Large eigenvalues and concentration

Bruno Colbois and Alessandro Savo

Abstract

Let $M^n = (M, g)$ be a compact, connected Riemannian manifold of dimension n and μ the measure $\mu = \sigma \mathrm{dvol}_g$ where $\sigma \in C^\infty(M)$ is a non-negative density. We first show that, under some mild metric conditions which do not involve the curvature, the presence of a large eigenvalue (or more precisely of a large gap in the spectrum) for the Laplacian associated to the density σ on M implies a strong concentration phenomenon for the measure μ. When the density is positive, we show that our result is optimal. Then we investigate the case of a Laplace-type operator $D = \nabla^*\nabla + T$ on a vector bundle E over M, and show that the presence of a large gap between the (k+1)-th eigenvalue λ_{k+1} and the k-th eigenvalue λ_k implies a concentration phenomenon for the eigensections associated to the eigenvalues $\lambda_1, ..., \lambda_k$ of the operator D.

2000 Mathematics Subject Classification. 58J50, 35P15.
Key words and phrases. Eigenvalues, upper bounds, Laplace type operators, concentration.

1 Introduction

The goal of this paper is to show that, under some mild metric conditions, the presence of a large eigenvalue of the Laplacian Δ on a compact Riemannian manifold M implies that the Riemannian volume concentrates around a finite set of points. Actually, we show that a similar phenomenon holds for any Laplace type operator D acting on sections of a vector bundle on M, if one replaces the Riemannian volume by the squared norm of a first eigensection of D.

Let us recall briefly the main known facts about concentration and the spectrum of the Laplace operator. In what follows, we number the eigenvalues of Δ so that $\lambda_1(M) = 0$ and $\lambda_2(M)$ is the first positive eigenvalue.

For a closed Riemannian manifold of dimension n whose Ricci curvature is bounded below: $\mathrm{Ric} \geq -(n-1)a^2$, we have the following well-known inequality due to Cheng [Che]:

$$\lambda_{k+1}(M) \leq \frac{(n-1)^2 a^2}{4} + \frac{c(n)k^2}{\mathrm{diam}(M)^2} \tag{1}$$

where $c(n)$ is a constant depending only on n. This shows that when the k-th eigenvalue is very large the whole manifold is contained in a small neighborhood of any of its points and so we have a strong concentration phenomenon.

At the other extreme, if we make no assumption other than compactness we still have a concentration phenomenon, first observed by Gromov and Milman (Theorem 4.1 in [GM]). It says that if A is a closed subset with *positive* normalized measure $\mu(A) = \alpha$ and $r > 0$, then:

$$\mu(A^r) \geq 1 - (1 - \alpha^2) \exp(-r\sqrt{\lambda_2(M)} \ln(1 + \alpha)), \tag{2}$$

where

$$A^r = \{x \in M : d(A, x) < r\}.$$

So, when the first (positive) eigenvalue is large, almost all relative volume of M lies in a small neighborhood of any set of fixed positive measure.

However, we need to stress that the assumption that $\mu(A)$ is positive is essential in the estimate; the sole assumption that $\lambda_2(M)$ is large does not guarantee that the volume concentrates around, say, a finite set of points. For example, take M_n to be the n-th dimensional unit sphere. Then $\lambda_2(M_n)$ (which is equal to n) tends to infinity with n; we have concentration in the sense of Gromov-Milman, and yet the volume of M_n is uniformly distributed and cannot concentrate around any finite set. In section 4.4 we will give another counterexample in which the dimension is fixed.

Inequality (2) can be generalized to the other eigenvalues using an interesting upper bound of $\lambda_k(M)$ due to Chung, Grigor'yan and Yau; the upper bound is given in terms of the least distance between k mutually disjoint subsets of fixed positive measure (see [CGY2] and also [FT] for a sharp estimate).

The present paper deals with concentration around a finite number of points, and with a simple metric condition which will imply this phenomenon. Namely, we require that the number of balls of radius r needed to cover a ball of radius $4r$ is uniformly bounded above by a constant C for $r \leq 1$. We then prove the following fact:

If the $(k+1)$-st eigenvalue of the Laplacian of M is large, then most of the volume of M concentrates near (at most) k points of the manifold.

However, we will prove a result (Theorem 4) which is much more general; in particular, it will imply the following fact. Consider a Laplace-type operator D acting on the sections of a smooth vector bundle on M (for example, the Laplacian on forms, the square of the Dirac operator or the Schroedinger operator). Then:

If the gap between the $(k+1)$-th and the first eigenvalue of D is large, then any first eigensection concentrates its L^2-norm near (at most) k points of the manifold.

Both the above estimates depend explicitly on the constant C.

In the rest of the Introduction we state the precise results: Theorems 1, 2 and 3.

1.1 Some definitions

In this paper we will consider metric measure spaces (M, μ, d) of the following type:

1) $M = (M^n, g)$ is a compact, connected Riemannian manifold of dimension n, possibly with non-empty boundary.

2) μ is the measure $\mu = \sigma \mathrm{dvol}_g$ where $\sigma \in C^\infty(M)$ is a non-negative density. We will also assume, without loss of generality, that μ is a probability measure, that is, $\int_M \sigma \mathrm{dvol}_g = 1$.

3) d is a distance function which is assumed to be Lipschitz, i.e. $|\nabla d| \leq 1$ a.e. with respect to μ.

For $r > 0$, define $C_d(M, r)$ to be the minimal number of balls of radius r in (M, d) needed to cover a ball of radius $4r$. Then $C_d(M, r)$ is finite for all r.

We will set:

$$C_d(M) = \sup_{r \in (0,1]} C_d(M, r), \tag{3}$$

and call it the *packing constant* of the pair (M, d): it is a metric invariant (it does not depend on the measure μ).

The packing constant is often used in similar contexts (see the survey [GNY], where it is used extensively). By the compactness of M, $C_d(M)$ is well defined.

Note that d is not necessarily the Riemannian distance. In fact, here are three typical situations in which it is easy to control the packing constant:

I. (M^n, g) is a closed Riemannian manifold and d is the intrinsic distance on M associated to the Riemannian metric g.

II. M^n is an immersed submanifold of another manifold X (for example, Euclidean space or hyperbolic space) and $d = d_{ext}$ is the extrinsic distance, that is, the restriction to M of the Riemannian distance on X.

III. M^n is a bounded domain with smooth boundary in a complete Riemannian manifold X and again $d = d_{ext}$ is the extrinsic distance.

In the first case we can easily estimate the packing constant in terms of a lower bound of the Ricci curvature and the dimension, using the Bishop-Gromov inequality (see [CM], Example 2.1). In cases **II** and **III**, a simple argument shows that $C_d(M) \leq C_d(X)^2$, and so the packing constant of an immersed submanifold of Euclidean or hyperbolic space is bounded above by an absolute constant depending only on the dimension of X; in particular, it is independent on the Ricci curvature of M. For example, if M is any submanifold of $\mathbf{R}^m$ then $C_d(M) \leq (1 + 3^{2m})^2$ (we stress that here d is the extrinsic distance; for the intrinsic distance this is no longer true in general).

