\left(\sum_{i=2}^{k+1} \beta_i^2\right) + a^2(n-1)\,\|\nabla f\|_{L^2}^2 \\
&\leq \Big(\mu_{k+1} + a^2(n-1)\Big)\,\|\nabla f\|_{L^2}^2 ,
\end{aligned}$$

where we have used that $\|\nabla f\|_{L^2}^2 = \sum_{i=2}^{k+1} \beta_i^2$. On the other hand, integration by parts gives (ν is the outward unit normal of ∂M)

$$\begin{aligned}\int_M \langle \nabla^*\nabla \mathrm{d}f, \mathrm{d}f\rangle \,\mathrm{dVol}_g &= \int_M |\nabla^2 f|^2 \,\mathrm{dVol}_g - \int_{\partial M} \langle \nabla_\nu \nabla f, \nabla f\rangle \,\mathrm{dVol}_{g,\partial M} \\ &= \int_M |\nabla^2 f|^2 \,\mathrm{dVol}_g - \int_{\partial M} \mathrm{Hess}^g f(\nu, \nabla f)\,\mathrm{dVol}_{g,\partial M} \ .\end{aligned}$$

In order to estimate the boundary integral, we note that along ∂M (keep in mind that f satisfies the Neumann boundary condition)

$$\mathrm{Hess}^g f(\nu, \nabla f) = \nabla f \cdot (\underbrace{\mathrm{d}f(\nu)}_{\equiv 0 \text{ on } \partial M}) - \mathrm{d}f(\nabla_{\nabla f}\nu) = -\ell(\nabla f, \nabla f)\ ,$$

where the second fundamental form is defined as $\ell = \nabla \nu$ with ν the outward unit normal to ∂M (this convention makes the unit Euclidean ball a domain with strictly convex boundary in the sense that its second fundamental form is positive). Thus, if ℓ is nonnegative we obtain

$$\begin{aligned}\int_M \langle \nabla^*\nabla \mathrm{d}f, \mathrm{d}f\rangle \,\mathrm{dVol}_g &= \int_M |\nabla^2 f|^2 \,\mathrm{dVol}_g + \int_{\partial M} \ell(\nabla f, \nabla f)\,\mathrm{dVol}_{g,\partial M} \\ &\geq \|\nabla^2 f\|_{L^2}^2 \ ,\end{aligned}$$

which reveals thanks to our first estimate

$$\|\nabla^2 f\|_{L^2}^2 \leq \|\nabla^2 f\|_{L^2}^2 + \int_{\partial M} \ell(\nabla f, \nabla f) = \int_M \langle \nabla^*\nabla \mathrm{d}f, \mathrm{d}f\rangle \leq \Big(\mu_{k+1} + a^2(n-1)\Big)\, \|\nabla f\|_{L^2}^2\ .$$

Therefore we control the Rayleigh quotient of $\mathrm{d}f$ since

$$R(\mathrm{d}f) = \frac{\int_M |\nabla^2 f|^2}{\int_M |\nabla f|^2} \leq \mu_{k+1} + a^2(n-1)\ .$$

As a conclusion, we control the Rayleigh quotient of any 1–form in the k–dimensional vector subspace of H^1 spanned by the family $(\alpha_i)_{i=2}^{k+1}$, and thanks to the MinMax principle (cf. (ii) of Theorem 4.5 Section 4 for details) we get that

$$\lambda_k(M,g) \leq \mu_{k+1}(M,g) + a^2(n-1)\ .$$

The last upper bound in the theorem follows from the Weyl–compatible estimate of μ_k in Theorem 1.3 proved in [10]. ■

Remark 3.2. *In* [13], *Gallot and Meyer proved a general (i.e. without any assumption on the curvature) bound from below for the spectrum of the rough Laplacian acting on any Riemannian vector bundle $E \longrightarrow M$, using the Neumann spectrum on the basis manifold (M^n, g). More precisely, they showed the following lower bound in case of $E = T^*M$*

$$\forall k \in \mathbb{N}^* \qquad \lambda_{k(n+1)} \geq \mu_{k+1}\ .$$

In particular, when the Ricci curvature is bounded below $\mathrm{Ric}^g \geq -(n-1)a^2$, $a \geq 0$, *this provides the inequality*

$$\forall k \in \mathbb{N}^* \qquad \mu_{k+1} \leq \lambda_{k(n+1)} \leq \mu_{k(n+1)+1} + a^2(n-1)\ .$$