1.2 Estimates for the Laplacian on functions

When the density $\sigma > 0$ we can consider the following operator L acting on $C^\infty(M)$:

$$Lu = \Delta u - \frac{1}{\sigma}\langle \nabla u, \nabla \sigma \rangle, \tag{4}$$

for all $u \in C^\infty(M)$. If $\partial M \neq \emptyset$, we assume Neumann boundary conditions. L is self-adjoint when acting on $L^2(M,\mu)$, where $\mu = \sigma \mathrm{dvol}_g$, and is associated to the quadratic form

$$u \mapsto \int_M |\nabla u|^2 \sigma \mathrm{dvol}_g.$$

The spectrum of L is discrete and will be denoted by $\{\lambda_k(L)\}_{k=1}^\infty$. Note that $\lambda_1(L) = 0$ and $\lambda_2(L) > 0$. If σ is constant (that is, μ is just a multiple of the Riemannian measure) one recovers the eigenvalues of the ordinary Laplacian on M. However, the generalization to Laplace-type operators will force us to consider non-constant densities.

Theorem 1. *Let $\mathcal{M} = (M,\mu,d)$ be a metric measured space as defined in Section 1.1 and assume that $\mu = \sigma \mathrm{dvol}_g$ with $\sigma > 0$ everywhere on M. Let L be the operator defined in* (4)*. Then, for all $k \geq 1$, there exists a set S of k points $x_1, \ldots, x_k \in M$ with the following property. If*

$$r = 8(k+1)C_d(M)^2 \cdot \frac{\log \lambda_{k+1}(L)}{\sqrt{\lambda_{k+1}(L)}}$$

then we have

$$\mu(S^r) \geq 1 - r.$$

provided that $\lambda_{k+1}(L) \geq e$. Here $C_d(M)$ is the packing constant defined in (3).

- The estimate is sharp, in the sense that the decay $\dfrac{\log \lambda}{\sqrt{\lambda}}$ is optimal as $\lambda = \lambda_{k+1}(L)$ tends to infinity, and cannot be replaced by a function with a faster rate of decrease. We refer to Section 4 for an explicit example.
- If the eigenvalue $\lambda_{k+1}(L)$ is large (so that r is small) then we see that almost all the measure μ is in the r-neighborhhood of k suitable points: this is the concentration property that we want to emphasize.
- It is perhaps worth mentioning that there is an equivalent formulation of our estimate in terms of the so-called Levy-Prokhorov distance between probability measures. Let us recall its definition. If (X,d) is a metric space, $\mathcal{B}(X)$ the borelian σ-algebra and $\mathcal{P}(X)$ the set of the probability measures on X, the *Levy-Prokhorov distance* d_P between two elements ν_1 and ν_2 of $\mathcal{P}(X)$ is:

$$d_P(\nu_1,\nu_2) = \inf\{r > 0 : \nu_1(C) \leq \nu_2(C^r) + r \text{ and } \nu_2(C) \leq \nu_1(C^r) + r \text{ for all } C \in \mathcal{B}(X)\}.$$

See for example [V] (6.5) page 97.

The following result is an equivalent formulation of Theorem 1.

Theorem 2. *In the hypothesis of Theorem 1 there exist k points $x_1, \ldots, x_k \in M$ and weights $p_1, \ldots, p_k \in [0,1)$ such that $\sum p_j = 1$ and*

$$d_P(\mu, \delta_S) \leq 8(k+1)C_d(M)^2 \cdot \frac{\log \lambda_{k+1}(L)}{\sqrt{\lambda_{k+1}(L)}},$$

where $\delta_S = \sum_{i=1}^k p_i \delta_{x_i}$ and δ_{x_i} is the Dirac measure concentrated at the point x_i.
In particular, for $k = 1$: there exists a point $x_1 \in M$ such that

$$d_P(\mu, \delta_{x_1}) \leq 16C_d(M)^2 \cdot \frac{\log \lambda_2(L)}{\sqrt{\lambda_2(L)}}.$$

The estimate is sharp: see Section 4.2.

In other words, when the eigenvalue is large, the measure μ is close, in the Levy-Prokhorov sense, to a weighted linear combination of the Dirac measures at the points $x_1, \ldots, x_k$.

The equivalence between the formulations in Theorem 1 and Theorem 2 will be proved in Section 4.1.

- Note that Theorems 1 and 2 apply obviously to the Laplacian acting on functions: it suffices to choose $\sigma = \frac{1}{\mathrm{Vol}(M)}$. In that case the concentration is relative to the (normalized) Riemannian volume.

1.3 Estimates for vector bundle Laplacians

The next task is to generalize Theorem 1 when the density σ is only assumed to be non-negative. For that purpose we introduce, in section 2, a weaker notion of spectrum and prove the relevant Theorem 4. Besides being interesting in itself, Theorem 4 will lead to a concentration phenomenon of eigensections in the context of Laplacians acting on sections of a vector bundle.

So, consider a vector bundle E over a compact Riemannian manifold (M^n, g) with empty boundary, and denote by ∇ a connection on E which is compatible with the metric g (see [B] for details). An operator D acting on sections of the bundle is said to be of *Laplace type* if it can be written

$$D = \nabla^* \nabla + T,$$

where T is a symmetric endomorphism of the fiber. Then, D is self-adjoint and elliptic. We list its eigenvalues as

$$\lambda_1(D) \leq \lambda_2(D) \leq \cdots \leq \lambda_k(D) \leq \dots$$

and denote by $\{\psi_1, \psi_2, \dots\}$ a corresponding orthonormal basis of eigensections.

Important examples of Laplace-type operators are given by the Laplacian acting on differential forms, by the square of the Dirac operator and by a Schroedinger operator acting on functions. In the first case T is the curvature term in the classical Bochner-Weitzenboeck formula, in the second case it is multiplication by a constant multiple of the scalar curvature and in the third case T is just the potential.

In the second main theorem we assume a large gap in the spectrum of D and prove that eigensections concentrate their norms near a finite set of points.

Theorem 3. *For each positive integer k there exist a set S of k points $x_1, \dots, x_k \in M$ with the following property. Let ψ be any unit L^2-norm linear combination of the first k eigensections of D, and $\mu = |\psi|^2 \mathrm{dvol}_g$. If $r = 25k \left(\frac{k^2(k+1)C_d(M)^2}{\lambda_{k+1}(D) - \lambda_k(D)} \right)^{1/3}$ then*

$$\mu(S^r) \geq 1 - r.$$

Equivalently, the Levy-Prokhorov distance between μ and a suitable linear combination of the Dirac measures at $x_1, \dots, x_k$ is bounded above by r.

Example. We take D to be the ordinary Laplacian on functions and assume that λ_{k+1} tends to infinity while λ_k is uniformly bounded. Then we know from Theorem 1 that the Riemannian volume concentrates around k suitable points $x_1, \dots, x_k$. Theorem 3 then says that any eigenfunction associated to eigenvalues less than λ_{k+1} will also concentrate its L^2-norm around $x_1, \dots, x_k$.

Example. We take D to be the Laplacian acting on p-forms and assume that the p-th Betti number of M is positive, say $b_p(M) = k > 0$. Then $\lambda_k(D) = 0$ and $\lambda = \lambda_{k+1}(D)$ is the first positive eigenvalue of D. Assume that λ is very large. Then the theorem gives the existence of $b_p(M)$ points such that all harmonic p-forms must concentrate their L^2-norms in a small neighborhood of the union of these points.

We also observe that, in general, a large gap in the spectrum of D does not necessarily imply concentration of the Riemannian volume unless, of course, D is the ordinary Laplacian, or there exist parallel sections (so that the density $\sigma = |\psi|^2$ is constant). We refer to Section 4.3 for an explicit example.