4. Appendix: Spectral Theorem for the rough Laplacian

In this section, we prove that the spectrum of the rough Laplacian $\Delta := \nabla^*\nabla$ (which is an order 2 elliptic operator) on (M^n, g) a compact manifold with or without boundary, is an unbounded sequence of real numbers $(\lambda_k)_{k\in\mathbb{N}}$ which can be increasingly ordered

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots \nearrow +\infty \quad ,$$

where λ_0 denotes the zero eigenvalue (which is possible only when (M^n, g) has non zero parallel 1–forms) with its multiplicity ($=\dim \mathrm{Ker}\, \nabla$). In case where $\dim \mathrm{Ker}\, \nabla = 0$, the spectrum conventionally starts with $\lambda_1 > 0$.

We have decided to give the proof of the spectral theorem for the rough Laplacian in this article since we could not find a good reference where the details were carried out when the boundary of M is not empty.

We will follow the arguments of [2] step by step in the context of the rough Laplacian on sections of any vector bundle E which is endowed with a metric compatible connection ∇.
The rough Laplacian acting on E is defined by the relation

$$\Delta = \nabla^*\nabla = -\sum_{i=1}^{n} \nabla^2_{e_i,e_i} = -\sum_{i=1}^{n} \left(\nabla_{e_i}\nabla_{e_i} - \nabla_{D_{e_i}e_i}\right) ,$$

where $(e_i)_{i=1}^n$ is a local orthonormal basis and D is the Levi-Civita connection of (M^n, g) (remark that $D = \nabla$ in the previous Sections 1, 2 and 3 since we worked on the tensorial bundle of 1–forms; it is not the case a priori on a general vector bundle E). Now for every sections $\sigma, \tau \in \Gamma(E)$, we can define a 1–form $\theta(X) = \langle \nabla_X \sigma, \tau\rangle$ for any vector field $X \in \Gamma(TM)$. In order to compute the divergence of θ, choose an orthonormal basis $(e_i)_{i=1}^n$ satisfying $D_{e_i}e_j = 0$ at the point of computation. Then

$$\begin{aligned} \operatorname{div}\theta &= -\sum_{i=1}^{n} e_i \cdot \langle \nabla_{e_i}\sigma, \tau\rangle = -\sum_{i=1}^{n} \langle \nabla_{e_i}\nabla_{e_i}\sigma, \tau\rangle + \langle \nabla_{e_i}\sigma, \nabla_{e_i}\tau\rangle \\ &= \langle \Delta\sigma, \tau\rangle - \langle \nabla\sigma, \nabla\tau\rangle \ . \end{aligned}$$

Applying Stokes' formula to θ, we get (remind that ν is the *outward* unit normal of ∂M)

$$\int_M \langle \Delta\sigma, \tau\rangle = \int_M \langle \nabla\sigma, \nabla\tau\rangle - \int_{\partial M} \langle \nabla_\nu \sigma, \tau\rangle \ .$$

The eigenvalue problem for the rough Laplacian consists on finding a couple $(\lambda, \sigma) \in \mathbb{R} \times \Gamma(E)$ such that

$$(P) \quad \begin{cases} \Delta\sigma = \lambda\sigma & \text{on } M \\ \nabla_\nu\sigma = 0 & \text{on } \partial M \ , \end{cases}$$

where the second equation (which is the analogous version of Neumann's boundary condition for sections of E) holds only if the boundary $\partial M \neq \emptyset$. When $\partial M = \emptyset$ we will talk about the closed eigenvalue problem, whereas if $\partial M \neq \emptyset$ we will talk about the Neumann eigenvalue problem.
The first result of this section deals with the good regularity property of the boundary problem (P) when $\partial M \neq \emptyset$.