Plan of the sections. The paper is structured as follows: in Section 2 we will prove Theorems 1 and a more general version of it, Theorem 4. In Section 3 we will establish the results for vector bundle Laplacians and prove Theorem 3. Sections 4 is devoted to the examples, in particular, the sharpness of the estimate given in Theorem 1 and Theorem 2.

2 Estimates for functions

2.1 A general estimate when the density is only non-negative

We consider a compact manifold M (with or without boundary) endowed with a distance function d and a measure $\mu = \sigma \mathrm{dvol}_g$ as in Section 1.1. We first consider the general case in which $\sigma \geq 0$: this will be needed to treat Laplace-type operators, where the density σ will be the squared norm of an eigensection, which can vanish at some points of M. However it is well-known from elliptic theory that eigensections can vanish only on sets of measure zero.

Let us then introduce the *weak spectrum* of the metric measured space $\mathcal{M} = (M, \mu, d)$ as follows. First, define the following Rayleigh quotient of the Lipschitz function f (such that $\int_M f^2 \mu > 0$):

$$R(f) = \frac{\int_M |\nabla f|^2 \mu}{\int_M f^2 \mu}.$$

Let us denote by W_k a vector space of Lipschitz functions on M of finite dimension k. Then, for all integers $k \geq 0$ we define:

$$\lambda_{k+1}(\mathcal{M}) \doteq \sup_{W_k} \inf\{R(f) : f \perp W_k\}. \tag{5}$$

It is clear that $\lambda_1(\mathcal{M}) = 0$. It is easy to check that the sequence $\lambda_j(\mathcal{M})$ is non-decreasing.

Having said that, we state the main theorem of this section.

Theorem 4. *Let $\mathcal{M} = (M, \mu, d)$ be as above, with $\mu = \sigma \mathrm{dvol}_g$ and $\sigma \geq 0$. Then, for each $k = 1, 2, \dots$ we can find a set S of k points $x_1, \dots, x_k \in M$ such that, if $r = 5\left(\frac{(k+1)C_d(M)^2}{\lambda_{k+1}(\mathcal{M})}\right)^{1/3}$, we have*

$$\mu(S^r) \geq 1 - r.$$

Remark. If the density σ is strictly positive on M, then it is clear by the max-min principle that the weak spectrum of $\mathcal{M}$ is equal to the spectrum of the self-adjoint elliptic operator L acting on $L^2(M, \sigma \cdot \mathrm{dvol}_g)$ and already defined in (4). That is, $\lambda_k(\mathcal{M}) = \lambda_k(L)$ for all k. In this case, using an upper bound of [CGY2] and an additional measure theoretic lemma proved in [CM] we can prove Theorem 1 in the Introduction, which is an improvement of Theorem 4 for large $\lambda = \lambda_{k+1}$ because $\frac{\log \lambda}{\sqrt{\lambda}}$ decays faster than $\lambda^{-1/3}$.

2.2 Preparatory results

In the next lemma we estimate the eigenvalues of $\mathcal{M}$ as defined in the previous section. The first part follows from a standard argument involving plateau functions, which applies to our case. The second part is an estimate due to Chung, Grigor'yan and Yau.

Lemma 5. a) *Let $\mathcal{M} = (M, \mu, d)$ and assume that $\mu = \sigma \cdot \mathrm{dvol}_g$ with $\sigma \geq 0$. Assume that there exist $k+1$ subsets of M, each of measure at least $\alpha > 0$, which are $2r$-separated (meaning that the distance between any two of the given sets is at least $2r$). Then:*

$$\lambda_{k+1}(\mathcal{M}) \leq \frac{1}{\alpha r^2}.$$

b) *If the density σ is strictly positive on M then:*

$$\lambda_{k+1}(\mathcal{M}) = \lambda_{k+1}(L) \leq \frac{1}{r^2}\left(\log\frac{2}{\alpha}\right)^2.$$

where L is the operator $Lu = \Delta u - \frac{1}{\sigma}\langle \nabla u, \nabla \sigma\rangle$ defined in (4).

Proof. a) Fix a subspace W of the space of Lipschitz functions on M, of finite dimension k. Let $A_1, \ldots, A_{k+1}$ be the subsets satisfying the assumptions, that is $\int_{A_j} \mu = \int_{A_j} \sigma \mathrm{dvol}_g \geq \alpha$ and $d(A_i, A_j) \geq 2r$ if $i \neq j$. For each $j = 1, \ldots k+1$ let ϕ_j be the following plateau function:

$$\phi_j(x) = \begin{cases} 1 \quad \text{on } A_j, \\ 1 - \dfrac{1}{r} d(x, A_j) \quad \text{on } \Omega_j = A_j^r \setminus A_j, \\ 0 \quad \text{on the complement of } A_j^r. \end{cases}$$

Note that the ϕ_j's are disjointly supported. Linear algebra shows that we can find numbers $a_1, \ldots, a_{k+1}$ such that the function

$$\phi = \sum_{j=1}^{k+1} a_j \phi_j$$

is Lipschitz, $L^2(\mu)$-orthogonal to W and non-zero. We can also assume that $\sum a_j^2 = 1$. The gradient of ϕ is supported on the union of the Ω_j's, and on Ω_j one has $|\nabla\phi| \leq |a_j|/r$ almost everywhere. Then:

$$\int_M |\nabla\phi|^2 \mu \leq \frac{1}{r^2}\int_M \mu = \frac{1}{r^2}$$

On the other hand:

$$\int_M \phi^2 \mu \geq \sum_j a_j^2 \int_{A_j} \mu \geq \alpha$$

Therefore $R(\phi) \leq 1/(\alpha r^2)$. As ϕ was orthogonal to W, we get:

$$\inf\{R(f) : f \perp W\} \leq \frac{1}{\alpha r^2}.$$

The right hand side is independent of the subspace W; hence taking the supremum over all k-dimensional subspaces W does not change the upper bound. Recalling the definition of λ_{k+1} one obtains the first part of the Lemma.

b) If the density σ is positive, we can use an estimate of Chung, Grigory'an and Yau [CGY1]. It says that, if the subsets $A_1, \ldots, A_{k+1}$ are at distance at least s from each other, then:

$$\lambda_{k+1}(L) \leq \frac{4}{s^2} \cdot \max_{i \neq j} \left(\log \frac{2}{\sqrt{\mu(A_i)\mu(A_j)}} \right)^2 .$$

The second inequality is now immediate by taking $s = 2r$. □

We will use a result of [CM] (corollary 2.3) which we state in a way more convenient to our purposes. Consider our metric space (M, d) and recall the packing constant $C_d(M)$. Let ν be any measure on M. Then we have

Proposition 6. *Let N be a positive integer. Suppose that for a given $s > 0$, we have for each $x \in M$*

$$\nu(B(x, s)) \leq \frac{\nu(M)}{4C_d(M)^2 N}.$$

Then, there exist N subsets $A_1, ..., A_N$ of M such that $\nu(A_i) \geq \dfrac{\nu(M)}{2C_d(M)N}$ for each i and, for each $i \neq j$: $d(A_i, A_j) \geq 3s$.

We will use the Proposition in the proof of Theorem 4 for ν given by the restriction of μ to a closed subset.

2.3 Proof of Theorem 4

Let $\lambda_{k+1}(\mathcal{M}) = \lambda$ and assume that it is positive. Let:

$$r = 5 \left(\frac{(k+1)C_d(M)^2}{\lambda} \right)^{1/3} . \tag{6}$$

We will prove that there exist a set S of suitably chosen points $x_1, \dots, x_k$ (not necessarily distinct) such that

$$\mu(S^r) \geq 1 - r. \tag{7}$$

We can suppose $r < 1$.

Let $\alpha = \dfrac{r}{4(k+1)C_d(M)^2}$. By the definitions of r and α one has:

$$\lambda = \frac{125}{4\alpha r^2}. \tag{8}$$

Step 1: construction of the points. We choose x_1 so that $\mu(B(x_1, \frac{r}{4})) \geq \mu(B(x, \frac{r}{4}))$ for all $x \in M$, and set:

$$X_1 = B(x_1, r)^c.$$

Next, we choose $x_2 \in X_1$ so that $\mu(B(x_2, \frac{r}{4})) \geq \mu(B(x, \frac{r}{4}))$ for all $x \in X_1$, and set:

$$X_2 = (B(x_1, r) \cup B(x_2, r))^c.$$

We continue in this way till we obtain k points $x_1, \dots, x_k$: to construct the j-th point $x_j \in X_{j-1}$, we demand that $\mu(B(x_j, \frac{r}{4})) \geq \mu(B(x, \frac{r}{4}))$ for all $x \in X_{j-1}$ and define

$$X_j = (B(x_1, r) \cup \dots \cup B(x_j, r))^c.$$

Note that if X_j is empty for some $j \leq k$ then $\mu\left(B(x_1, r) \cup \dots \cup B(x_j, r)\right) = 1 > 1 - r$, so we can take $S = \{x_1, ..., x_{j-1}\}$. We have $\mu(S^r) \geq 1 - r$ and the theorem is proved. So we can assume that

$$X_k = (B(x_1, r) \cup \dots \cup B(x_k, r))^c$$

is non-empty. Inequality (7) (and the theorem) follows if we show that

$$\mu(X_k) \leq r. \tag{9}$$

Step 2: proof of (9). We argue by contradiction and show that the inequality

$$\mu(X_k) > r \tag{10}$$

cannot occur. Let us then assume (10) and denote by B_i the ball $B(x_i, \frac{r}{4})$. By construction, the sets $B_1, \dots, B_k$ and X_k are $\frac{r}{2}$-separated and $\mu(B_1) \geq \mu(B_2) \geq \dots \geq \mu(B_k)$.

First case. Assume:

$$\mu(B_k) \geq \alpha.$$

Then $\mu(B_j) \geq \alpha$ for all j; moreover:

$$\mu(X_k) \geq r > \frac{r}{4(k+1)C_d(M)^2} = \alpha$$

simply because $C_d(M) \geq 1$. Therefore the sets $B_1, \ldots, B_k, X_k$ are $\frac{r}{2}$-separated and each of them has measure at least α. By Lemma 5:

$$\lambda = \lambda_{k+1}(\mathcal{M}) \leq \frac{16}{\alpha r^2} \tag{11}$$

which contradicts (8). Then the first case does not occur.

Second case. Assume:

$$\mu(B_k) < \alpha. \tag{12}$$

Consider the closed subset $X = X_{k-1}$. By the definition of x_k one has:

$$\mu(B(x, \frac{r}{4})) \leq \mu(B_k) \leq \alpha$$

for all $x \in X$. Recall that $X_k \subseteq X_{k-1} = X$.

We now consider the metric space (M, d) with the measure ν given by the restriction of μ to the closed subspace X, that is $\nu(A) = \mu(A \cap X)$. By (10) we have

$$r < \mu(X_k) \leq \mu(X) = \nu(M).$$

and therefore

$$\begin{aligned} \nu(B(x, \frac{r}{4}) &\leq \mu(B(x, \frac{r}{4}) \\ &\leq \alpha \\ &= \frac{r}{4(k+1)C_d(M)^2} \\ &\leq \frac{\nu(M)}{4(k+1)C_d(M)^2} \end{aligned}$$

By Proposition 6 applied for $s = \frac{r}{4}$ and $N = k+1$ we conclude that there exist $k+1$ subsets $A_1, \ldots, A_k$ which are $\frac{3r}{4}$-separated and such that

$$\nu(A_i) \geq \frac{\nu(M)}{2C_d(M)(k+1)} > \frac{r}{2C_d(M)(k+1)} \geq 2C_d(M)\alpha \geq 2\alpha.$$

for all i. Then $\mu(A_i) \geq 2\alpha$ for all i. Applying Lemma 5 one would obtain:

$$\lambda = \lambda_{k+1}(\mathcal{M}) \leq \frac{32}{9\alpha r^2} \tag{13}$$

which is again a contradiction with (8). The proof of Theorem 4 is now complete.

2.4 Proof of Theorem 1

Set $\lambda_{k+1}(\mathcal{M}) = \lambda$ and assume $\lambda \geq e$. Let:

$$r = \beta \frac{\log \lambda}{\sqrt{\lambda}}, \tag{14}$$

where $\beta = 8(k+1)C_d(M)^2$. We will find a set S of k points $x_1, ..., x_k$ such that

$$\mu(S^r) \geq 1 - r. \tag{15}$$

which is the statement of the theorem.
Set $\alpha = \dfrac{r}{4(k+1)C_d(M)^2}$. We first observe that

$$\lambda > \frac{256}{r^2} \left(\log \frac{2}{\alpha} \right)^2. \tag{16}$$

In fact (14) gives $\lambda = \dfrac{\beta^2}{r^2}(\log \lambda)^2 \geq \dfrac{\beta^2}{r^2}$, and substituting inside $\log \lambda$ we obtain (16) because $\frac{\beta}{r} = \frac{2}{\alpha}$ by the definitions of α and β and the fact that $\beta \geq 8$.
In order to show (15) we follow Step 1 and Step 2 exactly as in the proof of the previous theorem: we construct the points $x_1, \ldots, x_k$ as before and show that the inequality $\mu(X_k) > r$ leads to a contradiction with the inequality (16). The only change is to use the second inequality of Lemma 5 instead of the first, so that (11) becomes:

$$\lambda \leq \frac{16}{r^2} \left(\log \frac{2}{\alpha} \right)^2,$$

and (13) becomes:

$$\lambda \leq \frac{64}{9r^2} \left(\log \frac{2}{\alpha} \right)^2,$$

both of which contradicting (16).

Remark. It is not possible to replace the constant β in (14) by $\beta(\lambda)$ for a function $\beta(\lambda) \to 0$ as $\lambda \to \infty$. In fact, taking $\beta =$ constant is the optimal choice for the radius r (see Section 4.2).

3 The estimate for Laplace type operators

The scope of this section is to prove Theorem 3 stated in the Introduction. We start from the following:

Theorem 7. *Let M^n be a compact Riemannian manifold without boundary and D any Laplace-type operator on M. Fix integers j, k with $j \leq k$ and consider the m-m-space (M, μ_j, d), where $\mu_j = |\psi_j|^2 \cdot \mathrm{dvol}_g$ and ψ_j is a unit norm eigensection associated to $\lambda_j(D)$. Then there exists a set S_j of k points $x_1^j, \dots, x_k^j \in M$ such that if $r = 5\left(\frac{k(k+1)C_d(M)^2}{\lambda_{k+1}(D)-\lambda_j(D)}\right)^{1/3}$ then*

$$\mu_j(S_j^r) \geq 1 - r.$$

Of course, the result is significant only when the gap $\lambda_{k+1}(D) - \lambda_j(D)$ is large enough. As the gap $\lambda_{k+1}(D) - \lambda_k(D)$ increases to ∞, we see that any eigensection associated to $\lambda_j(D)$, with $j \leq k$, tends to concentrate its norm around at most k points $x_1^j, ..., x_k^j$, a priori depending on j. It is natural to ask if there is a relation between all these points for different eigenvalues. We can in fact show that, as the gap tends to infinity, all squared norms $|\psi_1|^2, ..., |\psi_k|^2$ will concentrate around a *common* set of k points. Actually, we will show that this also happens for the squared norm of any section in the direct sum of the first k eigenspaces: this is the statement of Theorem 3.