Lemma 4.1. *Let (M^n, g) be a compact and connected manifold with smooth and compact non–empty boundary ∂M. Then the boundary problem (P) is regular elliptic.*

Proof. We use the notations and terminology of [19], and apply Proposition 11.8 in [19] (p. 389). Let us fix some $p \in \partial M$ (if p lives on the interior of M then regularity is automatic) that we denote by $p = (x, y)$ where $x = (x_1, x_2, \cdots, x_{n-1}), y = x_n$ in normal coordinates. These normal coordinates are chosen such that $\nu = \partial_y$, and thereby the frozen coefficients problem associated to (P) is the following Cauchy ODE on some $\Phi \in C^\infty([0, \infty[, \mathbb{R}^r)$

$$(Frozen) \quad \begin{cases} \left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - g_p(\xi, \xi)\right)\Phi(y) = 0 \\ \left(\frac{\mathrm{d}\Phi}{\mathrm{d}y}\right)_{|y=0} = 0 \end{cases} ,$$

where r is the rank of E, ξ is any non–zero vector of $\mathbb{R}^{n-1}$, and $g_p(\xi, \xi) = \sum_{j,k=1}^{n-1} g^{jk}(p)\xi_j\xi_k$. Let us suppose that Φ is a bounded solution of $(Frozen)$, then any component Φ_s of Φ can be written as $\Phi_s : y \mapsto K_s \, \mathrm{e}^{-y\sqrt{g_p(\xi,\xi)}}$, for some constant $K_s \in \mathbb{R}$. The Cauchy data on Φ (i.e. its initial value) gives $-K_s\sqrt{g_p(\xi, \xi)} = 0$. But the metric g is positive definite and $\xi \neq 0$, which implies $K_s = 0$ for any s. We conclude that the frozen coefficients problem associated to (P) does not have non–zero bounded solution, which is a criterion for elliptic regularity. ■

Definition 4.2. *The functional space $H^1(E)$ or shortly H^1 is defined as the completion of $C^\infty(E) = \Gamma(E)$ (space of smooth sections of E) for the norm*

$$\|\sigma\|_1 := \left(\|\sigma\|^2_{L^2(M)} + \|\nabla\sigma\|^2_{L^2(M)}\right)^{1/2} \ .$$

H^1 is the so called Sobolev space of sections of E.

We naturally have a continuous inclusion $H^1 \hookrightarrow L^2$ since for every sections $\sigma \in H^1$ we have $\|\sigma\|_{L^2} \leq \|\sigma\|_1$. It will be a crucial point that this inclusion is in fact more than continuous.

Theorem 4.3. *The natural inclusion $H^1 \hookrightarrow L^2$ is compact.*

This fact is standard and has been proved for manifolds (possibly with boundary) in Section 4 of [19] for instance. Since Δ is formally self adjoint its spectrum is clearly contained in $[0,\infty[$. We now state the main result of this section (where the eigenvalues are *not* repeated with their multiplicity on the contrary of our usual convention).

Theorem 4.4 (Spectral Theorem)**.** *Let (M^n, g) be a compact manifold and E a vector bundle over M.*

(i) *The spectrum (i.e. set of eigenvalues) of Δ (for the closed eigenvalue problem or for the Neumann eigenvalue problem) is an unbounded sequence of real numbers $(\lambda_k)_{k\in\mathbb{N}}$ which can be increasingly ordered*

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \lambda_{k+1} < \cdots \nearrow +\infty \quad ,$$

with the following convention: λ_0 is the zero eigenvalue with multiplicity $\dim \operatorname{Ker} \nabla$; In case where there is no parallel section i.e. $\dim \operatorname{Ker} \nabla = 0$, the spectrum starts with the positive eigenvalue λ_1.

(ii) *Each eigenvalue λ_i has finite multiplicity and the eigenspaces E_{λ_i} corresponding to distinct eigenvalues are L^2–orthogonal.*

(iii) *The direct sum of the eigenspaces is dense in L^2.*

Proof. We consider $R(\sigma) = \frac{\int_M |\nabla\sigma|^2 \,\mathrm{dVol}_g}{\int_M |\sigma|^2 \,\mathrm{dVol}_g}$ the Rayleigh quotient on M with respect to the metric g which is defined for any section $\sigma \in H^1$ such that $\|\sigma\|_{L^2} \neq 0$. In order to prove our spectral theorem we consider the extrema of R on H^1 or equivalently on C^∞.