3.1 Proof of Theorem 7

The proof of Theorem 7 depends on the following two lemmas, in which we bound the gaps in the spectrum of D by the weak spectrum of the m-m-spaces $\mathcal{M}$ corresponding to the densities $\sigma = |\psi|^2$, where ψ is an eigensection of D. We then apply Theorem 4 to conclude.

Recall that $D = \nabla^\star \nabla + T$, where T is a symmetric endomorphism of the fiber. So the quadratic form associated to D is

$$\mathcal{Q}(\psi) = \int_M |\nabla \psi|^2 + \langle T\psi, \psi \rangle,$$

which is defined on the space of H^1-sections of the bundle (here integration is with respect to the Riemannian measure dvol_g). We fix an orthonormal basis of eigensections of D and denote it by $(\psi_1, \psi_2, \dots)$.

Lemma 8. *Let f be a Lipschitz function on M and ψ a smooth section of the bundle. Then:*

$$\mathcal{Q}(f\psi) = \int_M f^2 \langle D\psi, \psi \rangle + |\nabla f|^2 |\psi|^2.$$

Lemma 9. *Fix a positive integer k and let $j \leq k$. Let ψ_j be an eigensection associated to $\lambda_j(D)$, of unit L^2-norm, and consider the m-m-space $\mathcal{M}_j = (M, \mu_j, d)$ where $\mu_j = |\psi_j|^2 \mathrm{dvol}_g$. Then:*

$$\lambda_{k+1}(D) - \lambda_j(D) \leq k\lambda_{k+1}(\mathcal{M}_j).$$

Theorem 7 now follows immediately from Lemma 9 and Theorem 4 applied with the density $\sigma = |\psi_j|^2$.

Proof of Lemma 8. On the subset where ∇f exists (hence a.e. on M) one has:

$$|\nabla(f\psi)|^2 = |\nabla f|^2|\psi|^2 + f^2|\nabla\psi|^2 + 2f\langle\nabla_{\nabla f}\psi, \psi\rangle. \tag{17}$$

Now:

$$\int_M 2f\langle\nabla_{\nabla f}\psi, \psi\rangle = \int_M \frac{1}{2}\langle\nabla f^2, \nabla|\psi|^2\rangle = \int_M \frac{1}{2}f^2\Delta|\psi|^2$$

hence

$$\begin{aligned}\mathcal{Q}(f\psi) &= \int_M |\nabla(f\psi)|^2 + \langle T(f\psi), f\psi\rangle \\ &= \int_M f^2\left(|\nabla\psi|^2 + \frac{1}{2}\Delta|\psi|^2 + \langle T\psi, \psi\rangle\right) + |\nabla f|^2|\psi|^2\end{aligned}$$

Now recall the identity (Bochner formula): $\langle D\psi, \psi\rangle = |\nabla\psi|^2 + \frac{1}{2}\Delta|\psi|^2 + \langle T\psi, \psi\rangle$. The lemma follows.

Proof of Lemma 9. Given the metric-measure space $\mathcal{M} = (M, \mu, d)$ recall the definition of weak spectrum:

$$\lambda_{h+1}(\mathcal{M}) = \sup_{W_h}\inf\{R(f) : f \perp W_h\}.$$

where

$$R(f) = \frac{\int_M |\nabla f|^2\mu}{\int_M f^2\mu},$$

and W_h denotes a vector subspace of Lipschitz functions having dimension h. We will write for brevity $\lambda_i(\mathcal{M}) = \lambda_i$.

Fix $\epsilon > 0$. Then, for all integers $k \in \mathbf{N}$ we construct a $(k+1)$-dimensional subspace W_{k+1} of the space of Lipschitz functions on M such that, for all $f \in W_{k+1}$ one has:

$$R(f) \leq k(\lambda_{k+1} + \epsilon). \tag{18}$$

Set:

$$W_1 = \mathrm{span}(f_1),$$

where f_1 is the constant function 1. By definition, there exists a non-vanishing smooth function f_2 which is orthogonal to W_1 and satisfies:

$$R(f_2) \leq \lambda_2 + \epsilon.$$

We set

$$W_2 = \mathrm{span}(f_1, f_2).$$

We can assume that f_2 has unit L^2-norm. Continuing this process, we get

$$W_{k+1} = \mathrm{span}(f_1, \ldots, f_{k+1}),$$

where $(f_1, \ldots, f_{k+1})$ is an orthonormal set and, for all $j = 1, \ldots, k+1$:

$$R(f_j) \leq \lambda_j + \epsilon \leq \lambda_{k+1} + \epsilon. \tag{19}$$

Let us prove (18). Let $f = \sum_{j=1}^{k+1} a_j f_j$ be a function in W_{k+1}. We can assume that it has unit norm, so that $\sum_j a_j^2 = 1$. By the triangle inequality, since $\nabla f_1 = 0$, one has $|\nabla f| \leq \sum_{j=2}^{k+1} |a_j||\nabla f_j|$. By the Schwarz inequality $|\nabla f|^2 \leq \sum_{j=2}^{k+1} |\nabla f_j|^2$ and therefore, by (19):

$$R(f) \leq \sum_{j=2}^{k+1} R(f_j) \leq k(\lambda_{k+1} + \epsilon).$$

We can now prove the lemma. Fix $\epsilon > 0$ and consider the m-m-space $\mathcal{M}_j$ with measure $\mu_j = |\psi_j|^2 \mathrm{dvol}_g$, as in the statement of the Lemma. Let W_{k+1} be the subspace satisfying (18). By linear algebra, we can find a non-vanishing $f \in W_{k+1}$ such that the section $f\psi_j$ has unit norm and is orthogonal to the first k eigensections $\psi_1, \ldots, \psi_k$ of the spectrum of D. Using $f\psi_j$ as test-section for the eigenvalue $\lambda_{k+1}(D)$, we obtain, by Lemma 8:

$$\lambda_{k+1}(D) \leq \mathcal{Q}(f\psi_j) = \int_M f^2 \langle D\psi_j, \psi_j \rangle + |\nabla f|^2 |\psi_j|^2.$$

As $\langle D\psi_j, \psi_j \rangle = \lambda_j(D)|\psi_j|^2$ this becomes:

$$\lambda_{k+1}(D) - \lambda_j(D) \leq R(f) \leq k(\lambda_{k+1}(\mathcal{M}_j) + \epsilon),$$

by (18). Letting $\epsilon \to 0$ we obtain the assertion.

3.2 Proof of Theorem 3.

Let us start with the formal proof by considering an orthonormal basis $(\psi_1, \ldots, \psi_k)$ of the direct sum of the first k eigenspaces of D. Given $\mu_j = |\psi_j|^2 \cdot \mathrm{dvol}_g$, let us introduce the following auxiliary measure, which is just the average of the μ_j's:

$$\tilde{\mu} = \frac{1}{k} \sum_{j=1}^{k} \mu_j.$$

We also fix the radius

$$r = 5 \left(\frac{k^2(k+1)C_d(M)^2}{\lambda_{k+1}(D) - \lambda_k(D)} \right)^{1/3}. \tag{20}$$

We divide the proof in two steps.