Let us define $E_0 := \{\sigma \in H^1 \setminus \{0\} | R(\sigma) = 0\} \cup \{0\} = \operatorname{Ker} \nabla$ which is clearly a finite dimensional vectorial subspace of H^1. Obviously any section $\sigma \in E_0$ satisfies the zero eigenvalue problem. Notice that

$$\forall \sigma \in E_0 \; \forall \tau \in H^1 \quad \langle \sigma, \tau \rangle_1 = \langle \sigma, \tau \rangle_{L^2} \ , \tag{4.1}$$

where $\langle\ ,\ \rangle_1$ is the scalar product induced by the norm $\|\ \|_1$. The convention we have adopted allows E_0 to be trivial which means 0 is not an eigenvalue of the rough Laplacian.

We now have to define the orthogonal subspace $H_0 = \{\tau \in H^1 | \forall \sigma \in E_0 \ \langle \sigma, \tau \rangle_1 = 0\}$ and analogously $L_0 = \{\tau \in L^2 | \forall \sigma \in E_0 \ \langle \sigma, \tau \rangle_1 = 0\}$. Thanks to Equation (4.1), we have $H_0 = L_0 \cap H^1$ and so the inclusion $H_0 \hookrightarrow L_0$ is compact by Theorem 4.3. We set

$$\lambda_1 = \inf \{ R(\sigma) \,| \sigma \in H_0,\ \|\sigma\|_{L^2} \neq 0 \} \ .$$

This infimum exists since R is nonnegative on H_0 and we denote by $(\sigma_n)_{n\in\mathbb{N}}$ a minimizing sequence in H_0 for λ_1 which has unit L^2–norm. We clearly have by definition

$$\|\sigma_n\|_1^2 = R(\sigma_n) + 1 \overset{n\to\infty}{\longrightarrow} 1 + \lambda_1 \in [1, \infty[\ ,$$

and so $(\sigma_n)_{n\in\mathbb{N}}$ is bounded in H_0. Since the inclusion $H_0 \hookrightarrow L_0$ is compact, we can suppose (up to extract a subsequence) there exist some $\sigma \in H_0$ such that

$$\sigma_n \overset{L^2}{\longrightarrow} \sigma, \quad \text{and} \quad \sigma_n \overset{H^1}{\rightharpoonup} \sigma \ .$$

By using the Cauchy–Schwarz inequality for the scalar product $\langle\ ,\ \rangle_1$ we get

$$\forall \tau \in H^1 \quad \langle \sigma_n, \tau \rangle_1^2 \leq \|\sigma_n\|_1^2 \, \|\tau\|_1^2 = (R(\sigma_n) + 1) \, \|\tau\|_1^2 = (\lambda_1 + 1 + \varepsilon_n) \, \|\tau\|_1^2 \ ,$$

where $\varepsilon_n \overset{n\to\infty}{\longrightarrow} 0$. By weak convergence in H^1 of σ_n toward σ we obtain

$$\forall \tau \in H^1 \quad \langle \sigma, \tau \rangle_1^2 \leq (\lambda_1 + 1) \, \|\tau\|_1^2 \ ,$$

which leads for $\tau = \sigma$,

$$\|\sigma\|_1^2 \leq 1 + \lambda_1 \Longleftrightarrow R(\sigma) \leq \lambda_1 \ ,$$

and as a consequence $R(\sigma) = \lambda_1$. In particular λ_1 is achieved on H_0 and thereby is positive. We set

$$E_1 = \{ \sigma \in H_0 \setminus \{0\} \,| R(\sigma) = \lambda_1 \} \cup \{0\} \ .$$

Let us take some $\sigma \in E_1$ and any $\tau \in H_0$, then for any $t \in \mathbb{R}$ small enough we have $r(t) := R(\sigma + t\tau) \geq R(\sigma) = \lambda_1$. It follows

$$\begin{aligned} 0 = r'(0) &= 2\langle \sigma, \tau\rangle_1 \|\sigma\|_{L^2}^{-2} - 2\langle \sigma, \tau\rangle_{L^2} \|\sigma\|_1^2 \|\sigma\|_{L^2}^{-4} \\ &= 2\|\sigma\|_{L^2}^{-2}\Big(\langle \sigma, \tau\rangle_1 - \langle \sigma, \tau\rangle_{L^2}(R(\sigma)+1)\Big) \\ &= 2\|\sigma\|_{L^2}^{-2}\Big(\langle \sigma, \tau\rangle_1 - \langle \sigma, \tau\rangle_{L^2}(\lambda_1+1)\Big), \end{aligned}$$

and we deduce:

$$\forall \sigma \in E_1 \, \forall \tau \in H_0, \quad \langle \sigma, \tau\rangle_1 = (\lambda_1+1)\langle \sigma, \tau\rangle_{L^2} \ . \tag{4.2}$$