Step 1. There exists a set of points $Q = \{y_1, \dots, y_l\}$ with the property that $\tilde{\mu}(B(y_j, r)) \geq \frac{r}{k^2}$ for all j and $\tilde{\mu}(Q^r) \geq 1 - 2r$.

Step 2. There exists a subset $T = \{x_1, \dots, x_m\}$ of Q, with $m \leq k$, such that:

$$\tilde{\mu}(T^{5r}) \geq 1 - 5r.$$

(This gives a concentration result for the averaged measure $\tilde{\mu}$).

Thanks to Steps 1 and 2, we can conclude as follows: let $\psi = \sum_{j=1}^{k} a_j \psi_j$ be any unit norm section in the direct sum of the first k eigenspaces of D (so that $\sum_j a_j^2 = 1$) and let $\mu = |\psi|^2 \mathrm{dvol}_g$. By the Schwarz inequality we have, at any point:

$$|\psi|^2 \leq \left(\sum_j |a_j||\psi_j| \right)^2 \leq \sum_j |\psi_j|^2,$$

that is, $\mu \leq k\tilde{\mu}$. We deduce

$$\mu((T^{5kr})^c) \leq \mu((T^{5r})^c) \leq k\tilde{\mu}((T^{5r})^c \leq 5kr$$

by Step 2. We now take $S = T$: then $\mu(S^{5kr}) \geq 1 - 5kr$ and the theorem follows.

For the proof of the two steps we need the following lemma. Note that we can assume $r < 1/5$.

Lemma 10. *Assume that there exist $k + 1$ subsets $A_1, \dots, A_{k+1}$ which are $2r$-separated and have $\tilde{\mu}$-measure at least β. Then:*

$$\lambda_{k+1}(D) - \lambda_k(D) \leq \frac{k}{\beta r^2}.$$

Proof. As in the proof of Lemma 5, we can construct $k + 1$ disjointly supported, plateau functions $f_1, \dots, f_{k+1}$ with $R_{\tilde{\mu}}(f_j) \leq \frac{1}{\beta r^2}$ for each j, where $R_{\tilde{\mu}}$ is the Rayleigh quotient relative to the measure $\tilde{\mu}$. As $\tilde{\mu}$ is the average of the μ_j's, we see that for any non-negative function f there is an index i (depending on f) such that: $\int_M f\tilde{\mu} \leq \int_M f\mu_i$. Therefore, for each $j = 1, \dots, k+1$ there is an index $\alpha(j) = 1, \dots, k$ such that:

$$\begin{aligned} R_{\tilde{\mu}}(f_j) &= \frac{\int_M |\nabla f_j|^2 \tilde{\mu}}{\int_M f_j^2 \tilde{\mu}} \\ &\geq \frac{1}{k} \frac{\int_M |\nabla f_j|^2 \mu_{\alpha(j)}}{\int_M f_j^2 \mu_{\alpha(j)}} \\ &\geq \frac{1}{k} \, R_{\mu_{\alpha(j)}}(f_j) \end{aligned}$$

and then $R_{\mu_{\alpha(j)}}(f_j) \leq \dfrac{k}{\beta r^2}$ for all j. We consider the sections $s_j = f_j \psi_{\alpha(j)}$ for $j = 1, \dots, k+1$: they are disjointly supported and we can use them as test-sections for the eigenvalue $\lambda_{k+1}(D)$. Using Lemma 8 one sees that

$$\lambda_{k+1}(D) - \lambda_k(D) \leq \sup_j \{R_{\mu_{\alpha(j)}}(f_j)\} \leq \frac{k}{\beta r^2}.$$

□

Proof of Step 1. For all $j \leq k$ we observe from (20):

$$r \geq 5 \left(\frac{k(k+1) C_d(M)^2}{\lambda_{k+1}(D) - \lambda_j(D)} \right)^{1/3}.$$

So, by Theorem 7, there exist finite subsets $S_1, \dots, S_k \subseteq M$ of cardinality less than or equal to k such that, for all j:

$$\mu_j(S_j^r) \geq 1 - r.$$

We set $P = S_1 \cup \dots \cup S_k$ and observe that, by the definition of $\tilde{\mu}$:

$$\tilde{\mu}(P^r) \geq 1 - r. \tag{21}$$

We now consider the subset $Q = \{y_1, \dots, y_l\}$ formed by all points $y_j \in P$ such that $\tilde{\mu}(B(y_j, r)) \geq \dfrac{r}{k^2}$. Let $Q' = P \setminus Q$. Then by definition $\tilde{\mu}((Q')^r) \leq r$. Since, by (21), $\tilde{\mu}((Q')^r) + \tilde{\mu}(Q^r) \geq 1 - r$ we obtain

$$\tilde{\mu}(Q^r) \geq 1 - 2r \tag{22}$$

as asserted. Note in particular that Q is not empty because $r < 1/5$ by assumption.

Proof of Step 2. We construct the subset $T = \{x_1, \dots, x_m\}$ of Q as follows. Set:

$$x_1 = y_1.$$

If there exists some point $y_j \in Q$ in the complement of $B(x_1, 4r)$ we select it and denote it by x_2. Next, if there exists a point of Q in the complement of $B(x_1, 4r) \cup B(x_2, 4r)$ we select it and denote it by x_3, and so on. We iterate the process until it is possible, and obtain after $m \leq l$ steps the required subset T.

Assume that $m \geq k+1$. Then the balls $A_j = B(x_j, r)$ with $j = 1, \ldots, k+1$ are $2r$-separated by construction, and have $\tilde{\mu}$-measure at least equal to $\beta = \frac{r}{k^2}$. By Lemma 10 we see that:

$$\lambda_{k+1}(D) - \lambda_k(D) \leq \frac{k^3}{r^3}. \tag{23}$$

However, the definition (20) of r gives $\lambda_{k+1}(D) - \lambda_k(D) = \frac{c}{r^3}$ with the constant $c = 125k^2(k+1)C_d(M)^2 > k^3$ and we get a contradiction with (23).

Therefore $m \leq k$.

By the construction of T, every point $y_j \in Q$ is at distance not greater than $4r$ to some point of T, that is

$$Q \subseteq T^{4r}.$$

By the triangle inequality $Q^r \subseteq T^{5r}$ and therefore, by (22)

$$\tilde{\mu}(T^{5r}) \geq \tilde{\mu}(Q^r) \geq 1 - 2r > 1 - 5r,$$

and Step 2 follows.

4 Appendix

4.1 Facts about the Levy-Prokhorov distance

Recall that the Lévi-Prokhorov distance d_P between two probability mesaures defined on the same metric space (M, d) is:

$$d_P(\nu_1, \nu_2) = \inf\{r > 0 : \nu_1(C) \leq \nu_2(C^r) + r \text{ and } \nu_2(C) \leq \nu_1(C^r) + r \text{ for all } C\}.$$

Proposition 11. *Let (M, μ, d) be a m-m-space, $S = \{x_1, \ldots, x_k\}$ a set of k points in M and $r > 0$. Then $\mu(S^r) \geq 1 - r$ if and only if there exist weights $p_1, \ldots, p_k \in [0, 1)$ such that $\sum p_j = 1$ and*

$$d_P(\mu, \delta) \leq r,$$

where $\delta = \sum_{i=1}^k p_i \delta_{x_i}$ and δ_{x_i} is the Dirac measure concentrated at the point x_i.

Proof. Suppose first that $d_P(\mu, \delta) \leq r$. Then, choosing $C = S$ in the definition of d_P, we have

$$1 = \delta(S) \leq \mu(S^r) + r$$

and therefore $\mu(S^r) \geq 1 - r$.

To prove the converse, we assume $\mu(S^r) \geq 1 - r$. We first define the weights p_i.

Denote by B_i the ball $B(x_i, r)$ and consider the sets $\{A_i\}_{i=1}^k$ defined by:

$$\begin{cases} A_1 = B_1 \\ A_i = B_i \cap (B_1 \cup ... \cup B_{i-1})^c \quad \text{for } i \geq 2. \end{cases}$$

Then $A_i \subseteq B_i$ and and $A_i \cap A_j = \emptyset$ if $i \neq j$. Set $A = A_1 \cup ... \cup A_k$. Then $A = B_1 \cup ... \cup B_k = S^r$ so that

$$\mu(A) = \mu(S^r) \geq 1 - r.$$

- We now choose the weights $p_i = \dfrac{\mu(A_i)}{\mu(A)}$.

The proof is complete if we show that, for each Borel subset C, we have

$$\begin{cases} \delta(C) \leq \mu(C^r) + r \\ \mu(C) \leq \delta(C^r) + r. \end{cases} \tag{24}$$

We can order the points so that $x_1, \ldots, x_t \in C$ and $x_j \notin C$ for $j = t+1, \ldots, k$. Then $\delta(C) = p_1 + \cdots + p_t$. Now $B_1 \cup \cdots \cup B_t \subseteq C^r$; as $A_i \subseteq B_i$ and the A_i's are pairwise disjoint we have:

$$\mu(A_1) + \cdots + \mu(A_t) \leq \mu(B_1 \cup \cdots \cup B_t) \leq \mu(C^r).$$

Then:

$$\begin{aligned} \delta(C) &= p_1 + \cdots + p_t \\ &= \frac{\mu(A_1) + \cdots + \mu(A_t)}{\mu(A)} \\ &= \mu(A_1) + \cdots + \mu(A_t) + \frac{\mu(A_1) + \cdots + \mu(A_t)}{\mu(A)}(1 - \mu(A)) \\ &\leq \mu(C^r) + 1 - \mu(A) \\ &\leq \mu(C^r) + r \end{aligned}$$

which proves the first inequality in (24). For the second, write

$$\mu(C) = \mu(C \cap A_1) + ... + \mu(C \cap A_k) + \mu(C \cap A^c)$$

and note that, if $C \cap A_i \neq \emptyset$, then $x_i \in C^r$. As $\mu(C \cap A_i) \leq \mu(A_i) = p_i \mu(A) \leq p_i$ and $\mu(C \cap A^c) \leq \mu(A^c) \leq r$ we have

$$\begin{aligned} \mu(C) &\leq \sum_{i: x_i \in C^r} p_i + r \\ &\leq \delta(C^r) + r \end{aligned}$$

and the Proposition follows. □

4.2 Theorem 1 is sharp

For $R > 0$, let M_R be the surface of revolution in $\mathbf{R}^3$:

$$M_R = \{(x, y, z) \in \mathbf{R}^3 : y^2 + z^2 = \frac{1}{R^2} e^{-2Rx}, x \in [0, 1]\},$$

and consider the metric measure space (M_R, μ, d) where μ is the normalized Riemannian measure and d is the extrinsic distance inherited from $\mathbf{R}^3$. By a calculation in [FT] one knows that

$$\lambda_2(M_R) \geq \frac{1}{8} R^2 \tag{25}$$

(we take the Neumann boundary conditions). By the equivalent formulation of Theorem 1, given in Theorem 2, for each R there exists a point $p \in M_R$ such that:

$$d_P(\mu, \delta_p) \leq \gamma_R \frac{\log \lambda_R}{\sqrt{\lambda_R}}$$

for the constant $\gamma_R = 16 C_d(M_R)^2$, where we set $\lambda_R = \lambda_2(M_R)$. However, as we use the extrinsic distance, the constant γ_R admits a uniform upper bound by the packing constant of $\mathbf{R}^3$ (see section 1.1) hence:

$$d_P(\mu, \delta_p) \leq \gamma \frac{\log \lambda_R}{\sqrt{\lambda_R}}, \tag{26}$$

for some point $p \in M_R$ and for an absolute constant γ (we can take in fact $\gamma = 16(1+3^6)^2$).

Now, as $R \to \infty$ the first positive eigenvalue $\lambda_R \to \infty$ by (25). Therefore, by (26), the normalized Riemannian measure μ concentrates at some point of M_R: this is quite evident and can be verifed directly from the definition of M_R, because the limit metric measure space as $R \to \infty$ (in any reasonable sense) is the unit interval $[0, 1]$ endowed with its canonical distance and the Dirac measure supported at 0. In fact, one can check that the relative measure of a set at positive distance α from the circle $\{x = 0\}$ decreases to zero like $e^{-\alpha R}$.

The scope of this section is to show that, apart from the constant γ, the inequality (26) is actually sharp.

Theorem 12. *Let M_R and λ_R be as above. Then there exists R_0 such that, for all $R \geq R_0$ and for all $q \in M_R$ one has:*

$$d_P(\mu, \delta_q) \geq \frac{1}{48} \frac{\log \lambda_R}{\sqrt{\lambda_R}}.$$

For the proof, we use the following simple fact:

Lemma 13. *Assume that there exist two subsets A, B with relative volume at least s, and such that $d(A,B) \geq 2s$. Then $d_P(\mu, \delta_q) \geq s$ for all $q \in M_R$.*

Proof. Assume that there exists $q \in M_R$ such that $d_P(\mu, \delta_q) < s$. One sees from the definition of d_P that $\mu(B(q,s)) > 1 - s$ and therefore $\mu(B(q,s)) + \mu(A) > 1$. So A must intersect $B(q,s)$ and there exists $a \in A$ such that $d(a,q) < s$. Similarly, there exists $b \in B$ with $d(b,q) < s$. Applying the triangle inequality we get a contradiction with the assumption $d(A,B) \geq 2s$. □

We can now prove the theorem.

By (25) one has $\lambda_R > \dfrac{R^2}{9}$ hence, for R large, $\dfrac{1}{48}\dfrac{\log \lambda_R}{\sqrt{\lambda_R}} \leq \dfrac{1}{8}\dfrac{\log R}{R}$. So, it is enough to show that

$$d_P(\mu, \delta_q) \geq \frac{1}{8}\frac{\log R}{R} \tag{27}$$

for R large and for all $q \in M_R$.

For $L < L'$ in the interval $[0,1]$ consider the strip

$$M_{[L,L']} = \{(x,y,z) \in M_R : L \leq x \leq L'\}.$$

We will apply the Lemma taking:

$$A = M_{[0,\frac{1}{R}]}, \quad B = M_{[\frac{1}{2}\frac{\log R}{R},1]}, \quad s = \frac{1}{8}\frac{\log R}{R}.$$

We need the following simple volume estimate:

$$\mu(M_{[L,L']}) \geq \frac{e^{-LR} - e^{-L'R}}{2(1-e^{-R})}. \tag{28}$$

In fact, observe that M_R is obtained by rotating the curve $y = \frac{1}{R}e^{-Rx}$ around the x-axis. Then:

$$\mathrm{Vol}(M_{[L,L']}) = \frac{2\pi}{R}\int_L^{L'} e^{-Rx} ds,$$

with $ds = \sqrt{1+e^{-2Rx}}dx$. Inequality (28) now follows observing that $dx \leq ds < 2dx$ and recalling that $\mu(M_{[L,L']}) = \frac{\mathrm{Vol}(M_{[L,L']})}{\mathrm{Vol}(M_{[0,1]})}$.