Therefore, E_1 is a real vector space of finite dimension (from Theorem 4.3 the closed unit ball of E_1 is compact and the conclusion follows from the Riesz theorem). It is clear that any $\tau \in E_1$ is a weak solution of the eigenvalue problem, but by classical elliptic regularity result of Lemma 4.1, it comes out that $E_1 \subset C^\infty$, and moreover

1. when M is closed, we have $\Delta\tau = \lambda_1\tau$.
2. when $\partial M \neq \emptyset$, we have $\Delta\tau = \lambda_1\tau$ on M and $\nabla_\nu \tau = 0$ on ∂M.

Let us denote by L_1 and H_1 the orthogonal spaces of E_1 in respectively L^2 and H^1. By Equation (4.2), we have that $H_1 \hookrightarrow L_1$ is a compact inclusion (still because of Theorem 4.3). We now set

$$\lambda_2 = \inf\{R(\sigma)\,|\sigma \in H_1, \ \|\sigma\|_{L^2} \neq 0\} \ .$$

Following exactly the same arguments as for λ_1 we get that λ_2 is achieved on a finite dimensional vector subspace $E_2 \subset H_1$ which is characterized by the relation

$$\forall \sigma \in E_2 \, \forall \tau \in H_1, \quad \langle \sigma, \tau\rangle_1 = (\lambda_2+1)\langle \sigma, \tau\rangle_{L^2} \ . \tag{4.3}$$

By construction $\lambda_2 > \lambda_1$, and still by elliptic regularity argument we have $E_2 \subset C^\infty$. By induction, we obtain a sequence $(\lambda_i, E_i)_{i \geq 1}$ of eigenvalues and finite dimensional eigenspaces (which are L^2 orthogonal by construction).
Let us prove by contradiction that $\lim\limits_{k\to\infty} \lambda_k = +\infty$. So let us suppose that $(\lambda_i)_i$ is bounded by some $\lambda > 0$ then there exists an infinite sequence of L^2–orthonormal sections $(\sigma_i)_i$ such that: $\forall i \in \mathbb{N}^*, \quad \|\sigma_i\|_1^2 \leq 1+\lambda$. But it is not possible because of Theorem 4.3. We then have proved (i), (ii) and we let the proof of (iii) to the reader. ■

We have the following unique L^2–decomposition $H^1 = \widetilde{H^1} \oplus \left\{\sigma \in H^1\,|\nabla\sigma = 0\right\}$ (it is possible that $H^1 = \widetilde{H^1}$ depending on the vector bundle $E \to M$ under consideration). There exist some variational characterizations of the spectrum which can be proved exactly in the same way as in [2] (that is why we omit the proof of the following result).

Theorem 4.5 (MaxMin and MinMax)**.** *Let λ_k the k–th eigenvalue , $k \geq 1$ (here each eigenvalue is repeated with multiplicities) of the closed or the Neumann eigenvalue problem on a compact Riemannian manifold (M^n, g). Then the following variational characterizations hold:*

(i) ***MaxMin Principle:***

$$\lambda_k = \sup_{E_{k-1}} \inf\left\{R(\sigma)\,\middle|\sigma \neq 0, \ \sigma \in E_{k-1}^\perp\right\} ,$$

where E_{k-1} runs through the $(k-1)$–dimensional vectorial subspaces of $\widetilde{H^1}$ (or $\widetilde{C^\infty}$), and where $\perp$ means the L^2–orthogonal space.

(ii) ***MinMax Principle:***

$$\lambda_k = \inf_{L_k} \sup\{R(\sigma)\,|\sigma \neq 0, \ \sigma \in L_k\} ,$$

where L_k runs through the k–dimensional vectorial subspaces of $\widetilde{H^1}$ (or $\widetilde{C^\infty}$).

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