By the volume estimate in (28):

$$\begin{cases} \mu(A) \geq \dfrac{1-e^{-1}}{2(1-e^{-R})}, \\ \mu(B) \geq \dfrac{R^{-\frac{1}{2}}-e^{-R}}{2(1-e^{-R})}, \\ d(A,B) \geq \dfrac{1}{2}\dfrac{\log R}{R} - \dfrac{1}{R}. \end{cases}$$

It is now clear that, for $R \geq R_0$ sufficiently large, one has $\mu(A) \geq s, \mu(B) \geq s$ and $d(A,B) \geq 2s$. The Lemma gives $d_P(\mu, \delta_q) \geq s = \dfrac{1}{8}\dfrac{\log R}{R}$ and the theorem is proved.

4.3 Example for differential forms

We will now construct an example with a large gap on the spectrum of the Laplacian on p-forms, but in which there is no concentration of the Riemannian volume.

Indeed, the construction of large eigenvalues for p-forms is well-known, see [GP], [Gu], [CE]. We can easily adapt the construction described in [GP] for an hypersurface in $\mathbf{R}^{n+1}$, and we will only briefly sketch it.

We begin with a given hypersurface $M_0 \subset \mathbf{R}^{n+1}$, with p-th De Rham cohomology space of a given positive dimension. Then we deform M_0 by adding a long cylinder $[0, L] \times \mathbf{S}^{n-1}$ closed by a hemisphere. We denote by M_L this family of manifolds, whose volume is of the order of L as $L \to \infty$. It is shown in [GP] that, for $2 \leq p \leq n-2$, the nonzero p-forms spectrum of M_L is bounded below by a positive constant C not depending on L.

After renormalisation by a factor of order $L^{-1/n}$, we get a family of constant volume 1, with first nonzero eigenvalue for p-forms going to ∞ with L. Using the extrinsic Euclidean distance, we see that the packing constant is uniformly bounded, we can conclude that the L^2-norm of the harmonic forms have to concentrate, indeed on the part corresponding to M_0.

However, there is no concentration of the volume (the part M_0 concentrate to a point and the cylinder looks like a homogeneous 1-dimensional cylinder of length $L^{1-1/n}$).

4.4 Expanders

In this section we construct a family of manifolds $\bar{M}_i$ of fixed dimension n, such that $\lambda_2(\bar{M}_i) \to \infty$ but for which there is no concentration of the volume around any point.

We start from an n-dimensional compact, hyperbolic manifold M_i such that $\mathrm{Vol}(M_i) \to \infty$ as $i \to \infty$ and $\lambda_2(M_i) \geq C(n) > 0$, where $C(n)$ is a constant not depending on i. It is well known that such examples exist (see for example [Br]), even if their construction, related to the concept of expanders, is not easy. The M_i's can be realized as coverings of a fixed manifold. It is also known that the diameter of M_i is proportional to $\ln \mathrm{Vol}(M_i)$, hence tends to infinity as $i \to \infty$.

So, if we multiply the metric of M_i by $(\mathrm{diam}(M_i))^{-1}$, and denote by $\bar{M}_i$ the new family of Riemannian manifolds, it is clear that $\lambda_2(\bar{M}_i) \to \infty$ but $\mathrm{diam}\bar{M}_i = 1$. As $\bar{M}_i$ is a covering, the distribution of the volume is uniform, and we see that it cannot concentrate in a neighbourghood of a single point. It concentrates however in the sense described in [CGY1]: two sets $A_i, B_i \subset \bar{M}_i$ of volume $\geq \kappa \mathrm{Vol}(\bar{M}_i)$ (with a fixed $\kappa > 0$) have to be very close to each other, even if κ is small.

References

[B] Bérard, P. *From vanishing theorems to estimating theorems: the Bochner method revisited*, Bull. Amer. Math. Soc. 19 (1988), 371-406.

[Br] Brooks, R. *The spectral geometry of a tower of coverings*, J. Differential Geom. 23 (1986), no. 1, 97–107

[CE] Colbois, B.; El Soufi, A. *Eigenvalues of the Laplacian acting on p-forms and metric conformal deformations*, Proc. Amer. Math. Soc. 134 (2006), no. 3, 715–721

[CGY1] Chung, F. R. K.; Grigor'yan, A.; Yau, S.-T. *Upper bounds for eigenvalues of the discrete and continuous Laplace operators*, Adv. Math. 117 (1996), no. 2, 165–178.

[CGY2] Chung, F. R. K.; Grigor'yan, A.; Yau, S.-T. *Eigenvalues and diameters for manifolds and graphs*, Tsing Hua lectures on geometry and analysis (Hsinchu, 1990–1991), 79–105, Int. Press, Cambridge, MA, 1997.

[Che] Cheng, S.Y.; *Eigenvalue comparison theorems and its geometric applications*, Math. Z. 143 (1975), no. 3, 289–297.

[CM] Colbois B., Maerten D., *Eigenvalues estimate for the Neumann problem of a bounded domain* J. Geom. Analysis 18 (2008) 1022-1032

[FT] Friedmann J., Tillich J-P.; *Laplacian eigenvalues and distances between subsets of a manifold*, J. Differential Geom. 56 (2000), no. 2, 285–299.

[GM] Gromov, M.; Milman, V. D. *A topological application of the isoperimetric inequality*, Amer. J. Math. 105 (1983), no. 4, 843–854

[GNY] Grigor'yan, A; Netrusov, Y.; Yau, S-T; *Eigenvalues of elliptic operators and geometric applications*, Surveys in differential geometry. Vol. IX, 147–217 (2004).

[GP] Gentile, G.; Pagliara, V. *Riemannian metrics with large first eigenvalue on forms of degree p*, Proc. Amer. Math. Soc. 123 (1995), no. 12, 3855–3858.

[Gr] Gromov, M. *Metric structures for Riemannian and non-Riemannian spaces.* Progress in Mathematics, 152. Birkhuser Boston, Inc., Boston, MA, 1999.

[Gu] Guerini, P. *Prescription du spectre du laplacien de Hodge-de Rham*, Ann. Sci. Ecole Norm. Sup. (4) 37 (2004), no. 2, 270–303.

[GY] Grigor'yan, A.; Yau, S.-T. *Decomposition of a metric space by capacitors* Proc. Sympos. Pure Math. 65 (1996) 39-75.

[V] Villani, C.; *Optimal Transport, Old and New* Grundlehren der mathematischen Wissenschaften , Vol. 338 , Springer (2009).

Bruno Colbois
Université de Neuchâtel, Institut de Mathématiques, Rue Emile Argand 11, CH-2007, Neuchâtel, Suisse
bruno.colbois@unine.ch

Alessandro Savo
Dipartimento di Metodi e Modelli Matematici, Sapienza Università di Roma, Via Antonio Scarpa 16, 00161 Roma, Italy
savo@dmmm.uniroma1.it